ON CYCLIC HIGGS BUNDLES

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ABSTRACT. In this paper, we derive a maximum principle for a type of elliptic systems and apply it to analyze the Hitchin equation for cyclic Higgs bundles. We show several domination results on the pullback metric of the (possibly branched) minimal immersion f associated to cyclic Higgs bundles. Also, we obtain a lower and upper bound of the extrinsic curvature of the image of f. As an application, we give a complete picture for maximal $Sp(4, \mathbb{R})$ -representations in the 2g-3Gothen components and the Hitchin components.

1. INTRODUCTION

Let S be a closed, oriented surface of genus $g \ge 2$ and G be a reductive Lie group. Let Σ be a Riemann surface over S and denote its canonical line bundle by K_{Σ} . A G-Higgs bundle over Σ is a pair (E, ϕ) where E is a holomorphic vector bundle and ϕ is a holomorphic section of $End(E) \otimes K_{\Sigma}$ plus extra condition depending on G. The non-abelian Hodge theory developed by Corlette [9], Donaldson [12], Hitchin [14] and Simpson [22], provides a one-to-one correspondence between the moduli space of representations from $\pi_1(S)$ to G with the moduli space of G-Higgs bundles over Σ . The correspondence is through looking for an equivariant harmonic map from $\tilde{\Sigma}$ to the symmetric space G/K, where K is the maximal compact subgroup of G, for a given representation ρ or a given Higgs bundle (E, ϕ) .

In this paper, we are interested in the direction of the non-abelian Hodge correspondence from the moduli space of Higgs bundles to the space of equivariant harmonic maps. More explicitly, given a polystable G-Higgs bundle (E, ϕ) on Σ , there exists a unique Hermitian metric h compatible with G-structure satisfying the Hitchin equation

$$F^{\nabla^h} + \left[\phi, \phi^{*_h}\right] = 0,$$

called the harmonic metric, which gives the equivariant harmonic map from $\widetilde{\Sigma}$ to G/K. So for a given Higgs bundle (E, ϕ) , we would like to deduce geometric properties of the corresponding equivariant harmonic map: $\widetilde{\Sigma} \to G/K$.

We are particularly interested in the following $SL(n, \mathbb{C})$ -Higgs bundles

$$E = L_1 \oplus L_2 \oplus \dots \oplus L_n, \quad \phi = \begin{pmatrix} 0 & & \gamma_n \\ \gamma_1 & 0 & & \\ & \ddots & \ddots & \\ & & \gamma_{n-1} & 0 \end{pmatrix} : E \to E \otimes K_{\Sigma},$$

where L_k is a holomorphic line bundle and γ_k is a holomorphic section of $L_k^{-1}L_{k+1}K_{\Sigma}$, $k = 1, \dots, n$ $(L_{n+1} = L_1)$. Suppose det $E = \mathcal{O}$ and $\gamma_k \neq 0$, $k = 1, \dots, n-1$. Call such a Higgs bundle (E, ϕ) a cyclic Higgs bundle parameterized by $(\gamma_1, \gamma_2, \dots, \gamma_n)$. For G a subgroup of $SL(n, \mathbb{C})$, we call (E, ϕ) a cyclic G-Higgs bundle if it is a G-Higgs bundle and it is cyclic as a $SL(n, \mathbb{C})$ -Higgs bundle.

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The terminology "cyclic Higgs bundles" first appeared in [2]. Note that the notion here is a bit different from the one in [2], where the notion "cyclic" there is referred to the group G. One may also view cyclic Higgs bundles as a special type of quiver bundles in [1]. Cyclic Higgs bundles are special in G-Higgs bundles for G of higher rank. The harmonic metric for a cyclic Higgs bundle is diagonal, making it possible to analyze the solution to the Hitchin equation and hence the corresponding harmonic map. So studying cyclic Higgs bundles could give us hint on predicting what may happen to general Higgs bundles.

If a representation $\rho : \pi_1(S) \to SL(n, \mathbb{C})$ does not correspond to a cyclic Higgs bundle over one Riemann surface Σ , it is still possible that ρ corresponds to a cyclic Higgs bundle over another Riemann surface Σ' . By Labourie [17], any Hitchin representation for $SL(n, \mathbb{R})$ cannot correspond to a cyclic Higgs bundle over a deformation family of Riemann surfaces and later Collier [5] generalizes to a more general family of cyclic Higgs bundles.

If $n \ge 3$, the associated harmonic map for a cyclic Higgs bundle is conformal and hence is a (possibly branched) minimal immersion. In [10], the authors studied the pullback metric and curvature of the minimal immersion for cyclic Higgs bundles in the Hitchin component (the q_n case). In this paper, we derive a maximum principle for the elliptic systems. The maximum principle is very useful for the Toda-type equation with function coefficient, which appears in the Hitchin equation for cyclic Higgs bundles. With this powerful tool, we generalize and improve the results in [10] and discover some new phenomena.

1.1. Monotonicity of pullback metrics. Let (E, ϕ) be a cyclic $SL(n, \mathbb{C})$ -Higgs bundle parameterized by $(\gamma_1, \dots, \gamma_n), n \ge 3$. Let f be the corresponding harmonic map and it is in fact branched minimal. The Riemannian metric on $SL(n, \mathbb{C})/SU(n)$ is induced by the Killing form on $sl(n, \mathbb{C})$. Then the pullback metric of f is given by

$$g = 2n \operatorname{tr}(\phi \phi^{*_h}) dz \otimes d\bar{z},$$

where h is the harmonic metric. Though at branch points g = 0, we still call g a "metric".

There is a nature \mathbb{C}^* -action on the moduli space M_{Higgs} of $SL(n,\mathbb{C})$ -Higgs bundles:

$$\mathbb{C}^* \times \mathcal{M}_{Higgs} \longrightarrow \mathcal{M}_{Higgs} t \cdot (E, \phi) = (E, t\phi)$$

Theorem 1.1. Let (E, ϕ) be a cyclic Higgs bundle. Then along the \mathbb{C}^* -orbit of (E, ϕ) , outside the branched points, as |t| increases, the pullback metric g^t of the corresponding branched minimal immersions strictly increases.

If we integrate the pullback metric, it gives the Morse function f (up to a constant scalar) on the moduli space of Higgs bundles as the L^2 -norm of ϕ :

$$f(E,\phi) = \int_{\Sigma} \operatorname{tr}(\phi\phi^*) \sqrt{-1} dz \wedge d\bar{z}.$$

Corollary 1.2. Let (E, ϕ) be a cyclic Higgs bundle. Then along the \mathbb{C}^* -orbit of (E, ϕ) , the Morse function $f(E, t\phi)$ strictly increases as |t| increases.

Remark 1.3. The Morse function is the main tool to determine the topology of the moduli space of Higgs bundles, for example, in Hitchin [14, 15], Gothen [13]. The monotonicity in Corollary 1.2 is not new. In fact, Hitchin in [14] showed that with respect to the Kähler metric on the moduli space, the gradient flow of the Morse function is exactly the \mathbb{R}^* -part of \mathbb{C}^* -action. Hence, along \mathbb{C}^* -orbit of any Higgs bundles (E, ϕ) , the Morse function $f(E, t\phi)$ strictly increases as |t| increases. Here we improve the integral monotonicity to pointwise monotonicity along \mathbb{C}^* -orbit of cyclic Higgs bundles.

Consider the family of cyclic Higgs bundles (E, ϕ^t) parameterized by $(\gamma_1, \dots, t\gamma_n)$. For $t \in \mathbb{C}^*$, the family (E, ϕ^t) is gauge equivalent to $t^{\frac{1}{n}} \cdot (E, \phi) = (E, t^{\frac{1}{n}} \phi)$. If the cyclic Higgs bundle parameterized by $(\gamma_1, \dots, \gamma_{n-1}, 0)$ is again stable, in this case $\sum_{i=1}^k \deg(L_{n+1-k}) < 0$ for all $1 \le k \le n-1$, we extend the monotonicity of \mathbb{C}^* -family to the \mathbb{C} -family.

Theorem 1.4. Let (E, ϕ^t) be a cyclic Higgs bundle parameterized by $(\gamma_1, \dots, t\gamma_n)$ for $t \in \mathbb{C}$. If the cyclic Higgs bundle parameterized by $(\gamma_1, \dots, \gamma_{n-1}, 0)$ is stable, then outside the branched points, as |t| increases, the pullback metric g^t of corresponding branched minimal immersions for (E, ϕ^t) strictly increases. So does the Morse function.

Remark 1.5. If the cyclic Higgs bundle parameterized by $(\gamma_1, \dots, \gamma_{n-1}, 0)$ is stable, it lies in the moduli space of Higgs bundles and is fixed by the \mathbb{C}^* -action. Note that the fixed points of \mathbb{C}^* -action are exactly the critical points of the Morse function as shown in Hitchin [14].

1.2. Curvature of cyclic Higgs bundles in the Hitchin component. By Hitchin's description [15] of the Higgs bundle in the Hitchin component, the cyclic Higgs bundles in the Hitchin component are of the following form

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \dots \oplus K^{\frac{3-n}{2}} \oplus K^{\frac{1-n}{2}}, \quad \phi = \begin{pmatrix} 0 & q_n \\ 1 & 0 & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

where q_n is a holomorphic *n*-differential. We call such a Higgs bundle (E, ϕ) a cyclic Higgs bundle in the Hitchin component parameterized by q_n . If $q_n = 0$, the Higgs bundle is called Fuchsian.

The corresponding harmonic map $f: \widetilde{\Sigma} \to SL(n, \mathbb{R})/SO(n)$ is a minimal immersion for $n \geq 3$. We want to investigate that, as an immersed submanifold, how the image $f(\tilde{\Sigma})$ sits inside the symmetric space $SL(n,\mathbb{R})/SO(n)$.

Theorem 1.6. Let (E, ϕ) be a cyclic Higgs bundle in the Hitchin component parameterized by q_n . Let σ be the tangent plane of the image of f, then the curvature K_{σ} in $SL(n,\mathbb{R})/SO(n)$ satisfies

$$-\frac{1}{n(n-1)^2} \le K_{\sigma} < 0.$$

Remark 1.7. The sectional curvature K of $SL(n,\mathbb{R})/SO(n)$ and $SL(n,\mathbb{C})/SU(n)$ satisfies $-\frac{1}{n} \leq \frac{1}{n}$ $K \leq 0$ (see Proposition 5.1). For general Higgs bundles, one should not expect there is such a nontrivial lower bound at immersed points. For example, in the case of cyclic Higgs bundles parametrized by $(\gamma_1, \gamma_2, \dots, \gamma_n)$, if n-1 terms of γ_i 's have a common zero point, then the curvature of the tangent plane σ at that point achieves the most negative, i.e., $K_{\sigma} = -\frac{1}{n}$.

Remark 1.8. (1) The upper bound is shown in [10]. Here we give a new proof. As shown in [7], along the family of Higgs bundles parameterized by tq_n , K_{σ}^t approaches to 0 away from the zeros of $q_n \text{ as } |t| \to \infty$. So the upper bound $K_{\sigma} < 0$ is sharp. (2) The lower bound $-\frac{1}{n(n-1)^2}$ can only be achieved at some point in the case n = 2, 3.

(3) In the Fuchsian case, i.e. $q_n = 0$, the sectional curvature K_{σ} is $-\frac{6}{n^2(n^2-1)}$. However, it is strictly larger than the lower bound of K_{σ} for $q_n \neq 0$ case when n > 3. Hence, one cannot expect the curvature in Fuchsian case could serve as a lower bound of K_{σ} for general Hitchin representations.

For details one may see the remarks in the end of Section 5.

1.3. Comparison inside the real Hitchin fibers. Fix a Riemann surface Σ , the Hitchin fibration is a map from the moduli space of $SL(n,\mathbb{C})$ -Higgs bundles over Σ to the direct sum of the holomorphic differentials

$$h: M_{Higgs} \longrightarrow \bigoplus_{j=2}^{n} H^{0}(\Sigma, K^{j}) \ni (q_{2}, q_{3}, \cdots, q_{n}).$$
$$(E, \phi) \mapsto (\operatorname{tr}(\phi^{2}), \operatorname{tr}(\phi^{3}), \cdots, \operatorname{tr}(\phi^{n}))$$

Note that cyclic Higgs bundles (E, ϕ) lie in the Hitchin fiber at $(0, \dots, 0, n \cdot q_n)$, where $q_n = (-1)^{n-1} \det(\phi)$. There is one special point in each Hitchin fiber at $(0, \dots, 0, n \cdot q_n)$: the cyclic Higgs bundle in the Hitchin component parametrized by q_n .

In Proposition 6.1, we show that the harmonic metric in the cyclic Higgs bundle in the Hitchin component dominates the ones for other cyclic $SL(n, \mathbb{R})$ -Higgs bundles in the same Hitchin fiber in a certain sense.

As the applications in lower rank n = 2, 3, 4, we compare the pullback metric of the harmonic map for the cyclic Higgs bundle in the Hitchin component with the ones for other cyclic $SL(n, \mathbb{R})$ -Higgs bundles in the same Hitchin fiber at $(0, \dots, 0, n \cdot q_n)$.

Theorem 1.9. Let $(\tilde{E}, \tilde{\phi})$ be a cyclic Higgs bundle in the Hitchin component parameterized by q_n and (E, ϕ) be a distinct cyclic $SL(n, \mathbb{R})$ -Higgs bundle in Section 2.3 such that $\det \phi = (-1)^{n-1}q_n$. For n = 2, 3, 4, the pullback metrics g, \tilde{g} of the corresponding harmonic maps satisfy $g < \tilde{g}$.

Under the assumptions above, the Morse function satisfies $f(E, \phi) < f(\tilde{E}, \tilde{\phi})$.

By Hitchin's work in [14], all polystable $SL(2,\mathbb{R})$ -Higgs bundles with nonvanishing Higgs field are cyclic. We can then directly apply Theorem 1.9 to $SL(2,\mathbb{R})$ -representations, we recover the following result shown in [11].

Corollary 1.10. For any non-Fuchsian reductive $SL(2,\mathbb{R})$ -representation ρ and any Riemann surface Σ , there exists a Fuchsian representation j such that the pullback metric of the corresponding j-equivariant harmonic map $f_j: \widetilde{\Sigma} \to \mathbb{H}^2$ dominates the one for f_{ρ} .

Remark 1.11. Deroin and Tholozan in [11] show a stronger result by comparing Fuchsian representations with all $SL(2, \mathbb{C})$ -representations and the condition being reductive can be removed by separate consideration. Inspired by this result, they conjecture that in the Hitchin fiber, the Hitchin section maximizes the translation length. Our Theorem 1.9 here is exactly in the same spirit, but using the pullback metric rather than the translation length.

We expect that Theorem 1.9 holds for general Higgs bundles rather than just cyclic Higgs bundles.

Conjecture 1.12. Let (E, ϕ) be a Higgs bundle in the Hitchin component and (E, ϕ) be a distinct $SL(n, \mathbb{R})$ -Higgs bundle in the same Hitchin fiber at (q_2, q_3, \dots, q_n) . Then the pullback metrics g, \tilde{g} of corresponding harmonic maps satisfy $g < \tilde{g}$. As a result, the Morse function satisfies $f(E, \phi) < f(\tilde{E}, \tilde{\phi})$.

1.4. Maximal $Sp(4,\mathbb{R})$ -representations. For each reductive representation ρ into a Hermitian Lie group G, we can define a Toledo integer $\tau(\rho)$ satisfying the Milnor-Wood inequality $|\tau(\rho)| \leq \operatorname{rank}(G)(g-1)$. The representation ρ with $|\tau(\rho)| = \operatorname{rank}(G)(g-1)$ is called maximal. Maximal representations are Anosov [4] and hence discrete and faithful.

In the case for $Sp(4,\mathbb{R})$, there are $3 \cdot 2^{2g} + 2g - 4$ connected components of maximal representations containing 2^{2g} isomorphic components of Hitchin representations [15] and 2g - 3 exceptional components called Gothen components [13]. Labourie in [17] shows that any Hitchin representation corresponds to a cyclic Higgs bundle in the Hitchin component parametrized by q_4 over a unique Riemann surface. Together with the description in [13, 3] and Collier's work [5], any maximal representation for $Sp(4,\mathbb{R})$ in the Gothen components corresponds to a cyclic Higgs bundle over a unique Riemann surface Σ of the form

$$E = N \oplus NK^{-1} \oplus N^{-1}K \oplus N^{-1}, \quad \phi = \begin{pmatrix} 0 & & \nu \\ 1 & 0 & \\ & \mu & 0 \\ & & 1 & 0 \end{pmatrix},$$

where $g-1 < \deg N < 3g-3$. Note that if $N = K^{\frac{3}{2}}$, the above Higgs bundle corresponds to a Hitchin representation. As a result, for any $Sp(4,\mathbb{R})$ -representation in the Hitchin components or Gothen components, there is a unique ρ -equivariant minimal immersion of \widetilde{S} in $Sp(4,\mathbb{R})/U(2)$. Recently, this result is reproved and generalized to maximal SO(2,n)-representations in Collier-Tholozan-Toulisse [8].

For each Riemann surface, the above cyclic Higgs bundles with $\nu = 0$ play a similar role as the Fuchsian case: they are the fixed points of the \mathbb{C}^* -action. We call the corresponding representations μ -Fuchsian representations. The only difference with the Fuchsian case is that they form a subset inside each component rather than one single point since $\mu \in H^0(N^{-2}K^3)$ has many choices. As a corollary of Theorem 1.4, the space of μ -Fuchsian representations serves as the minimum set in its component of maximal $Sp(4,\mathbb{R})$ representations in the following sense.

Corollary 1.13. For any maximal $Sp(4, \mathbb{R})$ -representation ρ in the 2g-3 Gothen components (or the Hitchin components), there exists a μ -Fuchsian (or Fuchsian) representation j in the same component of ρ such that the pullback metric of the unique j-equivariant minimal immersion $f_j:$ $\tilde{S} \to Sp(4, \mathbb{R})/U(2)$ is dominated by the one for f_{ρ} .

To consider the curvature, as a corollary of Theorem 1.6, we have

Corollary 1.14. For any Hitchin representation ρ for $Sp(4,\mathbb{R})$, the sectional curvature K_{σ} in $Sp(4,\mathbb{R})/U(2)$ of the tangent plane σ of the unique ρ -equivariant minimal immersion satisfies (1) $K_{\sigma} = -\frac{1}{40}$, if ρ is Fuchsian; (2) $-\frac{1}{36} < K_{\sigma} < 0$ and $\exists p$ such that $K_{\sigma}(p) < -\frac{1}{40}$, if ρ is not Fuchsian.

Similarly, we also obtain an upper and lower bound on the curvature of minimal immersions for maximal representations.

Theorem 1.15. For any maximal representation ρ for $Sp(4, \mathbb{R})$ in each Gothen component, the sectional curvature K_{σ} in $Sp(4, \mathbb{R})/U(2)$ of tangent plane σ of the uniuqe ρ -equivariant minimal immersion satisfies

(1) $-\frac{1}{8} \leq K_{\sigma} < -\frac{1}{40}$ and the lower bound is sharp, if ρ is μ -Fuchsian; (2) $-\frac{1}{8} \leq K_{\sigma} < 0$, if ρ is not μ -Fuchsian.

Remark 1.16. As shown in [7],[19], along the family of $(E, t\phi)$, away from zeros of $det(\phi) \neq 0$, the sectional curvature goes to zero as $|t| \to \infty$. So the upper bounds in Part (2) in both Corollary 1.14 and Theorem 1.15 are sharp. The sectional curvature K in $Sp(4, \mathbb{R})/U(2)$ satisfies $-\frac{1}{4} \leq K \leq 0$. So the lower bounds in Corollary 1.14 and Theorem 1.15 are nontrivial.

As an immediate corollary of Theorem 1.9 for n = 4, comparing maximal representations in the Gothen components with Hitchin representations, we obtain

Corollary 1.17. For any maximal $Sp(4,\mathbb{R})$ -representation ρ in the 2g-3 Gothen components, there exists a Hitchin representation j such that the pullback metric of the unique j-equivariant minimal immersion $f_j: \widetilde{S} \to Sp(4,\mathbb{R})/U(2)$ dominates the one for f_{ρ} . 1.5. Maximum principle. We derive a maximum principle for the elliptic systems. It is the main tool we use throughout this paper.

Basically, we consider the following linear elliptic system

$$\Delta_g u_i + \langle X, \nabla u_i \rangle + \sum_{j=1}^n c_{ij} u_j = f_i, \quad 1 \leq i \leq n.$$

Roughly speaking, suppose the functions c_{ij} satisfy the following assumptions:

(a) cooperative: $c_{ij} \ge 0, i \ne j,$

(b) column diagonally dominant: $\sum_{i=1}^{n} c_{ij} \leq 0, \ 1 \leq j \leq n$,

(c) fully coupled: the index set $\{1, \dots, n\}$ cannot be split up in two disjoint nonempty sets α, β such that $c_{ij} \equiv 0$ for $i \in \alpha, j \in \beta$.

Then the maximum principle holds, that is, if $f_i \leq 0$ for $1 \leq i \leq n$, then $u_i \geq 0$ for $1 \leq i \leq n$. The precise statement is Lemma 3.1.

In the literature, it is common to require there exists a positive supersolution, which is equivalent to the maximum principle, see [18]. So for function coefficients, people usually suppose c_{ij} satisfy the row sum condition $\sum_{j=1}^{n} c_{ij} \leq 0$, $1 \leq i \leq n$, say [21]. The column sum condition $\sum_{i=1}^{n} c_{ij} \leq 0$, $1 \leq j \leq n$, or in other words column diagonally dominant condition, rarely appeared in the literature. The similar column sum condition first appeared in [18], Theorem 3.3.

To the knowledge of the authors, the maximum principles in the literature seem not to directly imply our maximum principle Lemma 3.1. We also remark that our proof is more elementary.

Structure of the article. The article is organized as follows. In Section 2, we recall some fundamental results about the Higgs bundle and introduce the cyclic Higgs bundles. In Section 3, we show a maximum principle for the elliptic systems, the main tool of this article. In Section 4, we show the monotonicity of the pullback metrics of the branched minimal immersions. In Section 5, we find out a lower and upper bound for the extrinsic curvature of the minimal immersions for cyclic Higgs bundles in the Hitchin component. In Section 6, we compare the harmonic metrics of cyclic Higgs bundle in the Hitchin component with other cyclic $SL(n,\mathbb{R})$ -Higgs bundles in the same Hitchin fiber. In Section 7, we apply our results to maximal $Sp(4,\mathbb{R})$ -representations.

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2. Preliminaries

In this section, we recall some facts in the theory of the Higgs bundles. One may refer [2][10][17]. Let Σ be a closed Riemann surface of genus ≥ 2 and $K = K_{\Sigma}$ be the canonical line bundle over Σ . For $p \in \Sigma$, let $\pi_1 = \pi_1(\Sigma, p)$ be the fundamental group of Σ . Let $\tilde{\Sigma}$ be the universal cover of Σ .

A $SL(n,\mathbb{C})$ -Higgs bundle over Σ is a pair (E,ϕ) , where E is a holomorphic vector bundle with det $E = \mathcal{O}$ and ϕ is a trace-free holomorphic section of $End(E) \otimes K$. We call (E,ϕ) is stable if for any proper ϕ -invariant holomorphic subbundle F, $\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E}$. We call (E,ϕ) is polystable if (E,ϕ) is a direct sum of stable Higgs bundles of degree 0.

2.1. Higgs bundles and harmonic maps.

Theorem 2.1. (Hitchin [14] and Simpson [22]) Let (E, ϕ) be a stable $SL(n, \mathbb{C})$ -Higgs bundle. Then there exists a unique Hermitian metric h on E compatible with $SL(n, \mathbb{C})$ structure, called the harmonic metric, solving the Hitchin equation

(1)
$$F^{\nabla^h} + [\phi, \phi^{*_h}] = 0,$$

where ∇^h is the Chern connection of h, in local holomorphic trivialization,

$$F^{\nabla^h} = \overline{\partial}(h^{-1}\partial h),$$

and ϕ^{*_h} is the adjoint of ϕ with respect to h, in the sense that

$$h(\phi(u), v) = h(u, \phi^{*_h}(v)) \in K, \quad u, v \in E$$

in local frame, $\phi^{*_h} = \bar{h}^{-1} \bar{\phi}^{\mathsf{T}} \bar{h}$.

Denote

$$G = SL(n, \mathbb{C}), \quad K = SU(n)$$

$$\mathfrak{g} = sl(n, \mathbb{C}), \quad \mathfrak{k} = su(n), \quad \mathfrak{p} = \{X \in sl(n, \mathbb{C}) : \overline{X}^t = X\}, \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

The Killing form $B(X, Y) = 2n \cdot \operatorname{tr}(XY)$

The harmonic metric h gives rise to a flat $SL(n, \mathbb{C})$ connection $D = \nabla^h + \Phi = \nabla^h + \phi + \phi^{*h}$. The holonomy of D gives a representation $\rho : \pi_1 \to SL(n, \mathbb{C})$ and the bundle (E, D) is isomorphic to $\tilde{\Sigma} \times_{\rho} \mathbb{C}^n$ with the associated flat connection. A Hermitian metric h on E is equivalent to a reduction $i : P_K \to P_G$ from unimodule frame bundle $P_G = \tilde{\Sigma} \times_{\rho} G$ of $E = \tilde{\Sigma} \times_{\rho} \mathbb{C}^n$ to the unitary frame bundle P_K of E with respect to h. Then it descends to be a section of $P_G/K = \tilde{\Sigma} \times_{\rho} G/K$ over Σ . Equivalently, it gives a ρ -equivariant map $f : \tilde{\Sigma} \to G/K$. Denote the bundle \tilde{P}_K be the pullback of the principle K-bundle $G \to G/K$ by f. Note that $\pi^* P_K = \tilde{P}_K$, where π is the covering map $\pi : \tilde{\Sigma} \to \Sigma$. The Maurer-Cartan form ω of G gives a flat connection on P_G , we still use ω to denote the connection. It coincides with the flat connection D. Consider $i^*\omega$, which is a \mathfrak{g} -value one form on P_K . Decomposing $i^*\omega = A + \Phi$ from $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where A is \mathfrak{k} -valued and Φ is \mathfrak{p} -valued. Then $A \in \Omega^1(P_K, \mathfrak{k})$ is a principal connection on P_K and Φ is a section of $T^*\Sigma \otimes (P_K \times_{Ad_K} \mathfrak{p})$. By complexification, Φ is also a section of

$$(T^*\Sigma\otimes\mathbb{C})\otimes(P_K\times_{Ad_K}\mathfrak{p}\otimes\mathbb{C}) = (T^*\Sigma\otimes\mathbb{C})\otimes(P_{K^{\mathbb{C}}}\times_{Ad_{K^{\mathbb{C}}}}\mathfrak{p}^{\mathbb{C}})$$
$$= (K\oplus\bar{K})\otimes(P_G\times_{Ad_G}\mathfrak{g}) = (K\oplus\bar{K})\otimes End_0(E)$$

where $End_0(E)$ the trace-free endormorphism bundle of E. With respect to the decomposition $(K \oplus \overline{K}) \otimes End_0(E), \Phi = \phi + \phi^*.$

With respect to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we can decompose $\omega = \omega^{\mathfrak{k}} + \omega^{\mathfrak{p}}$, where $\omega^{\mathfrak{k}} \in \Omega^1(G, \mathfrak{k}), \omega^{\mathfrak{p}} \in \Omega^1(G, \mathfrak{p})$. Moreover, $\omega^{\mathfrak{p}}$ descends to be an element in $\Omega^1(G/K, G \times_{Ad_K} \mathfrak{p})$. In fact, using the Maurer-Cartan form $\omega^{\mathfrak{p}} \in \Omega^1(G/K, G \times_{Ad_K} \mathfrak{p})$ over G/K: $T(G/K) \cong G \times_{Ad_K} \mathfrak{p}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ gives an Ad_K -invariant orthogonal decomposition and the Killing form B on \mathfrak{g} is positive on \mathfrak{p} . The Killing form B induces a Riemannian metric \tilde{B} on G/K: for two vectors $Y_1, Y_2 \in T_p(G/K)$,

$$\ddot{B}(Y_1, Y_2) = B(\omega^{\mathfrak{p}}(Y_1), \omega^{\mathfrak{p}}(Y_2)).$$

Then $f^*\omega^{\mathfrak{p}}$ is a section of $T^*\tilde{\Sigma}\otimes(\tilde{P}_K\times_{Ad_K}\mathfrak{p})$ over $\tilde{\Sigma}$.

By comparing the two decomposition of the Maurer-Cartan form ω , we obtain:

$$f^*\omega^{\mathfrak{p}} = \pi^*\Phi.$$

So for every tangent vector $X \in T\widetilde{\Sigma}$, under the isomorphism by the Maurer-Cartan form

$$\omega^{\mathfrak{p}}: T(G/K) \cong G \times_{Ad_K} \mathfrak{p},$$

we have

(2)
$$\omega^{\mathfrak{p}}(f_*(X)) = f^*\omega^{\mathfrak{p}}(X) = \pi^*\Phi(X) = \Phi(\pi_*(X))$$

We consider the pullback metric g on Σ , $g = \pi_* f^* \tilde{B}$. Since f is ρ -equivariant and \tilde{B} is G-invariant, g is well defined. Then $\forall X, Y \in T\Sigma$, locally choose any lift $\tilde{X}, \tilde{Y} \in T\tilde{\Sigma}$,

$$g(X,Y) = \tilde{B}(f_*(\tilde{X}), f_*(\tilde{Y})) = B(\omega^{\mathfrak{p}}(f_*(\tilde{X})), \omega^{\mathfrak{p}}(f_*(\tilde{Y})) = B(\Phi(X), \Phi(Y)).$$

Later in the paper, we may ignore this covering map π for short. Then we have

$$\operatorname{Hopf}(f) = g^{2,0} = 2n\operatorname{tr}(\phi\phi), \quad g^{1,1} = 2n\operatorname{tr}(\phi\phi^{*_h})dz \otimes d\bar{z}.$$

If Hopf(f) = 0, then as a section of $K \otimes \overline{K}$, the Hermitian metric is

$$g = g^{1,1} = 2n \operatorname{tr}(\phi \phi^{*_h}) dz \otimes d\bar{z}$$

The associated Riemannian metric of g is $g + \bar{g}$ on Σ , i.e., $2n \operatorname{tr}(\phi \phi^{*_h}) dz \cdot d\bar{z}$, where

$$dz \cdot d\overline{z} = dz \otimes d\overline{z} + d\overline{z} \otimes dz = 2|dz|^2 = 2(dx^2 + dy^2).$$

We focus on the cyclic Higgs bundles introduced below.

2.2. Cyclic Higgs bundles. A cyclic Higgs bundle is a $SL(n, \mathbb{C})$ -Higgs bundle (E, ϕ) of the following form

$$E = L_1 \oplus L_2 \oplus \dots \oplus L_n, \quad \phi = \begin{pmatrix} 0 & & \gamma_n \\ \gamma_1 & 0 & & \\ & \ddots & \ddots & \\ & & \gamma_{n-1} & 0 \end{pmatrix},$$

where L_k is a holomorphic line bundle over Σ and γ_k is a holomorphic section of $L_k^{-1}L_{k+1}K$, $k = 1, \dots, n$. The subscript is counted modulo n, i.e., $n + 1 \equiv 1$. Here det $E = \mathcal{O}$ and $\gamma_k \neq 0$, $k = 1, \dots, n - 1$. If $\gamma_n \neq 0$, (E, ϕ) is automatically stable, which implies the existence of the solution to the Hitchin equation (1). If $\gamma_n = 0$, (E, ϕ) stable in this case means $\sum_{i=1}^k \deg(L_{n+1-i}) < 0$ for all $1 \leq k \leq n - 1$.

Following the proof in Baraglia [2], Collier [5, 6], the harmonic metric is diagonal for cyclic Higgs bundles. We include the proof here for completeness.

Proposition 2.2. For a cyclic Higgs bundle (E, ϕ) , the harmonic metric h is diagonal, i.e.

$$h = diag(h_1, h_2, \cdots, h_n)$$

where each h_k is a Hermitian metric on L_k .

Proof. For $\omega = e^{\frac{2\pi i}{n}}$, consider the holomorphic $SL(n, \mathbb{C})$ -gauge transformation g_{ω} :

$$g_{\omega} = \begin{pmatrix} \omega^{\frac{n-1}{2}} & & \\ & \omega^{\frac{n-3}{2}} & \\ & & \ddots & \\ & & & \omega^{\frac{1-n}{2}} \end{pmatrix} : E \to E$$

It acts on the Higgs field ϕ as follows

$$g_{\omega} \cdot \phi = g_{\omega} \phi g_{\omega}^{-1} = \omega \cdot \phi$$

Then the metric $hg_{\omega}^{*h}g_{\omega}$ is a solution to the Higgs bundle $(g_{\omega}^{-1}\bar{\partial}_E g_{\omega}, g_{\omega}^{-1}\phi g_{\omega}) = (\bar{\partial}_E, \omega^{-1} \cdot \phi)$. Since U(1)-action does not change the harmonic metric, $hg_{\omega}^{*h}g_{\omega}$ is also the solution to the Higgs bundle (∂_E, ϕ) . Hence, by the uniqueness of harmonic metrics,

$$h = h g_{\omega}^{*_h} g_{\omega}.$$

Then h splits as (h_1, h_2, \dots, h_n) .

Denote $L \otimes \overline{L} = |L|^2$, then h_k is a smooth section of $|L_k|^{-2}$. Chosen a local holomorphic frame, we abuse γ_k to denote the local coefficient function of the section γ_k . Then locally the Hitchin equation is

$$\Delta \log h_k + |\gamma_k|^2 h_k^{-1} h_{k+1} - |\gamma_{k-1}|^2 h_{k-1}^{-1} h_k = 0, \quad k = 1, \cdots, n,$$

where $\Delta = \partial_z \bar{\partial}_z$, $|\gamma_k|^2 = \gamma_k \bar{\gamma}_k$ as a local function.

If $n \ge 3$, the Hopf differential of the harmonic map Hopf $(f) = tr(\phi^2) = 0$. And f is immersed at p if and only if $\phi(p) \ne 0$. At point p where $\phi(p) = 0$, f is branched at p. Then outside the branch points, the harmonic map is conformal, then minimal. The pullback metric is given by

$$g = 2n \operatorname{tr}(\phi \phi^{*_h}) = 2n (\sum_{k=1}^n |\gamma_k|^2 h_k^{-1} h_{k+1}) dz \otimes d\bar{z}.$$

Remark 2.3. For n = 2, we consider the (1,1) part of the pullback metric g instead.

2.3. Cyclic $SL(n, \mathbb{R})$ -Higgs bundles. A $SL(n, \mathbb{R})$ -Higgs bundle over Σ is a triple (E, ϕ, Q) , where (E, ϕ) is a $SL(n, \mathbb{C})$ -Higgs bundle and Q is a non-degenerate holomorphic quadratic form on E such that $Q(\phi u, v) = Q(u, \phi v)$ for $u, v \in E$. Such (E, ϕ, Q) corresponds to a representation

$$\rho: \pi_1 \to SL(n, \mathbb{R}) \hookrightarrow SL(n, \mathbb{C}).$$

Here we consider the holomorphic quadratic form

$$Q = \begin{pmatrix} & & 1 \\ & 1 & \\ & \ddots & \\ 1 & & \end{pmatrix} : E \xrightarrow{\cong} E^*.$$

For n = 2m, the cyclic $SL(n, \mathbb{R})$ -Higgs bundle is of the following form

$$E = L_1 \oplus \dots \oplus L_m \oplus L_m^{-1} \oplus \dots \oplus L_1^{-1}, \quad \phi = \begin{pmatrix} 0 & & & & \nu \\ \gamma_1 & \ddots & & & & \\ & \ddots & 0 & & & \\ & & \gamma_{m-1} & 0 & & \\ & & & \mu & 0 & & \\ & & & & \gamma_{m-1} & 0 & \\ & & & & \ddots & \ddots & \\ & & & & & & \gamma_1 & 0 \end{pmatrix}.$$

By the uniqueness of the solution, $h = \text{diag}(h_1, \dots, h_m, h_m^{-1}, \dots, h_1^{-1})$. Locally, the Hitchin equation is

$$\Delta \log h_1 + |\gamma_1|^2 h_1^{-1} h_2 - |\nu|^2 h_1^2 = 0,$$

$$\Delta \log h_k + |\gamma_k|^2 h_k^{-1} h_{k+1} - |\gamma_{k-1}|^2 h_{k-1}^{-1} h_k = 0, \quad k = 2, \cdots, m-1,$$

$$\Delta \log h_m + |\mu|^2 h_m^{-2} - |\gamma_{m-1}|^2 h_{m-1}^{-1} h_m = 0.$$

The pullback metric is $g = 2n(|\nu|^2 h_1^2 + |\mu|^2 h_m^{-2} + 2\sum_{k=1}^{m-1} |\gamma_k|^2 h_k^{-1} h_{k+1}) dz \otimes d\overline{z}.$

For n = 2m + 1, the cyclic $SL(n, \mathbb{R})$ -Higgs bundle is of the following form

$$E = L_{1} \oplus \dots \oplus L_{m} \oplus \mathcal{O} \oplus L_{m}^{-1} \oplus \dots \oplus L_{1}^{-1}, \quad \phi = \begin{pmatrix} 0 & & & & \nu \\ \gamma_{1} & \ddots & & & & \\ & \ddots & 0 & & & & \\ & & \gamma_{m-1} & 0 & & & \\ & & & \mu & 0 & & \\ & & & & \mu & 0 & & \\ & & & & & \gamma_{m-1} & 0 & \\ & & & & & & \gamma_{1} & 0 \end{pmatrix},$$

In this case, $h = \text{diag}(h_1, \dots, h_m, 1, h_m^{-1}, \dots, h_1^{-1})$. Locally, the Hitchin equation is

$$\Delta \log h_1 + |\gamma_1|^2 h_1^{-1} h_2 - |\nu|^2 h_1^2 = 0, \Delta \log h_k + |\gamma_k|^2 h_k^{-1} h_{k+1} - |\gamma_{k-1}|^2 h_{k-1}^{-1} h_k = 0, \quad k = 2, \cdots, m-1, \Delta \log h_m + |\mu|^2 h_m^{-1} - |\gamma_{m-1}|^2 h_{m-1}^{-1} h_m = 0.$$

The pullback metric is $g = 2n(|\nu|^2 h_1^2 + 2|\mu|^2 h_m^{-1} + 2\sum_{k=1}^{m-1} |\gamma_k|^2 h_k^{-1} h_{k+1}) dz \otimes d\overline{z}.$

2.4. Hitchin fibration and cyclic Higgs bundles in the Hitchin component. Fix a Riemann surface Σ , the Hitchin fibration is a map

$$h: M_{Higgs}(SL(n,\mathbb{C})) \longrightarrow \bigoplus_{j=2}^{n} H^{0}(\Sigma, K^{j}) \ni (q_{2}, q_{3}, \dots, q_{n})$$

given by $h([E, \phi]) = (\operatorname{tr}(\phi^2), \dots, \operatorname{tr}(\phi^n)).$

In [15], Hitchin defines a section s_h of this fibration whose image consists of stable Higgs bundles with corresponding flat connections having holonomy in $SL(n, \mathbb{R})$. Furthermore, the section s_h maps surjectively to the connected component (called Hitchin component) of the $SL(n, \mathbb{R})$ -Higgs bundle moduli space which naturally contains an embedded copy of Teichmüller space. The Teichmüller locus is corresponding to the image of $q_3 = \cdots = q_n = 0$. Such a (E, ϕ) corresponds to a representation ρ which can be factored through $SL(2, \mathbb{R})$,

$$\rho: \pi_1 \to SL(2,\mathbb{R}) \xrightarrow{\iota} SL(n,\mathbb{R}) \hookrightarrow SL(n,\mathbb{C}),$$

where ι is the canonical irreducible representation.

The cyclic Higgs bundles in the Hitchin component are corresponding to the image of s_h at $(0, \dots, 0, n \cdot q_n)$. More precisely

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \dots \oplus K^{\frac{3-n}{2}} \oplus K^{\frac{1-n}{2}}, \quad \phi = \begin{pmatrix} 0 & q_n \\ 1 & 0 & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

where q_n is a holomorphic *n*-differential.

If $q_n = 0$, the Higgs bundle is Fuchsian. For n = 2m,

$$h_k^{-1}h_{k+1} = \frac{1}{2}k(n-k)g_0, \quad 1 \le k \le m-1, \qquad h_m^{-2} = \frac{1}{2}m^2g_0;$$

for n = 2m + 1,

$$h_k^{-1}h_{k+1} = \frac{1}{2}k(n-k)g_0, \quad 1 \le k \le m-1, \qquad h_m^{-1} = \frac{1}{2}m(m+1)g_0.$$

Here g_0 is the hyperbolic metric such that $\Delta \log g_0 = g_0$.

3. MAXIMUM PRINCIPLE FOR SYSTEM

The main tool we use in this paper is the following maximum principle for system. We abuse the same notation g to denote both the metric $g(z)dz \otimes d\overline{z}$ and the local function g(z) on the surface. Define $\Delta_g = g^{-1}\Delta$, which is globally defined, called the Laplacian with respect to the metric $gdz \otimes d\overline{z}$.

Lemma 3.1. Let (Σ, g) be a closed Riemannian manifold. For each $1 \le i \le n$, let u_i be a C^2 function on $\Sigma \ P_i$, where P_i is an isolated subset of Σ (P_i can be empty). Suppose u_i approaches to $+\infty$ around P_i . Let $P = \bigcup_{i=1}^n P_i$. Let c_{ij} be continuous and bounded functions on $\Sigma \ P$, $1 \le i, j \le n$. Suppose c_{ij} satisfy the following assumptions: in $\Sigma \ P$, (a) cooperative: $c_{ij} \ge 0$, $i \ne j$,

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(b) column diagonally dominant: $\sum_{i=1}^{n} c_{ij} \leq 0, \ 1 \leq j \leq n$,

(c) fully coupled: the index set $\{1, \dots, n\}$ cannot be split up in two disjoint nonempty sets α, β such that $c_{ij} \equiv 0$ for $i \in \alpha, j \in \beta$.

Let f_i be non-positive continuous functions on $\Sigma \setminus P$, $1 \leq i \leq n$ and X be a continuous vector field on Σ . Suppose u_i satisfies

$$\Delta_g u_i + \langle X, \nabla u_i \rangle + \sum_{j=1}^n c_{ij} u_j = f_i \text{ in } \Sigma \smallsetminus P, \quad 1 \leq i \leq n.$$

Consider the following conditions:

Condition (1) $(f_1, \dots, f_n) \neq (0, \dots, 0)$, i.e., there exists $i_0 \in \{1, \dots, n\}$, $p_0 \in \Sigma \setminus P$, such that $f_{i_0}(p_0) \neq 0$; Condition (2) P is nonempty;

Condition (3) $\sum_{i=1}^{n} u_i \ge 0$.

Then either condition (1) or (2) imply $u_i > 0$, $1 \le i \le n$. And condition (3) implies either $u_i > 0$, $1 \le i \le n$ or $u_i \equiv 0$, $1 \le i \le n$.

Proof. Let
$$A = \{1, \dots, n\}$$
. For $S \subseteq A$, set $u_S = \sum_{i \in S} u_i$. If $S = \phi$, set $u_S = 0$. Let $P_S = \bigcup_{i \in S} P_i$. Then
 $\triangle_g u_S + \langle X, \nabla u_S \rangle + \sum_{i \in A} \sum_{l \in S} c_{li} u_i \leq 0$ in $\Sigma \smallsetminus P$.

Then

$$\Delta_g u_S + \langle X, \nabla u_S \rangle + \sum_{j \notin S} \sum_{l \in S} c_{lj} u_j + \sum_{k \in S} \sum_{l \in S} c_{lk} u_k \le 0 \text{ in } \Sigma \smallsetminus P.$$

Then for $S \neq \phi$,

$$\Delta_g u_S + \langle X, \nabla u_S \rangle + \sum_{j \notin S} \sum_{l \in S} c_{lj} (u_{\{j\} \cup S} - u_S) + \sum_{k \in S} (\sum_{l \in S} c_{lk}) (u_S - u_{S \setminus \{k\}}) \leq 0 \text{ in } \Sigma \setminus P.$$

Set

$$b_S = \min_{\Sigma} u_S, \quad \check{b}_S = \min_{j \notin S, k \in S} \{ b_{\{j\} \cup S}, b_{S \smallsetminus \{k\}} \}, \quad b = \min_{S \subseteq A} b_S,$$

Notice that all these constants are finite. By the assumptions (a)(b), in $\Sigma \setminus P$, $c_{lj} \ge 0$ for $l \in S, j \notin S$, and $\sum_{l \in S} c_{lk} \le 0$ for $k \in S$, then

$$\Delta_{g}\check{b}_{S} + \langle X, \nabla\check{b}_{S} \rangle + \sum_{j \notin S} \sum_{l \in S} c_{lj} (u_{\{j\} \cup S} - \check{b}_{S}) + \sum_{k \in S} (\sum_{l \in S} c_{lk}) (\check{b}_{S} - u_{S \setminus \{k\}}) \ge 0 \text{ in } \Sigma \setminus P_{S}$$

Then

$$\Delta_g(u_S - \check{b}_S) + \langle X, \nabla(u_S - \check{b}_S) \rangle + \left(-\sum_{j \notin S} \sum_{l \in S} c_{lj} + \sum_{k \in S} (\sum_{l \in S} c_{lk})\right) (u_S - \check{b}_S) \le 0 \text{ in } \Sigma \smallsetminus P.$$

Step 1: We show that under condition (1) and (2), $u_S \ge \dot{b}_S$ for any $S \subset A$; under condition (3), $u_S \ge \dot{b}_S$ for $S \not\subseteq A$. In particular, $b_S \ge \dot{b}_S$ for $S \subset A$ under condition (1) and (2) and for $S \not\subseteq A$ under condition (3).

If not, since $u_S - \check{b}_S$ approaches to $+\infty$ around P_S and continuous outside P_S , $u_S - \check{b}_S$ must attain a negative minimum in $\Sigma \setminus P_S$. First, we suppose $u_S - \check{b}_S$ is not a constant. By the assumptions (a)(b), in $\Sigma \setminus P$,

$$-\sum_{j\notin S}\sum_{l\in S}c_{lj} + \sum_{k\in S}\left(\sum_{l\in S}c_{lk}\right) \le 0.$$

Then by the strong maximum principle for the single equation (see [20]), the minimal point $p \notin \Sigma \setminus P$. So $p \in P \setminus P_S$. Since P is isolated, we consider $p_n \in \Sigma \setminus P$, $p_n \to p$. Then

$$\limsup_{p_n \to p} \left(\bigtriangleup_g (u_S - \check{b}_S) + \langle X, \nabla(u_S - \check{b}_S) \rangle + \left(-\sum_{\substack{j \notin S \ l \in S}} c_{lj} + \sum_{k \in S} (\sum_{l \in S} c_{lk}) \right) (u_S - \check{b}_S) \right) (p_n) \le 0.$$

By the continuity,

$$\lim_{p_n \to p} \left(\bigtriangleup_g (u_S - \check{b}_S) + \langle X, \nabla(u_S - \check{b}_S) \rangle \right) (p_n) = \left(\bigtriangleup_g (u_S - \check{b}_S) + \langle X, \nabla(u_S - \check{b}_S) \rangle \right) (p) \ge 0.$$

 So

$$\limsup_{p_n \to p} \left(\left(-\sum_{j \notin S} \sum_{l \in S} c_{lj} + \sum_{k \in S} (\sum_{l \in S} c_{lk}) \right) (u_S - \check{b}_S) \right) (p_n) \le 0.$$

If there exists a subsequence p_{n_k} such that $(-\sum_{j \notin S} \sum_{l \in S} c_{lj} + \sum_{k \in S} (\sum_{l \in S} c_{lk}))(p_{n_k})$ approaches to a negative number, then

$$\lim_{p_{n_k}\to p} \left(\left(-\sum_{j\notin S} \sum_{l\in S} c_{lj} + \sum_{k\in S} (\sum_{l\in S} c_{lk}) \right) (u_S - \check{b}_S) \right) (p_{n_k}) > 0.$$

Contradiction. Since P_S is isolated, we have $-\sum_{j \notin S} \sum_{l \in S} c_{lj} + \sum_{k \in S} (\sum_{l \in S} c_{lk})$ is continuous in $\Sigma \setminus P_S$. Then

$$\Delta_g(u_S - \check{b}_S) + \langle X, \nabla(u_S - \check{b}_S) \rangle + (-\sum_{j \notin S} \sum_{l \in S} c_{lj} + \sum_{k \in S} (\sum_{l \in S} c_{lk}))(u_S - \check{b}_S) \le 0 \text{ in } \Sigma \smallsetminus P_S,$$

Then by the strong maximum principle for the single equation, $u_S - \check{b}_S$ cannot achieve its negative minimum in $\Sigma \setminus P_S$ unless it is a constant. Contradiction. Second, if $u_S - \check{b}_S$ is a negative constant, then by the assumptions (a)(b), in $\Sigma \setminus P$,

$$-\sum_{j\notin S}\sum_{l\in S}c_{lj}+\sum_{k\in S}\left(\sum_{l\in S}c_{lk}\right)\equiv 0.$$

Then in $\Sigma \setminus P$, $\sum_{l \in S} c_{lk} \equiv 0$ for $k \in S$. Then by the assumptions (a)(b), $c_{ij} \equiv 0$ in $\Sigma \setminus P$, for $j \in S, i \notin S$, which is a contradiction to the assumption (c) unless S = A. If S = A, for condition (2), we have u_S cannot be a constant. And for condition $(1), u_S - \check{b}_S$ is a negative constant implies $\sum_{i \in A} f_i \equiv 0$, which also gives a contraction. So we obtain $u_S \ge \check{b}_S$ on the whole Σ . For condition (3), we obtain $u_S \ge \check{b}_S$ for $S \not\subseteq A$. So we finish the claim.

Step 2: We show b = 0.

Since $u_S = 0$ for $S = \phi$, we have $b \le 0$. If b < 0, suppose b is achieved by S_0 , and $|S_0|$ is the smallest among all minimizers. Then $S_0 \neq \phi$. Under condition (1) and (2), $u_{S_0} \ge \check{b}_{S_0}$ is automatically true. Under condition (3), we have $u_A \ge 0$ and hence $S_0 \not\subseteq A$, $u_{S_0} \ge \check{b}_{S_0}$.

Since c_{ij} are bounded, suppose $-\sum_{j \notin S_0} \sum_{l \in S_0} c_{lj} + \sum_{k \in S_0} (\sum_{l \in S_0} c_{lk}) \ge -M$, where M is a positive constant. Then in $\Sigma \setminus P$,

We have proved $u_{S_0} - \check{b}_{S_0} \ge 0$. Then by the continuity,

$$\Delta_g(u_{S_0} - \check{b}_{S_0}) + \langle X, \nabla(u_{S_0} - \check{b}_{S_0}) \rangle - M(u_{S_0} - \check{b}_{S_0}) \le 0 \text{ in } \Sigma \setminus P_{S_0}$$

Since $b \leq \check{b}_{S_0} \leq b_{S_0}$ and u_{S_0} achieves b, we have $\check{b}_{S_0} = b$. Then by the strong maximum principle, $u_{S_0} \equiv \check{b}_{S_0} = b$. Then

$$\Delta_g b + \langle X, \nabla b \rangle + \sum_{j \notin S_0} \sum_{l \in S_0} c_{lj} (u_{\{j\} \cup S_0} - b) + \sum_{k \in S_0} (\sum_{l \in S_0} c_{lk}) (b - u_{S_0 \setminus \{k\}}) \le 0 \text{ in } \Sigma \setminus P.$$

Then by the assumptions (a)(b),

$$\left(\sum_{l \in S_0} c_{lk}\right) (b - u_{S_0 \setminus \{k\}}) \equiv 0 \text{ in } \Sigma \setminus P, \text{ for } k \in S_0.$$

If $b - u_{S_0 \setminus \{k\}} = 0$ at one point, then $\dot{b}_{S_0 \setminus \{k\}} = b$, which is a contradiction since $|S_0|$ is the smallest. So in $\Sigma \setminus P$, $\sum_{l \in S_0} c_{lk} \equiv 0$ for $k \in S_0$. As the argument above, it is a contradiction to the assumption (c). Then we obtain b = 0, in particular, $u_i \ge 0$, $1 \le i \le n$.

Step 3: We finish the proof. Since $u_i \ge 0$, we have in $\Sigma \smallsetminus P$,

 $\Delta_q u_i + \langle X, u_i \rangle + c_{ii} u_i \leq 0, \quad 1 \leq i \leq n.$

Then as the argument above, by the strong maximum principle, there exists a subset $Z \subseteq A$, such that $u_i \equiv 0$ for $i \in Z$ and $u_j > 0$ for $j \notin Z$. Then for $i \in Z$, in $\Sigma \setminus P$, $0 \leq \sum_{j \notin Z} c_{ij} u_j = f_i \leq 0$. Since $u_j > 0$ for $j \notin Z$, $c_{ij} \equiv 0$ for $i \in Z$, $j \notin Z$. Suppose condition (1) $(f_1, \dots, f_n) \notin (0, \dots, 0)$ or condition (2) P is nonempty holds, we can rule out the possibility Z = A. Suppose condition (3) $\sum_{i=1}^n u_i \geq 0$ holds, Z must be empty or A. So either $u_i > 0$, $1 \leq i \leq n$ or $u_i \equiv 0$ for $1 \leq i \leq n$.

Remark 3.2. Let λ_i be positive numbers, $i = 1, \dots, n$. Let $u'_i = \lambda_i u_i$, $c'_{ij} = c_{ij}\lambda_i\lambda_j^{-1}$. If c'_{ij} satisfy the assumptions (a)(b)(c), then we still obtain the same results for u_i .

Remark 3.3. The assumption (c) is easy to check by the following procedure. If $1 \in \alpha$, consider $\beta_1 = \{j : c_{1j} \equiv 0\}$, $\alpha_1 = \{1, \dots, n\} \setminus \beta_1$. Then $\alpha_1 \cap \beta = \phi$. Then $\alpha_1 \subseteq \alpha$. Denote $\alpha_0 = \{1\}$. If $\alpha_1 \subseteq \alpha_0$, then $\alpha = \alpha_0$ gives such a partition. If $\alpha_1 \notin \alpha_0$, consider $\beta_2 = \{j : c_{ij} \equiv 0, i \in \alpha_0 \cup \alpha_1\}$, $\alpha_2 = \{1, \dots, n\} \setminus \beta_2$. Then $\alpha_2 \subseteq \alpha$. If $\alpha_2 \subseteq \alpha_0 \cup \alpha_1$, then $\alpha = \alpha_0 \cup \alpha_1$ gives such a partition. If $\alpha_2 \notin \alpha_0 \cup \alpha_1$, consider $\beta_3 = \{j : c_{ij} \equiv 0, i \in \bigcup_{k=0}^2 \alpha_k\}$, $\alpha_3 = \{1, \dots, n\} \setminus \beta_3$. Repeat this procedure, then either we obtain a partition α, β such that $c_{ij} \equiv 0$ for $i \in \alpha, j \in \beta$ or we show that $1 \notin \alpha$. If $1 \notin \alpha$, repeat the procedure above for $2, 3, \dots, n$. Then we can show whether such a partition exists or not.

Remark 3.4. The maximum principle above may be applied to the non-linear version under certain assumptions, by using the linearization

$$F(u_1, \dots, u_n, x) - F(v_1, \dots, v_n, x) = \sum_{j=1}^n (u_j - v_j) \int_0^1 \frac{\partial F}{\partial u_j} (tu_1 + (1-t)v_1, \dots, tu_n + (1-t)v_n, x) dt.$$

For the problems involving poles, we need to check whether the coefficient after linearization is bounded.

4. MONOTONICITY OF PULLBACK METRICS

In this section, we first consider the family of the cyclic Higgs bundles (E, ϕ^t) parametrized by $(\gamma_1, \dots, \gamma_{n-1}, t\gamma_n), n \ge 3$ for $t \in \mathbb{C}$. We show the monotonicity of the pullback metrics of the corresponding branched minimal immersions along the family ϕ^t .

Proposition 4.1. Let (E, ϕ^t) be a family of cyclic Higgs bundles parametrized by $(\gamma_1, \dots, \gamma_{n-1}, t\gamma_n)$, $n \ge 3, \gamma_n \ne 0, t \in \mathbb{C}^*$ and h^t be the corresponding harmonic metrics on E. Then as |t| increases, $h_k^{-1}h_{k+1}$, $k = 1, \dots, n-1$ and $t^2h_n^{-1}h_1$ strictly increase. As a result, outside the branch points, the pullback metric g^t of the corresponding branched minimal immersions strictly increases.

Proof. We show that for 0 < |t'| < |t|, all the terms for t dominate the corresponding terms for t'. Let $u_k = h_k^{-1} h_{k+1}$, $k = 1, \dots, n-1$, $u_n = |t|^2 h_n^{-1} h_1$. Then

$$\Delta \log u_k + |\gamma_{k+1}|^2 u_{k+1} - 2|\gamma_k|^2 u_k + |\gamma_{k-1}|^2 u_{k-1} = 0, \quad k = 1, \dots, n,$$

And \tilde{u}_k are similarly defined for t', satisfying

$$\log \tilde{u}_k + |\gamma_{k+1}|^2 \tilde{u}_{k+1} - 2|\gamma_k|^2 \tilde{u}_k + |\gamma_{k-1}|^2 \tilde{u}_{k-1} = 0, \quad k = 1, \cdots, n,$$

Let $v_k = \log(u_k \tilde{u}_k^{-1})$, then

$$\Delta v_k + |\gamma_{k+1}|^2 \tilde{u}_{k+1} (e^{v_{k+1}} - 1) - 2|\gamma_k|^2 \tilde{u}_k (e^{v_k} - 1) + |\gamma_{k-1}|^2 \tilde{u}_{k-1} (e^{v_{k-1}} - 1) = 0, \quad k = 1, \dots, n,$$

Let

$$c_k = g_0^{-1} |\gamma_k|^2 \tilde{u}_k \int_0^1 e^{(1-t)(v_k)} dt, \ k = 1, \cdots, n.$$

Then v_k 's satisfy

It is easy to check that the above system of equations satisfies the assumptions in Lemma 3.1 and condition (3), since $\sum_{k=1}^{n} v_k = 2\log(\frac{|t|}{|t'|}) > 0$. One can apply the maximum principle Lemma 3.1, then $v_k > 0$, $k = 1, \dots, n$. Then we obtain $u_k > \tilde{u}_k$, $k = 1, \dots, n$. Finally, the monotonicity of g^t follows from $g^t = 2n(\sum_{k=1}^{n-1} |\gamma_k|^2 h_k^{-1} h_{k+1} + |\gamma_n|^2 t^2 h_n^{-1} h_1) dz \otimes d\bar{z}$.

For $t \in \mathbb{C}^*$, the family (E, ϕ^t) is gauge equivalent to $t^{\frac{1}{n}}(E, \phi) = (E, t^{\frac{1}{n}}\phi)$ by the gauge transformation $\psi_t = \text{diag}(t^{\frac{n-1}{2n}}, t^{\frac{n-3}{2n}}, \dots, t^{\frac{3-n}{2n}}, t^{\frac{1-n}{2n}})$, since

$$t^{\frac{1}{n}} \begin{pmatrix} 0 & & \gamma_n \\ \gamma_1 & 0 & & \\ & \ddots & \ddots & \\ & & \gamma_{n-1} & 0 \end{pmatrix} = \psi_t^{-1} \begin{pmatrix} 0 & & t\gamma_n \\ \gamma_1 & 0 & & \\ & \ddots & \ddots & \\ & & \gamma_{n-1} & 0 \end{pmatrix} \psi_t$$

Then we obtain the following results.

Corollary 4.2. Let (E, ϕ) be a cyclic Higgs bundle parametrized by $(\gamma_1, \dots, \gamma_n), n \ge 3$. Let g^t be the pullback metric corresponding to $t\phi$ for $t \in \mathbb{C}^*$. Then outside the branch points, along the \mathbb{C}^* -orbit, q^t strictly increases as |t| increases.

Consider the Morse function f on the moduli space of Higgs bundles as the L^2 -norm of ϕ :

$$f(E,\phi) = \int_{\Sigma} \operatorname{tr}(\phi\phi^*) \sqrt{-1} dz \wedge d\bar{z}.$$

Corollary 4.3. Let (E, ϕ) be a cyclic Higgs bundle. Then along the \mathbb{C}^* -orbit of (E, ϕ) , the Morse function $f(E, t\phi)$ strictly increases as |t| increases.

Applying Proposition 4.1 to $SL(n,\mathbb{R})$ case, we obtain the monotonicity of the harmonic metric.

Corollary 4.4. Let (E, ϕ) be a cyclic $SL(n, \mathbb{R})$ Higgs bundle parameterized by $(\nu, \gamma_1, \dots, \gamma_{m-1}, \mu)$, $\nu \neq 0$. Denote $\nu = \gamma_0$, $\mu = \gamma_m$. Consider a family of $SL(n,\mathbb{R})$ cyclic Higgs bundles parameterized by $(\gamma_0, \dots, t\gamma_l, \dots, \gamma_m)$, $l = 0, \dots, m$ for $t \in \mathbb{C}^*$. Let $h^{t\gamma_l}$ be the corresponding harmonic metrics. Then as |t| increases, $h_k^{t\gamma_l}$ strictly increases for $k = 1, \dots, l$ and $h_k^{t\gamma_l}$ strictly decreases for $k = l + 1, \dots, m$.

If the cyclic Higgs bundles parametrized by $(\gamma_1, \dots, \gamma_{n-1}, 0)$ is stable, we can extend the monotonicity of the pullback metric of \mathbb{C}^* -family to \mathbb{C} -family.

Proposition 4.5. Let (E, ϕ) be a family of cyclic Higgs bundles parametrized by $(\gamma_1, \dots, \gamma_{n-1}, \gamma_n)$, $n \geq 3$, $\gamma_n \neq 0$ and h be the corresponding harmonic metrics on E. If $(E, \tilde{\phi})$ be a family of cyclic Higgs bundles parametrized by $(\gamma_1, \dots, \gamma_{n-1}, 0)$ is stable, then $h_k^{-1}h_{k+1}$, $k = 1, \dots, n-1$ and $h_n^{-1}h_1$ for (E,ϕ) strictly dominate the items for (E,ϕ) . As a result, outside the branch points, the pullback metric g of the corresponding branched minimal immersions for (E, ϕ) strictly dominates the one for (E, ϕ) .

Proof. Set n + 1 = 1, then the equation for h_k is

$$\Delta \log h_k + |\gamma_k|^2 h_k^{-1} h_{k+1} - |\gamma_{k-1}|^2 h_{k-1}^{-1} h_k = 0, \quad k = 1, \cdots, n.$$

Let $u_k = h_k^{-1} h_{k+1}, k = 1, \dots, n$. Then

$$\Delta \log u_1 + |\gamma_2|^2 u_2 - 2|\gamma_1|^2 u_1 = -|\gamma_n|^2 u_n \le 0,$$

$$\Delta \log u_k + |\gamma_{k+1}|^2 u_{k+1} - 2|\gamma_k|^2 u_k + |\gamma_{k-1}|^2 u_{k-1} = 0, \quad k = 2, \cdots, n-2,$$

$$\Delta \log u_{n-1} - 2|\gamma_{n-1}|^2 u_{n-1} + |\gamma_{n-2}|^2 u_{n-2} = -|\gamma_n|^2 u_n \le 0.$$

And \tilde{h}_k, \tilde{u}_k are similarly defined for t = 0.

Let $v_k = \log(u_k \tilde{u}_k^{-1}), k = 1, \dots, n-1$. Then

$$\Delta v_1 + |\gamma_2|^2 \tilde{u}_2(e^{v_2} - 1) - 2|\gamma_1|^2 \tilde{u}_1(e^{v_1} - 1) \leq 0,$$

$$\Delta v_k + |\gamma_{k+1}|^2 \tilde{u}_{k+1}(e^{v_{k+1}} - 1) - 2|\gamma_k|^2 \tilde{u}_k(e^{v_k} - 1) + |\gamma_{k-1}|^2 \tilde{u}_{k-1}(e^{v_{k-1}} - 1) = 0, \quad k = 2, \dots, n-2,$$

$$\Delta v_{n-1} - 2|\gamma_{n-1}|^2 \tilde{u}_{n-1}(e^{v_{n-1}} - 1) + |\gamma_{n-2}|^2 \tilde{u}_{n-2}(e^{v_{n-2}} - 1) \leq 0.$$

Let $c_k = g_0^{-1} |\gamma_k|^2 \tilde{u}_k \int_0^1 e^{(1-t)v_k} dt$, $k = 1, \dots, n-1$. Then v_k 's satisfy

$$\begin{array}{rcl} & & & & & & \\ & & & & \\ & & & \\ & & & \\$$

It is easy to check that the above system of equations satisfies the assumptions in Lemma 3.1 and condition (1), since $\gamma_n \neq 0$. Applying the maximum principle Lemma 3.1, $v_k > 0$, $k = 1, \dots, n-1$. Then we obtain $u_k > \tilde{u}_k$, $k = 1, \dots, n-1$.

Remark 4.6. Proposition 4.1, 4.5 is a generalization of the metric domination theorem in [10] in two aspects: (1) from dominating the Fuchsian case to monotonicity along the \mathbb{C} -family; (2) from cyclic Higgs bundles in the Hitchin component to general cyclic Higgs bundles.

5. Curvature of cyclic Higgs bundles in the Hitchin component

In this section, we would like to obtain a lower and upper bound for the extrinsic curvature of the branched minimal immersion associated to cyclic Higgs bundles. Let's first get to know how big the range of the sectional curvature of the symmetric space is.

Proposition 5.1. Let $G = SL(n, \mathbb{C}), SL(n, \mathbb{R}), Sp(2m, \mathbb{R}) (n = 2m)$, the maximal compact subgroup K = SU(n), SO(n), U(m) respectively. For any tangent plane σ in G/K, the sectional curvature K_{σ} for the associated symmetric space G/K satisfies

$$-\frac{1}{n} \le K_{\sigma} \le 0,$$

where (1) for $SL(n,\mathbb{C}), SL(n,\mathbb{R}), -\frac{1}{n}$ can be achieved by the tangent plane spanned by

$$E_{ij} + E_{ji}, E_{ii} - E_{jj} \quad for \ any \ 1 \le i < j \le n.$$

(2) for $Sp(2m,\mathbb{R})$ where n = 2m, $-\frac{1}{n}$ can be achieved by the tangent plane spanned by

$$E_{i,m+i} + E_{m+i,i}, E_{ii} - E_{m+i,m+i} \quad for \ any \ 1 \le i \le m.$$

Proof. Suppose the Cartan decomposition of the Lie algebra is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The sectional curvature of the plane spanned by the vectors $Y_1, Y_2 \in T_p(G/K)$ is (see [16] for reference)

$$K(Y_1 \wedge Y_2) = \frac{B([\omega^{\mathfrak{p}}(Y_1), \omega^{\mathfrak{p}}(Y_2)], [\omega^{\mathfrak{p}}(Y_1), \omega^{\mathfrak{p}}(Y_2)])}{B(\omega^{\mathfrak{p}}(Y_1), \omega^{\mathfrak{p}}(Y_1)) \cdot B(\omega^{\mathfrak{p}}(Y_2), \omega^{\mathfrak{p}}(Y_2)) - B(\omega^{\mathfrak{p}}(Y_1), \omega^{\mathfrak{p}}(Y_2))^2}.$$

So it is enough by only checking $Y_1, Y_2 \in T_{eK}(G/K) = \mathfrak{p}$. The upper bound is obvious since B is negative definite on \mathfrak{k} , where $[\omega^{\mathfrak{p}}(Y_1), \omega^{\mathfrak{p}}(Y_2)]$ lies.

Now we show the lower bound. Let σ be the plane $\sigma = \operatorname{span}\{Y, Z\}$ where $Y, Z \in \mathfrak{p}$ satisfying $\operatorname{tr}(YZ) = 0, \operatorname{tr}(Y^2) = \operatorname{tr}(Z^2)$. The Killing form $B(Y, Z) = 2n \cdot \operatorname{tr}(YZ)$. Define U = Y + iZ, V = Y - iZ,

then the sectional curvature of the plane $\sigma = \operatorname{span}\{Y, Z\}$ is

$$K_{\sigma} = -\frac{\operatorname{tr}(UVUV) - \operatorname{tr}(U^{2}V^{2})}{n \cdot \operatorname{tr}(UV)^{2}}$$

$$\geq -\frac{\operatorname{tr}(UVUV)}{n \cdot \operatorname{tr}(UV)^{2}} \quad \text{using } \operatorname{tr}(U^{2}V^{2}) = \operatorname{tr}(U^{2}\overline{U^{2}}^{T}) \geq 0$$

$$\geq -\frac{1}{n} \quad \text{using } \operatorname{tr}(A^{2}) \leq \operatorname{tr}(A)^{2}.$$

The equality holds if and only if $U^2 = 0$ and $UV = U\overline{U}^T$ has only one nonzero eigenvalue. In terms of Y, Z, the equality holds if and only if $Y^2 = Z^2$, YZ + ZY = 0, and $Y^2 + Z^2 + i(ZY - YZ)$ has only one nonzero eigenvalue. The rest is by direct calculation.

For general cyclic Higgs bundles, one should not expect a nontrivial lower bound of the extrinsic curvature at immersed points since it could achieve the plane of the most negative curvature in $SL(n, \mathbb{C})/SU(n)$.

Proposition 5.2. For cyclic Higgs bundles parametrized by $(\gamma_1, \gamma_2, \dots, \gamma_n)$, if there exists a point such that n-1 terms of γ_i 's are equal to zero, the sectional curvature of the tangent plane of the associated harmonic map at this point is $-\frac{1}{n}$.

Proof. Firstly, n = 2 case is obvious. Let $n \ge 3$. The associated harmonic map is a possibly branched minimal immersion. The tangent plane σ of the minimal immersion at f(p) inside G/K is spanned by $Y_{f(p)} = f_*(\frac{\partial}{\partial x})$ and $Z_{f(p)} = f_*(\frac{\partial}{\partial y})$. Using the formula (2) in Section 2,

$$\omega^{\mathfrak{p}}(Y) = \Phi(\frac{\partial}{\partial z}) = (\phi + \phi^{*})(\frac{\partial}{\partial x}) = \phi(\frac{\partial}{\partial z}) + \phi^{*}(\frac{\partial}{\partial \overline{z}}),$$

$$\omega^{\mathfrak{p}}(Z) = \Phi(\frac{\partial}{\partial y}) = (\phi + \phi^{*})(\frac{\partial}{\partial y}) = \sqrt{-1}\phi(\frac{\partial}{\partial z}) - \sqrt{-1}\phi^{*}(\frac{\partial}{\partial \overline{z}})$$

One may refer the details in Section 2 in [10]. Hence

$$[\omega^{\mathfrak{p}}(Y),\omega^{\mathfrak{p}}(Z)] = -2\sqrt{-1}[\phi(\frac{\partial}{\partial z}),\phi^{*}(\frac{\partial}{\partial \bar{z}})] = -2\sqrt{-1}[\phi,\phi^{*}](\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}})$$

Since f is conformal, we have $Y \perp Z$. Then the sectional curvature of the plane σ is

$$K_{\sigma} = K(Y \wedge Z) = \frac{B([\omega^{\mathfrak{p}}(Y), \omega^{\mathfrak{p}}(Z)], [\omega^{\mathfrak{p}}(Y), \omega^{\mathfrak{p}}(Z)])}{B(\omega^{\mathfrak{p}}(Y), \omega^{\mathfrak{p}}(Y))B(\omega^{\mathfrak{p}}(Z), \omega^{\mathfrak{p}}(Z))}$$

$$(3) = -\frac{B([\phi, \phi^{*}], [\phi, \phi^{*}])}{B(\phi, \phi^{*})B(\phi, \phi^{*})} = -\frac{\operatorname{tr}([\phi, \phi^{*}][\phi, \phi^{*}])}{2n \cdot \operatorname{tr}(\phi\phi^{*})^{2}}$$

$$= -\frac{(h_{n}^{-1}h_{1}|\gamma_{n}|^{2} - h_{1}^{-1}h_{2}|\gamma_{1}|^{2})^{2} + (h_{1}^{-1}h_{2}|\gamma_{1}|^{2} - h_{2}^{-1}h_{3}|\gamma_{2}|^{2})^{2} + \dots + (h_{n-1}^{-1}h_{n}|\gamma_{n-1}|^{2} - h_{n}^{-1}h_{1}|\gamma_{n}|^{2})^{2}}{2n(h_{n}^{-1}h_{1}|\gamma_{n}|^{2} + h_{1}^{-1}h_{2}|\gamma_{1}|^{2} + \dots + h_{n-1}^{-1}h_{n}|\gamma_{n-1}|^{2})^{2}}$$

In particular, if at point p, there exists k_0 such that $\gamma_i = 0$, for $i \neq k_0$, and $\gamma_{k_0} \neq 0$. Then

$$K_p = -\frac{2(h_{k_0-1}^{-1}h_{k_0}|\gamma_{k_0}|^2)^2}{2n \cdot (h_{k_0-1}^{-1}h_{k_0})^2|\gamma_{k_0}|^2)^2} = -\frac{1}{n}.$$

Remark 5.3. For example, consider the cyclic Higgs bundle $(L \oplus \mathcal{O} \oplus L^{-1}, \begin{pmatrix} 0 & 0 & \beta \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix})$, where $\deg L < \deg K, 0 \neq \alpha \in H^0(L^{-1}K), 0 \neq \beta \in H^0(L^2K)$. Suppose in addition, zeros of β do not contain

all zeros of α . Then at any point where $\alpha = 0, \beta \neq 0$, the map is an immersion and the extrinsic curvature is $-\frac{1}{3}$.

So instead, we restrict ourselves to cyclic Higgs bundles in the Hitchin components. In this case, we obtain a nontrivial lower and upper bound on the extrinsic curvature of the associated minimal immersion into G/K.

Let (E, ϕ) be a cyclic Higgs bundle in the Hitchin component parameterized by $q_n \neq 0$ and (E, ϕ) be the Fuchsian case. Let h, \tilde{h} be the corresponding harmonic metrics. For n = 2m even, define

$$\nu_1 = \frac{h_1^2 |q_n|^2}{h_1^{-1} h_2}, \quad \nu_k = \frac{h_{k-1}^{-1} h_k}{h_k^{-1} h_{k+1}}, \quad k = 2, \cdots, m-1, \quad \nu_m = \frac{h_{m-1}^{-1} h_m}{h_m^{-2}}.$$

Similarly, define

$$\tilde{\nu}_1 = 0, \quad \tilde{\nu}_k = \frac{\tilde{h}_{k-1}^{-1} \tilde{h}_k}{\tilde{h}_k^{-1} \tilde{h}_{k+1}}, \quad k = 2, \cdots, m-1, \quad \tilde{\nu}_m = \frac{\tilde{h}_{m-1}^{-1} \tilde{h}_m}{\tilde{h}_m^{-2}}.$$

By the explicit description of \tilde{h} , $\tilde{h}_k^{-1}\tilde{h}_{k+1} = \frac{1}{2}k(n-k)g_0$ for $k = 1, \dots, m-1$ and $\tilde{h}_m^{-2} = \frac{m(n-m)}{2}g_0$. Here g_0 is the hyperbolic metric.

For n = 2m + 1 odd, $\nu_k, \tilde{\nu}_k, \tilde{h}_k$ are as above except $\nu_m = \frac{h_{m-1}^{-1}h_m}{h_m^{-1}}, \ \tilde{\nu}_m = \frac{\tilde{h}_{m-1}^{-1}\tilde{h}_m}{\tilde{h}_m^{-1}}, \ \tilde{h}_m^{-1} = \frac{m(n-m)}{2}g_0.$

Proposition 5.4. In the above settings,

$$\frac{(k-1)(n-k+1)}{k(n-k)} = \tilde{\nu}_k < \nu_k < 1, \qquad k = 1, \cdots, m$$

Remark 5.5. The inequality $\nu_k < 1$ recovers Lemma 5.3, q_n case in [10]. Here we give a new proof using the maximum principle Lemma 3.1 directly.

Proof. We only prove the case for n = 2m. The proof is similar for n = 2m + 1. The equation system for h_k is

$$\Delta \log h_1 + h_1^{-1} h_2 - |q_n|^2 h_1^2 = 0, \Delta \log h_k + h_k^{-1} h_{k+1} - h_{k-1}^{-1} h_k = 0, \quad k = 2, \dots, m-1, \Delta \log h_m + h_m^{-2} - h_{m-1}^{-1} h_m = 0.$$

Let

$$u_0 = \log(|q_n|^2 h_1^2), \quad u_k = \log(h_k^{-1} h_{k+1}), \quad 1 \le k \le m-1, \quad u_m = \log(h_m^{-2}).$$

By the holomorphicity, $\Delta \log |q_n| = 0$ outside the zeros of q_n . Then outside the zeros of q_n ,

$$\Delta u_0 + 2e^{u_1} - 2e^{u_0} = 0,$$

$$\Delta u_k + e^{u_{k+1}} - 2e^{u_k} + e^{u_{k-1}} = 0, \quad k = 1, \dots, m-1,$$

$$\Delta u_m - 2e^{u_m} + 2e^{u_{m-1}} = 0.$$

To prove $\nu_k < 1$, let $v_k = u_{k+1} - u_k$, $c_k = \int_0^1 e^{tu_{k+1} + (1-t)u_k} dt$, $k = 0, \dots, m-1$. Then outside the zeros of q_n ,

$$\begin{split} & \Delta v_0 - 3c_0v_0 + c_1v_1 &= 0, \\ & \Delta v_k + c_{k-1}v_{k-1} - 2c_kv_k + c_{k+1}v_{k+1} &= 0, \quad k = 1, \cdots, m-2, \\ & \Delta v_{m-1} + c_{m-2}v_{m-2} - 3c_{m-1}v_{m-1} &= 0. \end{split}$$

Note that only v_0 has poles at zeros of q_n . To apply Lemma 3.1, we check that c_0 is bounded. In fact, around the zeros of q_n ,

$$c_0 = \int_0^1 e^{tu_1 + (1-t)u_0} dt = \int_0^1 (|q_n|^2 h_1^2)^{1-t} e^{tu_1} dt \le C.$$

It is then easy to check that the above system of equations satisfies the assumptions in Lemma 3.1 and condition (2), since the set of poles (i.e. the set of zeros of q_n) is nonempty. Applying the maximum principle Lemma 3.1, $v_k > 0$, $k = 1, \dots, m - 1$. Then we obtain $\nu_k < 1$, $k = 1, \dots, m - 1$.

To prove $\nu_k > \tilde{\nu}_k$, define

$$u_k = \log(h_k^{-1}h_{k+1}), \quad 1 \le k \le m-1, \quad u_m = \log(h_m^{-2}).$$

Then

$$\Delta(u_2 - u_1) + e^{u_3} - 3e^{u_2} + 3e^{u_1} = |q_n|^2 h_1^2 \ge 0,$$

$$\Delta(u_{k+1} - u_k) + e^{u_{k+2}} - 3e^{u_{k+1}} + 3e^{u_k} - e^{u_{k-1}} = 0, \quad k = 2, \cdots, m - 2,$$

$$\Delta(u_m - u_{m-1}) - 3e^{u_m} + 4e^{u_{m-1}} - e^{u_{m-2}} = 0.$$

And \tilde{u}_k are similarly defined for the Fuchsian case, satisfying

$$\Delta(\tilde{u}_2 - \tilde{u}_1) + e^{u_3} - 3e^{u_2} + 3e^{u_1} = 0,$$

$$\Delta(\tilde{u}_{k+1} - \tilde{u}_k) + e^{\tilde{u}_{k+2}} - 3e^{\tilde{u}_{k+1}} + 3e^{\tilde{u}_k} - e^{\tilde{u}_{k-1}} = 0, \quad k = 2, \dots, m - 2,$$

$$\Delta(\tilde{u}_m - \tilde{u}_{m-1}) - 3e^{\tilde{u}_m} + 4e^{\tilde{u}_{m-1}} - e^{\tilde{u}_{m-2}} = 0.$$

To estimate $(u_{k+1} - u_k) - (\tilde{u}_{k+1} - \tilde{u}_k)$, we have for $k = 2, \dots, m-2$,

$$(e^{u_{k+2}} - 3e^{u_{k+1}} + 3e^{u_k} - e^{u_{k-1}}) - (e^{u_{k+2}} - 3e^{u_{k+1}} + 3e^{u_k} - e^{u_{k-1}})$$

$$= e^{\tilde{u}_{k+2}} (e^{u_{k+2} - \tilde{u}_{k+2}} - e^{u_{k+1} - \tilde{u}_{k+1}}) - 2e^{\tilde{u}_{k+1}} (e^{u_{k+1} - \tilde{u}_{k+1}} - e^{u_k - \tilde{u}_k}) + e^{\tilde{u}_k} (e^{u_k - \tilde{u}_k} - e^{u_{k-1} - \tilde{u}_{k-1}})$$

$$+ (e^{\tilde{u}_{k+2}} - e^{\tilde{u}_{k+1}}) (e^{u_{k+1} - \tilde{u}_{k+1}} - 1) - 2(e^{\tilde{u}_{k+1}} - e^{\tilde{u}_k}) (e^{u_k - \tilde{u}_k} - 1) + (e^{\tilde{u}_k} - e^{\tilde{u}_{k-1}}) (e^{u_{k-1} - \tilde{u}_{k-1}} - 1)$$

$$= e^{\tilde{u}_{k+2}} (e^{u_{k+2} - \tilde{u}_{k+2}} - e^{u_{k+1} - \tilde{u}_{k+1}}) - 2e^{\tilde{u}_{k+1}} (e^{u_{k+1} - \tilde{u}_{k+1}} - e^{u_k - \tilde{u}_k}) + e^{\tilde{u}_k} (e^{u_k - \tilde{u}_k} - e^{u_{k-1} - \tilde{u}_{k-1}})$$

$$+ (e^{u_{k+1} - \tilde{u}_{k+1}} - e^{u_k - \tilde{u}_k}) (e^{\tilde{u}_{k+2}} - e^{\tilde{u}_{k+1}}) - (e^{u_k - \tilde{u}_k} - e^{u_{k-1} - \tilde{u}_{k-1}}) (e^{\tilde{u}_k} - e^{\tilde{u}_{k-1}})$$

$$+ (e^{u_k - \tilde{u}_k} - 1) (e^{\tilde{u}_{k+2}} - 3e^{\tilde{u}_{k+1}} + 3e^{\tilde{u}_k} - e^{\tilde{u}_{k-1}}).$$

Since $\tilde{u}_{k+1} - \tilde{u}_k$ is a globally defined constant function, the equation of $\tilde{u}_{k+1} - \tilde{u}_k$ gives

$$e^{\tilde{u}_{k+2}} - 3e^{\tilde{u}_{k+1}} + 3e^{\tilde{u}_k} - e^{\tilde{u}_{k-1}} = 0.$$

Then

$$(e^{u_{k+2}} - 3e^{u_{k+1}} + 3e^{u_k} - e^{u_{k-1}}) - (e^{\tilde{u}_{k+2}} - 3e^{\tilde{u}_{k+1}} + 3e^{\tilde{u}_k} - e^{\tilde{u}_{k-1}})$$

= $e^{\tilde{u}_{k+2}}(e^{u_{k+2}-\tilde{u}_{k+2}} - e^{u_{k+1}-\tilde{u}_{k+1}}) + (e^{\tilde{u}_{k+2}} - 3e^{\tilde{u}_{k+1}})(e^{u_{k+1}-\tilde{u}_{k+1}} - e^{u_k-\tilde{u}_k}) + e^{\tilde{u}_{k-1}}(e^{u_k-\tilde{u}_k} - e^{u_{k-1}-\tilde{u}_{k-1}}).$

Similarly, for k = 1,

$$(e^{u_3} - 3e^{u_2} + 3e^{u_1}) - (e^{\tilde{u}_3} - 3e^{\tilde{u}_2} + 3e^{\tilde{u}_1}) = e^{\tilde{u}_3}(e^{u_3 - \tilde{u}_3} - e^{u_2 - \tilde{u}_2}) + (e^{\tilde{u}_3} - 3e^{\tilde{u}_2})(e^{u_2 - \tilde{u}_2} - e^{u_1 - \tilde{u}_1}),$$

for $k = m - 1$,

$$(-3e^{u_m} + 4e^{u_{m-1}} - e^{u_{m-2}}) - (-3e^{\tilde{u}_m} + 4e^{\tilde{u}_{m-1}} - e^{\tilde{u}_{m-2}}) = -3e^{\tilde{u}_m}(e^{u_m - \tilde{u}_m} - e^{u_{m-1} - \tilde{u}_{m-1}}) + (4e^{\tilde{u}_{m-1}} - 3e^{\tilde{u}_m})(e^{u_{m-1} - \tilde{u}_{m-1}} - e^{u_{m-2} - \tilde{u}_{m-2}}).$$

Let $v_k = (u_{k+1} - u_k) - (\tilde{u}_{k+1} - \tilde{u}_k), k = 1, \dots, m-1$. Let $c_k = \int_0^1 e^{t(u_{k+1} - \tilde{u}_{k+1}) + (1-t)(u_k - \tilde{u}_k)} dt, k = 1, \dots, m-1$. Then

$$\Delta v_1 + e^{\tilde{u}_3} c_2 v_2 + (e^{\tilde{u}_3} - 3e^{\tilde{u}_2}) c_1 v_1 \ge 0 \Delta v_k + e^{\tilde{u}_{k+2}} c_{k+1} v_{k+1} + (e^{\tilde{u}_{k+2}} - 3e^{\tilde{u}_{k+1}}) c_k v_k + e^{\tilde{u}_{k-1}} c_{k-1} v_{k-1} = 0, \quad k = 2, \dots, m-2, \Delta v_{m-1} - 3e^{\tilde{u}_m} c_{m-1} v_{m-1} + (4e^{\tilde{u}_{m-1}} - 3e^{\tilde{u}_m}) c_{m-2} v_{m-2} = 0.$$

To apply the maximum principle, we need to check

$$e^{\tilde{u}_{k+2}} - 2e^{\tilde{u}_{k+1}} + e^{\tilde{u}_k} \le 0, \quad k = 1, \cdots, m - 2.$$

This is from the equation of \tilde{u}_{k+1} and the fact $\tilde{u}_{k+1} = \text{const} + \log g_0$, $\Delta \log g_0 = g_0$. Other conditions to apply the maximum principle hold clearly (for $e^{\tilde{u}_{m-1}} \leq e^{\tilde{u}_m}$, it is from Lemma 5.4), so we obtain the desired result.

The cyclic Higgs bundles in the Hitchin component for $n \ge 3$ induce minimal immersions $f: \Sigma \to \infty$ $SL(n,\mathbb{R})/SU(n)$. We want to investigate that, as an immersed submanifold, how $f(\tilde{\Sigma})$ sits in the symmetric space.

Theorem 5.6. Let $f: \widetilde{\Sigma} \to SL(n,\mathbb{R})/SU(n)$ be the harmonic map associated to Higgs bundles in the Hitchin component parameterized by q_n . Then the sectional curvature K_{σ} of the tangent plane σ of the image of f in G/K satisfies

$$-\frac{1}{n(n-1)^2} \le K_{\sigma} < 0.$$

The equality can be achieved only if n = 2, 3.

Proof. In the case n = 2, the extrinsic curvature is constantly $-\frac{1}{2}$. Now we consider $n \ge 3$ case. We only prove the case for n = 2m. The proof is similar for n = 2m + 1. Using the curvature formula (3), the sectional curvature of the plane σ is

$$K_{\sigma} = -\frac{(h_1^2|q_n|^2 - h_1^{-1}h_2)^2 + (h_1^{-1}h_2 - h_2^{-1}h_3)^2 + \dots + (h_{m-1}^{-1}h_m - h_m^{-2})^2}{n(h_1^2|q_n|^2 + 2h_1^{-1}h_2 + 2h_2^{-1}h_3 + \dots + 2h_{m-1}^{-1}h_m + h_m^{-2})^2}.$$

Then $K_{\sigma} < 0$ follows from Proposition 5.4. To show $K_{\sigma} \ge -\frac{1}{n(n-1)^2}$, let $\mu_k = \nu_k^{-1}$, then

$$K_{\sigma} \geq -\frac{(h_{1}^{-1}h_{2})^{2} + (h_{1}^{-1}h_{2} - h_{2}^{-1}h_{3})^{2} + \dots + (h_{m-1}^{-1}h_{m} - h_{m}^{-2})^{2}}{n(2h_{1}^{-1}h_{2} + 2h_{2}^{-1}h_{3} + \dots + 2h_{m-1}^{-1}h_{m} + h_{m}^{-2})^{2}}$$
$$= -\frac{1 + (1 - \mu_{2})^{2} + (1 - \mu_{3})^{2}\mu_{2}^{2} + \dots + (1 - \mu_{m})^{2}\mu_{2}^{2}\dots\mu_{m-1}^{2}}{n(2 + 2\mu_{2} + 2\mu_{2}\mu_{3} + \dots + \mu_{2}\dots\mu_{m})^{2}}$$

Define the functions G_k, H_k for $3 \le k \le m+1$ as follows. For $3 \le k \le m-1$,

$$G_k = (1 - \mu_k)^2 + (1 - \mu_{k+1})^2 \mu_k^2 + \dots + (1 - \mu_m)^2 \mu_k^2 \dots \mu_{m-1}^2$$

$$H_k = 2 + 2\mu_k + 2\mu_k \mu_{k+1} + \dots + 2\mu_k \dots \mu_{m-1} + \mu_k \dots \mu_m$$

and

$$G_m = (1 - \mu_m)^2$$
, $H_m = 2 + \mu_m$, $G_{m+1} = 0$, $H_{m+1} = 1$.

The derivatives in μ_k for $3 \le k \le m$ are,

$$(G_k)_{\mu_k} = 2\mu_k(1+G_{k+1}) - 2, \quad (H_k)_{\mu_k} = H_{k+1}$$

Define F_k as a function of μ_k , for $3 \le k \le m+1$,

$$F_k(\mu_k) = \frac{1 + G_k}{(2(k-2) + H_k)^2}$$

So $K_{\sigma} \geq -\frac{1}{n}F_2$. For $3 \leq k \leq m$,

$$F_k(\mu_k) = \frac{1+G_k}{(2(k-2)+H_k)^2} = \frac{1+(1-\mu_k)^2+\mu_k^2G_{k+1}}{(2(k-1)+\mu_kH_{k+1})^2}.$$

We claim:

Lemma 5.7. $F_2 < F_3$. **Lemma 5.8.** $F_k < F_{k+1}$, for $3 \le k \le m$. Therefore, combining Lemma 5.7 and 5.8, the sectional curvature

$$K_{\sigma} \ge -\frac{1}{n}F_2 > -\frac{1}{n}F_{m+1} = \frac{-1}{n(n-1)^2}, \quad \text{for } m > 1.$$

Proof. (of Lemma 5.7) The derivative of F_2 in μ_2 is

$$(F_2)_{\mu_2} = \frac{2(2(\mu_2(1+G_3)-1)-(2-\mu_2)H_3)}{(1+(1-\mu_2)^2+\mu_2^2G_3)(2+\mu_2H_3)^3} = \frac{2F}{(1+(1-\mu_2)^2+\mu_2^2G_3)(2+\mu_2H_3)^3},$$

where $F = 2(\mu_2(1+G_3)-1) - (2-\mu_2)H_3$. Then

$$F < 2(\tilde{\mu}_{2}(1+G_{3})-1) - (2-\tilde{\mu}_{2})H_{3}$$

$$= 2(\frac{2(n-2)}{n-1}(1+G_{3})-1) - (2-\frac{2(n-2)}{n-1})H_{3}$$

$$= \frac{2}{n-1}(n-3+2(n-2)G_{3}-H_{3})$$

$$= \frac{2}{n-1}(n-3+2(n-2)((1-\mu_{3})^{2}+\dots+(1-\mu_{m})^{2}\mu_{3}^{2}\dots\mu_{m-1}^{2}) - (2+2\mu_{3}+\dots+2\mu_{3}\dots\mu_{m-1}) - \mu_{3}\dots\mu_{m})$$

$$= \frac{2}{n-1}(n-3+P_{m}),$$

where $P_k = 2(n-2)((1-\mu_3)^2 + \dots + (1-\mu_k)^2 \mu_3^2 \dots \mu_{k-1}^2) - (2+2\mu_3 + \dots + 2\mu_3 \dots \mu_{k-1}) - (n+1-2k)\mu_3 \dots \mu_k$, for $3 \le k \le m$. Claim: $P_{k+1} < P_k$, for $3 \le k \le m-1$.

$$P_{k+1} = 2(n-2)((1-\mu_3)^2 + \dots + (1-\mu_{k+1})^2 \mu_3^2 \dots \mu_k^2) - (2+2\mu_3 + \dots + 2\mu_3 \dots \mu_k) - (n-1-2k)\mu_3 \dots \mu_{k+1}$$

= $2(n-2)((1-\mu_3)^2 + \dots (1-\mu_k)^2 \mu_3^2 \dots \mu_{k-1}^2) - (2+2\mu_3 + \dots + 2\mu_3 \dots \mu_k)$
 $+ 2(n-2)(1-\mu_{k+1})^2 \mu_3^2 \dots \mu_k^2 - (n-1-2k)\mu_3 \dots \mu_{k+1}$

The last term $2(n-2)(1-\mu_{k+1})^2\mu_3^2\cdots\mu_k^2 - (n-1-2k)\mu_3\cdots\mu_{k+1}$ satisfies

$$2(n-2)(1-\mu_{k+1})^{2}\mu_{3}^{2}\cdots\mu_{k}^{2} - (n-1-2k)\mu_{3}\cdots\mu_{k+1}$$

$$= \mu_{3}\cdots\mu_{k}(2(n-2)\mu_{3}\cdots\mu_{k}(1-\mu_{k+1})^{2} - (n-1-2k)\mu_{k+1})$$

$$< \mu_{3}\cdots\mu_{k}(2(n-2)\tilde{\mu}_{3}\cdots\tilde{\mu}_{k}(1-\mu_{k+1})^{2} - (n-1-2k)\mu_{k+1})$$

$$= \mu_{3}\cdots\mu_{k}(k(n-k)(1-\mu_{k+1})^{2} - (n-1-2k)\mu_{k+1})$$

$$= \mu_{3}\cdots\mu_{k}k(n-k)(\mu_{k+1}^{2} - (2+\frac{n-1-2k}{k(n-k)})\mu_{k+1} + 1)$$
Since $1 < \mu_{k+1} < \tilde{\mu}_{k+1} = \frac{(k+1)(n-1-k)}{k(n-k)} = 1 + \frac{n-1-2k}{k(n-k)}$, by Proposition 5.4.
$$< -(n-1-2k)\mu_{3}\cdots\mu_{k}.$$

Hence $P_{k+1} < P_k$. So

$$P_m < P_3 = 2(n-2)(1-\mu_3)^2 - 2 - (n-5)\mu_3$$

$$\leq 2(n-2)(\mu_3^2 - (2+\frac{n-5}{2(n-2)})\mu_3 + 1) < -(n-3)$$

Hence F < 0 and then $(F_2)_{\mu_2} < 0$. Therefore $F_2(\mu_2) < F_2(1) = F_3$. *Proof.* (of Lemma 5.8) The derivative of F_k with respect to μ_k is

$$(F_k)_{\mu_k} = \frac{2(2(k-1)(\mu_k + \mu_k G_{k+1} - 1) - (2 - \mu_k)H_{k+1})}{(2(k-1) + \mu_k H_{k+1})(1 + (1 - \mu_k)^2 + \mu_k^2 G_{k+1})^3}$$

By Proposition 5.4, $\mu_k < \tilde{\mu}_k = \frac{k(n-k)}{(k-1)(n+1-k)}$,

$$G_k < (1 - \tilde{\mu}_k)^2 + (1 - \tilde{\mu}_{k+1})^2 \tilde{\mu}_k^2 + \dots + (1 - \tilde{\mu}_m)^2 \tilde{\mu}_k^2 \cdots \tilde{\mu}_{m-1}^2$$

= $\frac{1^2 + 3^2 + \dots + (n+1-2k)^2}{(k-1)^2 (n+1-k)^2} = \frac{(n+1-2k)(n+2-2k)(n+3-2k)}{6(k-1)^2 (n+1-k)^2}.$

By Proposition 5.4, $\mu_k > 1$, then $H_k > n + 3 - 2k$. The term $2(k-1)(\mu_k - 1 + \mu_k G_{k+1}) - (2 - \mu_k)H_{k+1}$ satisfies

$$\begin{aligned} &\lim 2(k-1)(\mu_{k}-1+\mu_{k}G_{k+1})-(2-\mu_{k})H_{k+1} \text{ satisfies} \\ &2(k-1)((\mu_{k}-1)+\mu_{k}G_{k+1})-(2-\mu_{k})H_{k+1} \\ &< 2(k-1)((\mu_{k}-1)+\mu_{k}G_{k+1})-(2-\mu_{k})H_{k+1} \\ &< \frac{2(k-1)}{(k-1)(n+1-k)}((n+1-2k)+\frac{(n-1-2k)(n-2k)(n+1-2k)}{6k(n-k)}) \\ &-(2-\frac{k(n-k)}{(k-1)(n+1-k)})(n+1-2k) \\ &= \frac{n+1-2k}{(k-1)(n+1-k)}(\frac{(n-1-2k)(n-2k)(k-1)}{3k(n-k)}-(k-2)(n-k)) \\ &= \frac{(n+1-2k)(n-k)}{(k-1)(n+1-k)}(\frac{(n-1-2k)(n-2k)(k-1)}{3(n-k)^{2}k}-(k-2)) \\ &< \frac{(n+1-2k)(n-k)}{(k-1)(n+1-k)}(\frac{1}{3}-(k-2)) < 0, \quad \text{for } k \ge 3. \end{aligned}$$

Hence F_k decreases as μ_k increases. Then $F_k(\mu_k) < F_k(1) = F_{k+1}$.

Remark 5.9. As shown in [7], along the family of Higgs bundles parameterized by tq_n $(q_n \neq 0)$ for $t \in \mathbb{C}$, as $|t| \rightarrow +\infty$, away from zeros of q_n , the curvature K_{σ}^t approaches to 0.

Remark 5.10. The sectional curvature of $SL(n, \mathbb{R})/SO(n)$ satisfies $-\frac{1}{n} \leq K \leq 0$. So the lower bound $-\frac{1}{n(n-1)^2}$ is nontrivial.

Remark 5.11. (1) In the Fuchsian case, i.e. $q_n = 0$, the sectional curvature K_{σ} is $-\frac{6}{n^2(n^2-1)}$. Note that $-\frac{6}{n^2(n^2-1)} \ge -\frac{1}{n(n-1)^2}$ and equality holds for n = 2, 3. (2) At the zeros p of q_n , $K_{\sigma} \le -\frac{6}{n^2(n^2-1)}$ and equality holds if and only if n = 2, 3. For example, in the case $n = 2m \ge 4$,

$$\begin{split} K_{\sigma}(p) &= -\frac{(h_{1}^{-1}h_{2})^{2} + (h_{1}^{-1}h_{2} - h_{2}^{-1}h_{3})^{2} + \dots + (h_{m-1}^{-1}h_{m} - h_{m}^{-2})^{2}}{n(2h_{1}^{-1}h_{2} + 2h_{2}^{-1}h_{3} + \dots + 2h_{m-1}^{-1}h_{m} + h_{m}^{-2})^{2}} \\ &= -\frac{(h_{1}^{-1}h_{2})^{2} + (h_{1}^{-1}h_{2} - h_{2}^{-1}h_{3})^{2} + \dots + (h_{m-1}^{-1}h_{m} - h_{m}^{-2})^{2}}{n((2m-1)(h_{1}^{-1}h_{2}) - (2m-3)(h_{1}^{-1}h_{2} - h_{2}^{-1}h_{3}) - \dots - (h_{m-1}^{-1}h_{m} - h_{m}^{-2}))^{2}} \\ &= by using the Cauchy-Schwarz inequality and \nu_{k} > \tilde{\nu}_{k} \text{ for } k = 2, \dots, m \\ &< -\frac{1}{n((2m-1)^{2} + (2m-3)^{2} + \dots + 1^{2})} \\ &= -\frac{6}{n^{2}(n^{2}-1)}. \end{split}$$

The case n = 2m + 1 is similar.

6. Comparison inside the real Hitchin fibers at $(0, \dots, 0, q_n)$

Fix a Riemann surface Σ , the Hitchin fibration is a map from moduli space of Higgs bundles to the direct sum of holomorphic differentials. We restrict to the $SL(n, \mathbb{R})$ -Higgs bundles.

We first compare the harmonic metrics for cyclic $SL(n, \mathbb{R})$ -Higgs bundles (E, ϕ) in the Hitchin fiber at $(0, \dots, 0, n \cdot q_n)$, that is, det $\phi = (-1)^{n-1}q_n$.

Proposition 6.1. Let $(\tilde{E}, \tilde{\phi})$ be a cyclic Higgs bundle in the Hitchin component parameterized by q_n and (E, ϕ) be a distinct cyclic $SL(n, \mathbb{R})$ -Higgs bundle in Section 2.3 satisfying det $\phi = (-1)^{n-1}q_n$. Let h, \tilde{h} be the corresponding harmonic metrics.

(1) For n = 2m, suppose $\gamma_1^2 \gamma_2^2 \cdots \gamma_{m-1}^2 \mu \nu = q_n$, then

$$\begin{aligned} h_{k} &> |\gamma_{k}|^{2} \cdots |\gamma_{m-1}|^{2} |\mu| \tilde{h}_{k}, \ k = 1, \cdots, m-1, \\ h_{m} &> |\mu| \tilde{h}_{m}. \\ h_{m+1-k}^{-1} &> |\nu| |\gamma_{1}|^{2} \cdots |\gamma_{m-k}|^{2} \tilde{h}_{k}, \ k = 1, \cdots, m-1, \\ \end{aligned}$$

$$(2) \ For \ n = 2m+1, \ suppose \ \gamma_{1}^{2} \gamma_{2}^{2} \cdots \gamma_{m-1}^{2} \mu^{2} \nu = q_{n}, \ then \\ h_{k} &> |\gamma_{k}|^{2} \cdots |\gamma_{m-1}|^{2} |\mu|^{2} \tilde{h}_{k}, \ k = 1, \cdots, m-1, \\ \end{aligned}$$

$$h_{m} &> |\mu|^{2} \tilde{h}_{m}. \end{aligned}$$

Proof. We only prove the inequalities on the first line for n = 2m. For other cases, the proofs are similar. Define a new Hermitian metric on each L_k ,

$$\hat{h}_{k} = |\gamma_{k}|^{2} \cdots |\gamma_{m-1}|^{2} |\mu| \tilde{h}_{k}, \ k = 1, \cdots, m-1, \quad \hat{h}_{m} = |\mu| \tilde{h}_{m}$$

By the holomorphicity, $\Delta \log |\gamma_k| = 0$ outside the zeros of γ_k (similar for μ, ν). Then \hat{h} satisfies, outside the zeros of q_n , locally

$$\Delta \log \hat{h}_1 + |\gamma_1|^2 \hat{h}_1^{-1} \hat{h}_2 - |\nu|^2 \hat{h}_1^2 = 0,$$

$$\Delta \log \hat{h}_k + |\gamma_k|^2 \hat{h}_k^{-1} \hat{h}_{k+1} - |\gamma_{k-1}|^2 \hat{h}_{k-1}^{-1} \hat{h}_k = 0, \quad k = 2, \cdots, m - 1,$$

$$\Delta \log \hat{h}_m + |\mu|^2 \hat{h}_m^{-2} - |\gamma_{m-1}|^2 \hat{h}_{m-1}^{-1} \hat{h}_m = 0.$$

Notice that \hat{h} satisfies the same equation system as h, but have zeros.

Define $u_i = \log(h_i/\hat{h}_i)$ and u_i goes to $+\infty$ around the set P_i , the zeros of \hat{h}_i . Let

$$c_{1} = g_{0}^{-1} |\nu|^{2} \hat{h}_{1}^{2} \int_{0}^{1} e^{(1-t)u_{1}} dt,$$

$$c_{k} = g_{0}^{-1} |\gamma_{k}|^{2} \hat{h}_{k-1}^{-1} \hat{h}_{k} \int_{0}^{1} e^{(1-t)u_{k}} dt, \quad k = 2, \cdots, m$$

$$c_{m+1} = g_{0}^{-1} |\mu|^{2} \hat{h}_{m}^{-2} \int_{0}^{1} e^{(1-t)u_{m}} dt.$$

Then u_i 's satisfy

$$\Delta_{g_0} u_1 - (c_2 + 2c_1)u_1 + c_2 u_2 = 0, \Delta_{g_0} u_k + c_{k+1}u_{k+1} - (c_k + c_{k+1})u_k + c_k u_{k-1} = 0, \quad k = 2, \dots, m-1, \Delta_{g_0} u_m - (2c_{m+1} + c_m)u_m + c_m u_{m-1} = 0.$$

We need to check the coefficients are bounded. The c_i 's are indeed bounded from the fact $\int_0^1 x^{1-t} dt \leq C$ around x = 0. It is then easy to check that the above system of equations satisfies the assumptions in Lemma 3.1 and condition (2), since the set $P = \bigcup_i P_i$ of poles is nonempty. Applying Lemma 3.1 (the maximum principle), we obtain $u_k > 0$, $k = 1, \dots, m$.

Concerning the associated harmonic maps $f: \widetilde{\Sigma} \to G/K$. We show that the pullback metric of the harmonic map for the cyclic Higgs bundle in the Hitchin component parameterized by q_n dominates the ones for other cyclic $SL(n, \mathbb{R})$ -Higgs bundles in the Hitchin fiber at $(0, \dots, 0, n \cdot q_n)$ for n = 2, 3, 4.

Theorem 6.2. Let $(\tilde{E}, \tilde{\phi})$ be a cyclic Higgs bundle in the Hitchin component parameterized by q_n and (E, ϕ) be a distinct cyclic $SL(n, \mathbb{R})$ -Higgs bundle in Section 2.3 such that det $\phi = (-1)^{n-1}q_n$. In the case (1) $n = 2, \mu\nu = q_2$, (2) $n = 3, \mu^2\nu = q_3$, (3) $n = 4, \gamma^2\mu\nu = q_4$, the pullback metrics g, \tilde{g} of corresponding harmonic maps satisfy $g < \tilde{g}$.

Proof. For n = 2, locally

$$\frac{1}{4}g = q_2 dz^2 + (|\nu|^2 h^2 + |\mu|^2 h^{-2}) dz \cdot d\bar{z} + \bar{q}_2 d\bar{z}^2$$

$$\frac{1}{4}\tilde{g} = q_2 dz^2 + (|\mu|^2 |\nu|^2 \tilde{h}^2 + \tilde{h}^{-2}) dz \cdot d\bar{z} + \bar{q}_2 d\bar{z}^2$$

So

$$\frac{1}{4}g(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}) = (|\nu|h - |\mu|h^{-1})^2 + 2|\mu\nu|, \quad \frac{1}{4}\tilde{g}(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}}) = (|\mu||\nu|\tilde{h} - \tilde{h}^{-1})^2 + 2|\mu\nu|.$$

From Proposition 6.1, $|\mu||\nu|\tilde{h} \leq |\mu|h^{-1} < \tilde{h}^{-1}$, $|\mu||\nu|\tilde{h} \leq |\nu|h < \tilde{h}^{-1}$. Then

$$(|\nu|h - |\mu|h^{-1})^2 < (|\mu||\nu|\tilde{h} - \tilde{h}^{-1})^2,$$

which implies $g < \tilde{g}$.

For n = 3, we claim $|\nu|^2 \tilde{h} h^2 < 1$. The Hitchin equation is reduced to

$$\Delta \log(|\nu|^2 \tilde{h} h^2) + \tilde{h}^{-1} - |\mu|^4 |\nu|^2 \tilde{h}^2 + 2|\mu|^2 h^{-1} - 2|\nu|^2 h^2 = 0.$$

Let $u = |\nu|^2 \tilde{h} h^2$, $a = |\mu|^2 \tilde{h} h^{-1}$. Then

$$\Delta \log u + \tilde{h}^{-1}(1 + 2a - (2 + a^2)u) = 0.$$

Notice that $u \equiv 1$ is a supersolution, then by the maximum principle, u < 1. For the pullback metric g, \tilde{g} , locally,

$$\frac{1}{6}g = |\nu|^2 h^2 + 2|\mu|^2 h^{-1}, \quad \frac{1}{6}\tilde{g} = |\nu|^2 |\mu|^4 \tilde{h}^2 + 2\tilde{h}^{-1}.$$

Let $x = |\nu|h$, $\tilde{x} = |\mu||\nu|^2 \tilde{h}$, $A = |q_3| = |\nu||\mu|^2$. Outside the zeros of $\mu\nu$, from Proposition 6.1, $x < \tilde{x}$. Then

$$\frac{1}{6}(g-\tilde{g}) = (x^2 + \frac{2A}{x}) - (\tilde{x}^2 + \frac{2A}{\tilde{x}}) = \frac{(x-\tilde{x})}{x\tilde{x}}((x+\tilde{x})x\tilde{x} - 2A)$$

$$< \frac{(x-\tilde{x})}{x\tilde{x}}(2x^2\tilde{x} - 2A) = \frac{2|\nu||\mu|^2(x-\tilde{x})}{x\tilde{x}}(|\nu|^2\tilde{h}h^2 - 1) < 0.$$

So outside the zeros of $q_3 = \mu^2 \nu$, we obtain $g < \tilde{g}$. We can easily see it also holds at the zeros of q_3 .

For n = 4, locally

$$\frac{1}{8}g = |\nu|^2 h_1^2 + 2|\gamma|^2 h_1^{-1} h_2 + |\mu|^2 h_2^{-2} = (|\nu|h_1 - |\mu|h_2^{-1})^2 + 2|\nu||\mu|h_1 h_2^{-1} + 2|\gamma|^2 h_1^{-1} h_2,$$

$$\frac{1}{8}\tilde{g} = |\mu|^2 |\nu|^2 |\gamma|^4 \tilde{h}_1^2 + 2\tilde{h}_1^{-1} \tilde{h}_2 + \tilde{h}_2^{-2} = (|\mu||\nu||\gamma|^2 \tilde{h}_1 - \tilde{h}_2^{-1})^2 + 2|\nu||\mu||\gamma|^2 \tilde{h}_1 \tilde{h}_2^{-1} + 2\tilde{h}_1^{-1} \tilde{h}_2.$$

From Proposition 6.1, $|\mu||\nu||\gamma|^2 \tilde{h}_1 \le |\mu|h_2^{-1} < \tilde{h}_2^{-1}, |\mu||\nu||\gamma|^2 \tilde{h}_1 \le |\nu|h_1 < \tilde{h}_2^{-1}$. Then $(|\mu||\nu||\gamma|^2 \tilde{h}_1 - \tilde{h}_2^{-1})^2 > (|\nu|h_1 - |\mu|h_2^{-1})^2.$

Let $x = |\gamma|^2 h_1^{-1} h_2$, $\tilde{x} = \tilde{h}_1^{-1} \tilde{h}_2$, $A = |q_4| = |\mu| |\nu| |\gamma|^2$.

Claim: $x < \tilde{x}$ and $x\tilde{x} > A$, outside the zeros of γ . Then the desired result follows from the basic identity $x + \frac{A}{x} - \tilde{x} - \frac{A}{\tilde{x}} = (x - \tilde{x})(1 - \frac{A}{x\tilde{x}})$.

To show $x < \tilde{x}$, let $u = \frac{x}{\tilde{x}} = |\gamma|^2 h_1^{-1} h_2 \tilde{h}_1 \tilde{h}_2^{-1}$. Then u satisfies

$$\Delta \log u - 2\tilde{h}_1^{-1}\tilde{h}_2(u-1) + 2|\mu||\nu||\gamma|^2\tilde{h}_1\tilde{h}_2^{-1}(u^{-1}-1) + (|\nu|h_1 - |\mu|h_2^{-1})^2 - (|\mu||\nu||\gamma|^2\tilde{h}_1 - \tilde{h}_2^{-1})^2 = 0.$$

Then

$$\Delta \log u - 2\tilde{h}_1^{-1}\tilde{h}_2(u-1) + 2|\nu||\mu||\gamma|^2\tilde{h}_1\tilde{h}_2^{-1}(u^{-1}-1) > 0.$$

Notice that 1 is a subsolution, then by the maximum principle, u < 1.

To show $x\tilde{x} > A$, let $u = \frac{A}{x\tilde{x}} = |\mu||\nu|h_1h_2^{-1}\tilde{h}_1\tilde{h}_2^{-1}$. Then u satisfies

$$\Delta \log u + (2\tilde{h}_1^{-1}\tilde{h}_2 + 2|\gamma|^2 h_1^{-1}h_2)(1-u) - (|\nu|h_1 - |\mu|h_2^{-1})^2 - (|\mu||\nu||\gamma|^2\tilde{h}_1 - \tilde{h}_2^{-1})^2 = 0.$$

Then

$$\Delta \log u + (2\tilde{h}_1^{-1}\tilde{h}_2 + 2|\gamma|^2 h_1^{-1}h_2)(1-u) > 0.$$

Notice that $u \equiv 1$ is a solution, then by the maximum principle, u < 1. At the zeros of γ , we can also obtain $g < \tilde{g}$ from $|\mu||\nu|h_1h_2^{-1}\tilde{h}_1\tilde{h}_2^{-1} < 1$. So we finish the proof.

By integration, we obtain

Corollary 6.3. The Morse function achieves the maximum in the Hitchin point in the above cases.

As an immediate corollary in terms of representations for n = 2, we recover the following result shown in [11].

Corollary 6.4. For any non-Fuchsian reductive $SL(2,\mathbb{R})$ -representation ρ and any Riemann surface Σ , there exists a Fuchsian representation j such that the pullback metric of the corresponding j-equivariant harmonic map $f_j: \widetilde{\Sigma} \to \mathbb{H}^2$ dominates the one for f_{ρ} .

Proof. For any reductive $SL(2, \mathbb{R})$ -representation ρ , if it is into the compact subgroup $SO(2, \mathbb{R})$, the associated harmonic map is constant. In this case, the statement is clear. Given any Riemann surface Σ , if the representation ρ is not into the compact group $SO(2, \mathbb{R})$, it corresponds to a cyclic Higgs bundle parametrized by (α, β) over Σ by [14]. Then we choose the Fuchsian representation j corresponding to the cyclic Higgs bundle parametrized by $q_2 = \alpha\beta$ over Σ . The statement follows from Theorem 6.2.

7. Maximal $Sp(4, \mathbb{R})$ -representations

For each reductive representation ρ into $Sp(2n,\mathbb{R})$, we can define a Toledo integer $\tau(\rho) \coloneqq \frac{2}{\pi} \int_S f^* \omega$ where f is any ρ -equivariant continuous map $f : \widetilde{S} \to Sp(2n,\mathbb{R})/U(n)$ and ω is the normalized $Sp(2n,\mathbb{R})$ -invariant Kähler 2-form on $Sp(2n,\mathbb{R})/U(n)$. It is well-known that $|\tau(\rho)| \leq n(g-1)$. The representation ρ with $|\tau(\rho)| = n(g-1)$ is called maximal.

A $Sp(4,\mathbb{R})$ -Higgs bundle over Σ is a pair $(V \oplus V^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})$ where V is a rank 2 holomorphic vector bundle over $\Sigma, \beta \in H^0(S^2V \otimes K_{\Sigma})$ and $\gamma \in H^0(\Sigma, S^2V^* \otimes K_{\Sigma})$. The Toledo integer of the $Sp(4,\mathbb{R})$ -Higgs bundle is the integer deg(V). There are $3 \cdot 2^{2g} + 2g - 4$ components of maximal $Sp(4,\mathbb{R})$ -representations shown in [13] containing 2^{2g} Hitchin components isomorphic to each other and 2g - 3 exceptional components called Gothen components.

Labourie in [17] shows that any $Sp(4,\mathbb{R})$ Hitchin representation corresponds to a cyclic Higgs bundle in the Hitchin components over a unique Riemann surface. As a result, there is a unique ρ equivariant minimal immersion of \widetilde{S} into $Sp(4,\mathbb{R})/U(2)$ for any Hitchin representation for $Sp(4,\mathbb{R})$. For each Riemann surface Σ , each Gothen component is explicitly described in [3] as the moduli space of Higgs bundles of the following form

$$E = N \oplus NK^{-1} \oplus N^{-1}K \oplus N^{-1}, \quad \phi = \begin{pmatrix} 0 & q_2 & 0 & \nu \\ 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & q_2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $g-1 < \deg(N) < 3g-3$, $\nu \in H^0(N^2K)$, $\mu \in H^0(N^{-2}K^3)$, and $q_2 \in H^0(K^2)$. Here $V = N \oplus N^{-1}K$.

By Collier's work [5], we can replace the variation of q_2 with a variation of base Riemann surface structure. That is, any maximal $Sp(4,\mathbb{R})$ -representation in the Gothen components corresponds to a Higgs bundle over a unique Riemann surface Σ of the form

$$E = N \oplus NK^{-1} \oplus N^{-1}K \oplus N^{-1}, \quad \phi = \begin{pmatrix} 0 & 0 & 0 & \nu \\ 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $g-1 < \deg(N) < 3g-3$, $\mu \neq 0$ and ν can be zero. The 2g-3 Gothen components are indexed by the degree of N. These are cyclic $SL(4,\mathbb{C})$ -Higgs bundles. Note that If $N = K^{\frac{3}{2}}$, this gives the Hitchin representation. As a result, for any $Sp(4,\mathbb{R})$ -representation in the Gothen components, there is a unique ρ -equivariant minimal immersion of \widetilde{S} into $Sp(4,\mathbb{R})/U(2)$.

The above cyclic Higgs bundles with $\nu = 0$ are stable and play a similar role as the Fuchsian case. We call the corresponding representations μ -Fuchsian representations. The space of μ -Fuchsian representations serves as the minimum in its component of maximal $Sp(4, \mathbb{R})$ representations in the following sense.

Corollary 7.1. For any maximal representation $\rho : \pi_1(S) \to Sp(4,\mathbb{R})$ in the 2g-3 Gothen components, there exists a μ -Fuchsian representation j of $\pi_1(S)$ such that the pullback metric of the unique j-equivariant minimal immersion $f_j : \widetilde{S} \to Sp(4,\mathbb{R})/U(2)$ is dominated by the one for f_ρ .

Proof. For any maximal representation in the Gothen component, we can realize it as a cyclic Higgs bundle parametrized by $(1, \mu, 1, \nu)$ over some Riemann surface Σ . Then we choose the μ -Fuchian representation corresponding to cyclic Higgs bundle parametrized by $(1, \mu, 1, 0)$ over Σ . Then the statement follows from Theorem 4.5.

Since any Hitchin representation for $Sp(4,\mathbb{R})$ corresponds to a cyclic Higgs bundle over some Riemann surface Σ , we obtain bounds on the extrinsic curvature of minimal immersions for maximal representations in the Hitchin component as an immediate corollary of Theorem 5.6.

Corollary 7.2. For any Hitchin representation ρ for $Sp(4,\mathbb{R})$, the sectional curvature K_{σ} in $Sp(4,\mathbb{R})/U(2)$ of the tangent plane σ of the uniuge ρ -equivariant minimal immersion satisfies (1) $K_{\sigma} = -\frac{1}{40}$, if ρ is Fuchsian; (2) $-\frac{1}{36} < K_{\sigma} < 0$ and $\exists p$ such that $K_{\sigma}(p) < -\frac{1}{40}$, if ρ is not Fuchsian.

Remark 7.3. The lower bound $-\frac{1}{36}$ is nontrivial, since the sectional curvature K in $Sp(4, \mathbb{R})/U(2)$ satisfies that $-\frac{1}{4} \leq K \leq 0$.

Similarly, we also obtain estimates on the extrinsic curvature of minimal immersions for maximal representations in 2g - 3 Gothen component.

Theorem 7.4. For any maximal representation ρ for $Sp(4,\mathbb{R})$ in each Gothen component, the sectional curvature K_{σ} in $Sp(4,\mathbb{R})/U(2)$ of the tangent plane σ of the unique ρ -equivariant minimal immersion satisfies

(1) $-\frac{1}{8} \leq K_{\sigma} < -\frac{1}{40}$ and the lower bound is sharp, if ρ is μ -Fuchsian; (2) $-\frac{1}{8} \leq K_{\sigma} < 0$, if ρ is not μ -Fuchsian.

Proof. It is sufficient to work with cyclic Higgs bundle parameterized by $(1, \mu, 1, \nu)$ of the above form. The Hitchin equation in this case is

$$\Delta \log h_1 + h_1^{-1} h_2 - |\nu|^2 h_1^2 = 0, \Delta \log h_2 + |\mu|^2 h_2^{-2} - h_1^{-1} h_2 = 0$$

Using the curvature formula (3), the sectional curvature of the tangent plane σ of the minimal immersion is

$$K_{\sigma} = -\frac{(h_1^2|\nu|^2 - h_1^{-1}h_2)^2 + (h_1^{-1}h_2 - h_2^{-2}|u|^2)^2}{4 \cdot (h_1^2|\nu|^2 + 2h_1^{-1}h_2 + h_2^{-2}|\mu|^2)^2}.$$

For the right inequality, outside zeros of $\mu\nu$,

=

$$\Delta \log h_1^2 h_2^{-2} |\mu\nu| - (|\mu|^2 h_2^{-2} + |\nu|^2 h_1^2) + 2h_1^{-1} h_2 = 0$$

$$\Rightarrow \quad \Delta \log h_1^2 h_2^{-2} |\mu\nu| - 2|\mu\nu| h_1 h_2^{-1} - 2h_1^{-1} h_2 \ge 0$$

$$\Rightarrow \quad \Delta \log h_1^2 h_2^{-2} |\mu\nu| - 2(h_1^2 h_2^{-2} |\mu\nu| - 1) h_1^{-1} h_2 \ge 0$$

So at the maximum of $h_1^2 h_2^{-2} |\mu\nu|$, $h_1^2 h_2^{-2} |\mu\nu| - 1 \le 0$. Hence $h_1^2 h_2^{-2} |\mu\nu| \le 1$ on the whole surface. By the strong maximum principle, we obtain that $h_1^2 h_2^{-2} |\mu\nu| < 1$. So $h_1^{-1} h_2 = |\nu|^2 h_1^2$ and $|\mu|^2 h_2^{-2} = h_1^{-1} h_2$ cannot hold at any point p simultaneously, since it would imply that $h_1^2 h_2^{-2} |\mu\nu| = 1$ at point p, contradiction. Therefore $K_{\sigma} < 0$.

For the left inequality. Let $f_1 = \frac{h_1^2 |\nu|^2}{h_1^{-1} h_2}, f_2 = \frac{h_2^{-2} |\mu|^2}{h_1^{-1} h_2}$. Claim: $f_1, f_2 < \frac{4}{3}$. The equation for f_1 is, outside zeros of ν ,

So at the maximum of f_1 , $3(1-f_1)+1 \le 0$, hence $f_1 \le \frac{4}{3}$. Use the strong maximum principle, $f_1 < \frac{4}{3}$. It is similar for f_2 . The claim is proven.

Using $0 \le f_1, f_2 < \frac{4}{3}$,

$$K_{\sigma} = -\frac{\operatorname{tr}([\phi, \phi^*][\phi, \phi^*])}{2n \cdot \operatorname{tr}(\phi\phi^*)^2} = -\frac{(f_1 - 1)^2 + (f_2 - 1)^2}{4(2 + f_1 + f_2)^2} \ge -\frac{1 + 1}{16} = -\frac{1}{8}.$$

Note that K_{σ} only achieves $-\frac{1}{8}$ if $f_1 = f_2 = 0$. This only happens at common zeros of μ and ν . In the μ -Fuchsian case, $\nu = 0$. So $f_1 = 0$ and again $f_2 < \frac{4}{3}$. Then using $(f_2 + 2)^{-1} \in (\frac{3}{10}, \frac{1}{2}]$,

$$K_{\sigma} = -\frac{1 + (f_2 - 1)^2}{4(f_2 + 2)^2} = -\frac{(f_2 + 2 - 3)^2 + 1}{4(f_2 + 2)^2} = -\frac{10}{4} (((f_2 + 2)^{-1} - \frac{3}{10})^2 - \frac{1}{40} < -\frac{1}{40}.$$

Note that at zeros of μ in μ -Fuchsian case, the curvature $K_{\sigma} = -\frac{1}{8}$.

Remark 7.5. As shown in [19], along the family of $(E, t\phi)$, as $|t| \to \infty$, away from zeros of $\mu\nu \neq 0$, the sectional curvature goes to zero. So the upper bound in Part (2) is sharp.

We compare the Gothen components with the Hitchin components.

Corollary 7.6. For any maximal representation $\rho: \pi_1(S) \to Sp(4,\mathbb{R})$ in the 2q-3 Gothen components, there exists a Hitchin representation j of $\pi_1(S)$ such that the pullback metric of the unique *j*-equivariant minimal immersion $f_j: \widetilde{S} \to Sp(4, \mathbb{R})/U(2)$ dominates the one for f_{ρ} .

Proof. For any maximal representation ρ in the Gothen components, it corresponds to a cyclic Higgs bundle parametrized by $(1, \mu, 1, \nu)$ over some Riemann surface Σ . Then we choose the Hitchin representation j corresponding to cyclic Higgs bundle in the Hitchin component parametrized by $q_4 = \mu\nu$ over Σ . The statement then follows from Theorem 6.2 for n = 4.

References

- L. Alvarez-Cónsul, O. García-Prada, Hitchin-Kobayashi correspondence, quivers and vortices, Comm. Math. Phys. 238 (2003), 1–31
- [2] D. Baraglia, G₂ Geometry and integrable system, thesis, arXiv:1002.1767v2, 2010.
- [3] S. B. Bradlow, O. García-Prada, and P. B. Gothen, Deformations of maximal representations in Sp(4, ℝ), Q. J. Math. 63 (2012), no. 4, 795–843. MR 2999985.
- [4] M. Burger, A. Iozzi, F. Labourie, and A. Wienhard, Maximal representations of surface groups: symplectic Anosov structures, Pure Appl. Math. Q. 1 (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 543–590. MR 2201327 (2007d:53064)
- [5] B. Collier, Maximal Sp(4,ℝ) surface group representations, minimal surfaces and cyclic surfaces, Geometriae Dedicata 180 (2015), no. 1, 241–285.
- [6] B. Collier, Finite order automorphism of Higgs bundles: theory and application, thesis, 2016.
- B. Collier, Q. Li, Asymptotics of Higgs bundles in the Hitchin component, Adv. Math. 307 (2017), 488–558, MR3590524, Zbl 06670884.
- [8] B. Collier, Nicolas Tholozan, Jérémy Toulisse, The geometry of maximal representations of surface groups into SO(2,n), arxiv 1702.08799.
- [9] K. Corlette, Flat G-bundles with canonical metrics, J. Diff. Geom. 28(1988), no. 3, 361-382, MR965220, Zbl 0676.58007.
- [10] S. Dai, Q. Li, Minimal surfaces for Hitchin representations, to appear in Journal of Differential Geometry.
- [11] B. Deroin, N. Tholozan, Dominating surface group representations by Fuchsian ones, Int. Math. Res. Not. IMRN 2016, no. 13, 4145–4166, MR3544632.
- [12] S. K. Donaldson, Twisted harmonic maps and the self-duality equations, Proc. London Math. Soc. (3) 55 (1987), no. 1, 127–131, MR0887285, Zbl 0634.53046.
- [13] P. B. Gothen, Components of spaces of representations and stable triples, Topology 40 (2001), 823–850.
- [14] N. J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) 55 (1987), no. 1, 59–126, MR0887284, Zbl 0634.53045.
- [15] N. J. Hitchin, Lie groups and Teichmüller space, Topology 31 (1992), no. 3, 449–473, MR1174252, Zbl 0769.32008.
- [16] J. Jost, Riemannian Geometry and Geometric Analysis, Third edition, Universitext, Springer-Verlag, New York, 2002.
- [17] F. Labourie, Cyclic surfaces and Hitchin components in rank 2, Ann. of Math. (2) 185 (2017), no. 1, 1–58, MR3583351, Zbl 06686583.
- [18] J. López-Gómez, M. Molina-Meyer, The maximum principle for cooperative weakly coupled elliptic systems and some applications, Differential Integral Equations 7 (1994), no. 2.
- [19] T. Mochizuki, Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces, J. Topol. 9 (2016), no. 4, 1021–1073. MR3620459.
- [20] J. Jost, Partial differential equations, Second edition. Graduate Texts in Mathematics, Springer, New York, 2007.
- [21] M. H. Protter, H. F. Weinberger, Maximum Principles in Differential Equations, Springer-Verlag, New York, 1984.
- [22] C. Simpson, Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization, J. Amer. Math. Soc. 1 (1988), no. 4, 867–918. MR0944577, Zbl 0669.58008.

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