Preface

The studies on time optimal controls started with finite dimensional systems in 1950's, and then extended to infinite dimensional systems in 1960's. Time optimal control is a very important field in optimal control theory, from perspectives of both application and mathematics. Optimal control theory can be viewed as a branch of optimization theory or variational theory. In general, an optimal control problem is to ask for a control from an available set so that a certain optimality criterion, related to a given dynamic system, is achieved. When the above-mentioned criterion is elapsed time, such an optimal control problem is called a time optimal control problem. From perspective of application, there are many practical examples which can be described as time optimal control problems. For instance, reaching the top fastest by an elevator; warming the room by the stove quickly and stopping the harmonic oscillator in the shortest time. Systematic studies on time optimal control problems may help us to design the best strategy to solve these practical problems. Theoretically, time optimal control problems connect with several other fields of control theory, such as controllability, dynamic programming and so on. The studies on time optimal controls may develop studies on these related fields. These can be viewed as the motivations to study time optimal controls.

The time optimal control problems studied in this monograph can be stated roughly as follows: Given a controlled system over the right half time interval, we ask for such a control (from a constraint set) that satisfies the following three conditions: (i) It is only active from a starting time point to an ending time point; (ii) It drives the solution of the controlled system from an initial state set (at the zero time) to a target set (at the ending time); (iii) The length of the interval, where the control is active, is the shortest among all such candidates. Thus, each time optimal control problem has four undetermined variables: starting time, initial states, ending time and controls. They form a family of tetrads. When the starting time point is fixed, such a time optimal control problem is called a minimal time control problem. It is to initiate control (in a constraint set) from the beginning so that the corresponding solution (of a controlled system) reaches a given target set in the shortest time. When the ending time point is fixed, such a time optimal control problem is called a maximal time control problem. It is to delay the initiation of active control (in a constraint set) as late as possible, so that the corresponding solution (of a controlled system) reaches a given target set in a fixed ending time.

Most parts of this monograph focus on the above-mentioned two kinds of time optimal control problems for linear controlled evolution equations. In this monograph, we set up a general mathematical framework on time optimal control problems governed by some evolution equations. This framework has four ingredients: a controlled system, a control constraint set, a starting set and an ending set. We introduce five themes on time optimal control problems under this framework: the existence of admissible controls and optimal controls, Pontryagin's maximum principle for optimal controls, the equivalence of several different kinds of optimal control problems, and bang-bang properties. The aim of this monograph is to summarize our, our seniors' and our collaborators', ideas, methods and results in the aforementioned themes. Many of these ideas, methods and results are the recent advances in the field of time optimal control. These may lead us to comprehensive understanding of the field on time optimal control. We tried our best to make the monograph self-contained and wish that it could interest specialists and graduate students in this field, as well as in other related fields. Indeed, the prerequisite is to know a little about the functional analysis and differential equations. There are a lot of literatures on time optimal control problems for evolution systems. We have not attempted to give a complete list of references. All the references in this monograph are closely related to the materials introduced here. It may possibly happen that some important works in this field have been overlooked.

More than thirteen years ago, H. O. Fattorini in his well-known book:
H. O. Fattorini, Infinite dimensional linear control systems, the time optimal and norm optimal problems, North-Holland Mathematics Studies, 201, Elsevier Science B. V., Amsterdam, 2005.
introduced objects: minimal time control problems and minimal norm control problems. The controlled system there reads: $y' = Ay + u$, where A generates a C_0-semigroup in a Banach space. (Hence, the state space and the control space are the same, and the control operator is the identity.) The book is mainly concerned with three topics: First, the Pontryagin Maximum Principle with multipliers in different spaces which are called the regular space and the singular space. Second, the bang-bang property of minimal time and the minimal norm controls. Third, connections between minimal time and minimal norm control problems. Also, the existence of optimal controls, based on the assumption that there is an admissible control, is presented. In the period after the book, the studies on time optimal control problems were developed greatly. We believe that it is the right time to summarize some of these developments. Differing from the book, the objects in the current monograph are the minimal time and the maximal time control problems; the controlled system in the current monograph reads: $y' = Ay + Bu$, where A generates a C_0-semigroup in a Hilbert space (which is the state space), B is a linear and bounded operator from the state space to another Hilbert space (which is the control space). Though many topics in the current monograph, such as the Pontryagin Maximum Principle, the bang-bang property and the equivalence of different optimal control problems, are quite similar to those in the book, we study these issues from different perspectives

and by using different ways. Moreover, studies on some topics are developed a lot, for instance, the equivalence of the minimal time control problems and the minimal norm control problems; the bang-bang property for minimal time controls. The topic on the existence of admissible controls was not touched upon in the book, while in the current monograph, we introduce several ways to study it.

Let us now be more precise on the contents of the different chapters of this monograph.

Chapter 1. This chapter introduces some very basic elements of preliminaries such as functional analysis, evolution equations, controllability and observability estimates.

Chapter 2. In this chapter, we first set up a general mathematical framework of time optimal control problems for evolution equations, and present four ingredients of this framework; we next introduce connections between minimal and maximal time control problems; we then show main subjects (on time optimal control problems) which will be studied in this monograph. Several examples are given to illustrate the framework, its ingredients and main subjects.

What deserves to be mentioned is as follows: At the end of Section 2.1, we present a minimal blowup time control problem for some nonlinear ODEs with the blowup behaviour. It is a special time optimal control problem (where the target is outside of the state space) and is not under our framework.

Chapter 3. The subject of this chapter is the existence of admissible controls and optimal controls for time optimal control problems. The key of this subject is the existence of admissible controls. We study it from three perspectives: the controllability, minimal norm problems and reachable sets. Consequently, we provide three ways to obtain the existence of admissible controls. We also show how to derive the existence of optimal controls from that of admissible controls.

We end this chapter with the existence of optimal controls for a minimal blowup time control problem governed by nonlinear ODEs with the blowup behaviour. This problem is very interesting from the viewpoints of application and mathematical theory. However, the studies of such a problem are more difficult than those of problems under our framework.

Chapter 4. This chapter is devoted to the Pontryagin Maximum Principle, which is indeed a kind of first-order necessary conditions on time optimal controls. In general, there are two methods to derive the Pontryagin Maximum Principle. They are the analytical method and the geometric method. We focus on the second one, and show how to use it to derive the Pontryagin Maximum Principle for both the minimal time control problems and the maximal time control problems.

Three different kinds of maximum principles are introduced. They are the classical Pontryagin Maximum Principle, the local Pontryagin Maximum Principle, and the weak Pontryagin Maximum Principle. We find a way consisting of two steps to derive these maximum principles. In step 1, using the Hahn-Banach Theorem for different cases, we separate different objects in different spaces. In step 2, we build up different representation formulas for different cases. Besides, we give connections among the above-mentioned maximum principles.

Chapter 5. This chapter develops two equivalence theorems for several different kinds of optimal control problems. The first one is concerned with the equivalence of the minimal time control problem and the minimal norm control problem. The second one deals with the equivalence of the maximal time control problem, the minimal norm control problem and the optimal target control problem. Some applications of these equivalence theorems are referred to in some examples and the miscellaneous note.

The above equivalence theorems allow us to obtain desired properties for one optimal control problem through studying another simpler one. Two facts deserve to be mentioned: First, between the minimal norm control problem and the minimal time control problem, the second one is simpler. Second, among the maximal time control problem, the minimal norm control problem and the optimal target control problem, the first one is the most complicated one and the last one is the simplest one.

Chapter 6. In this chapter, we present the bang-bang property for time optimal control problems. In plain language, this property means that each optimal control reaches the boundary of a control constraint set at almost every time. It can be compared with such a property of a function that its extreme points stay only on the boundary of its domain. We begin with finite dimensional cases and then turn to infinite dimensional cases. We introduce two ways to approach the bang-bang property. The first way is the use of the Pontryagin Maximum Principle, together with a certain unique continuation. The second way is the use of a certain null controllability from measurable sets in time.

In this monograph, we give some miscellaneous notes at the end of each chapter. In these notes, we review the history and the related works of the involved study, or point out some open problems in the related fields. Besides, after most theorems, we provide examples to help readers to understand the theorems better.

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