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Tempered Fractional Multistable Motion and Tempered Multifractional Stable Motion

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Abstract This work defines two classes of processes, that we term *tempered fractional multistable motion* and *tempered multifractional stable motion*. They are extensions of fractional multistable motion and multifractional stable motion, respectively, obtained by adding an exponential tempering to the integrands. We investigate certain basic features of these processes, including scaling property, tail probabilities, absolute moment, sample path properties, pointwise Hölder exponent, Hölder continuity of quasi norm, (strong) localisability and semi-long-range dependence structure. These processes may provide useful models for data that exhibit both dependence and varying local regularity/intensity of jumps.

Keywords Stable processes · Multistable processes · Multifractional processes · Sample paths · Long-range dependence · Localisability

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1 Introduction

Linear fractional stable motion (LFSM) can be represented by the stochastic integral of a symmetric α-stable random measure $dZ_\alpha(x)$, that is

$$X(t) = \int_{-\infty}^{\infty} \Big[(t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}\Big] dZ_\alpha(x), \qquad t \in \mathbf{R}, \tag{1.1}$$

where $0 < \alpha \leq 2, 0 < H < 1$, $(x)_+ = \max\{x, 0\}$ and $0^0 = 0$. See for example Samorodnitsky and Taqqu [11]. This stochastic process has two important features. It is self-similar with Hurst parameter H, i.e. for any $c > 0$, $t_1, ..., t_d \in \mathbf{R}$,

$$\big(X(c\,t_1), ..., X(c\,t_d)\big) \stackrel{d}{=} \big(c^H X(t_1), ..., c^H X(t_d)\big),$$

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and it has stationary increments, i.e., for any $\tau \in \mathbf{R}$, $\big(X(t) - X(0),\ -\infty < t < \infty\big) \overset{d}{=} \big(X(\tau + t) - X(\tau),\ -\infty < t < \infty\big)$, where $\overset{d}{=}$ indicates equality in distribution. Because its increments can exhibit the heavy-tailed analog of long-range dependence (see Watkins et al. [14]), the model is useful in practice to model, for example, financial data, internet traffic, noise on telephone line, signal processing and atomospheric noise, see Nolan [10] for many references.

There exist at least three extensions of LFSM, i.e., linear multifractional stable motion (LmFSM), linear fractional multistable motion (LFmSM) and linear tempered fractional stable motion (LTFSM). Stoev and Taqqu [12,13] first introduced LmFSM by replacing the self-similarity parameter H in the integral representation of the LFSM by a time-varying function H_t. Stoev and Taqqu have examined the effect of the regularity of the function H_t on the local structure of the process. They also showed that under certain Hölder regularity conditions on the function H_t, the LmFSM is locally equivalent to a LFSM, in the sense of finite-dimensional distributions. Thus LmFSM is a locally self-similar stochastic process. Whereas the LFSM is always continuous in probability, this is not in general the case for LmFSM. Stoev and Taqqu have obtained necessary and sufficient conditions for the continuity in probability of the LmFSM. Falconer and Lévy Véhel [6] defined the second model extension of LFSM, called LFmSM. LFmSM behaves locally like linear fractional $\alpha(t)$-stable motion close to time t, in the sense that the local scaling limits are linear fractional $\alpha(t)$-stable motions, but where the stability index $\alpha(t)$ varies with t. This extension allows one to account for the fact that the nature of irregularity, including the stability level, may vary in time. See also Falconer and Liu [7] where the α-stable random measure in (1.1) has been replaced by a time-varying $\alpha(t)$-multistable random measure. Recently, Meerschaert and S-abzikar [9] defined the third extension, termed LTFSM, by adding an exponential tempering to the power-law kernel in a LFSM. They showed that the LTFSM exhibits semi-long-range dependence, and therefore provides a useful alternative model for data that exhibit strong dependence.

In view of trying to combine the properties of both LFmSM and LTFSM, we define in this work a new stochastic process by adding an exponentiel tempering to the power-law kernel of LFmSM. Our *linear tempered fractional multistable motion* (LTFmSM) is thus an extension of LFmSM and LTFSM. In particular, linear tempered fractional multistable motion behaves locally like the linear fractional $\alpha(t)$-stable motion with stability index $\alpha(t)$ that varies in time t, and it exhibits semi-long-range dependence structure as LTFSM does. Similarly, to combine the properties of both LmFSM and LTFSM, we define another new stochastic process, called *linear tempered multifractional stable motion* (LTmFSM), by adding an exponentiel tempering to the power-law kernel of LmFSM. This new process is also of semi-long-range dependence structure. We investigate basic properties of the two new processes, including scaling properties, tail probabilities, absolute moment, sample path properties, pointwise Hölder exponent, Hölder continuity of quasi norm and (strong) localisability. Such properties are important and have been widely studied. For instance, Falconer and Liu [7] have investigated sample path properties, localisability and strong localisability of LFmSM; Le Guével and Lévy Véhel [8] have investigated the pointwise Hölder exponent of LFmSM; Ayache and Hamonier [2] have examined the fine path properties of LmFSM; Meerschaert and Sabzikar [9] have studied scaling properties, sample path properties and Hölder continuity of quasi norm of LTFSM.

The reader will note that, in this work, our emphasis is on the properties that set apart LTFmSM and LTmFSM, rather than on their common ones. Further work is needed to introduce and study linear tempered multifractional multistable motion (LTmFmSM). We believe that studying the specific properties of LTFmSM and LTmFSM will be helpful for future investigation of LTmFmSM.

The remainder of this paper is organized as follows. In Section 2, we define the linear tempered fractional multistable motion and the linear tempered multifractional stable motion. In Section 3, we elucidate the dependence structure of the two stochastic processes. In Sections 4 - 8, we analyze their properties.

2 Definitions of LTFmSM and LTmFSM

Throughout this paper, for given $0 < a \leq b \leq 2$, the function $\alpha : \mathbf{R} \longrightarrow [a, b]$ will be a Lebesgue measurable function that will play the role of a varying stability index. We recall the definition of variable exponent Lebesgue space:

$$\mathcal{F}_\alpha := \{f : f \text{ is measurable with } ||f||_\alpha < \infty\}$$

where

$$||f||_\alpha := \left\{\lambda > 0 : \int_{-\infty}^{\infty} \left|\frac{f(x)}{\lambda}\right|^{\alpha(x)} dx = 1\right\}. \tag{2.2}$$

Then $||\cdot||_\alpha$ is a quasinorm.

Falconer and Liu [7] defined the *multistable stochastic integral* $I(f) := \int f(x) dM_\alpha(x), f \in \mathcal{F}_\alpha$, by specifying the finite-dimensional distribution of I. Here and after, $dM_\alpha(x)$ stands for the multistable measure, which is an independently scattered symmetric random measure. Assume $\alpha(x) \in [a, b] \subset (0, 2]$. Given $f_1, f_2, ..., f_d \in \mathcal{F}_\alpha$, Falconer and Liu defined a probability distribution on the vector $\big(I(f_1), I(f_2), ..., I(f_d)\big) \in \mathbf{R}^d$ by the following characteristic function

$$\mathbf{E}\Big[e^{i\sum_{k=1}^d \theta_k I(f_k)}\Big] = \exp\left\{-\int_{-\infty}^{\infty}\Big|\sum_{k=1}^n \theta_k f_k(x)\Big|^{\alpha(x)} dx\right\}.$$

The essential point is that $\alpha(x)$ may vary with x. With the definition of multistable stochastic integral, Falconer and Liu [7] (cf. Proposition 4.3 therein) defined linear fractional multistable motion (LFmSM)

$$X(t) = \int_{-\infty}^{\infty}\Big[(t-x)_+^{H-\frac{1}{\alpha(x)}} - (-x)_+^{H-\frac{1}{\alpha(x)}}\Big] dM_\alpha(x). \tag{2.3}$$

They also investigated some basic properties of LFmSM, such as localisability and strong localisability.

By adding an exponential tempering to the power-law kernel in LFSM (1.1), that is

$$X_{H,\alpha,\lambda}(t) := \int_{-\infty}^{\infty}\Big[e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{\alpha}}\Big] dZ_\alpha(x), \tag{2.4}$$

$\lambda > 0, 0 < \alpha < 2$ and $H > 0$, Meerschaert and Sabzikar [9] recently defined the so-called linear tempered fractional stable motion (LTFSM). They showed that LTFSM is short memory, but its increments behave like long memory when λ is very small. Thus LTFSM exhibits semi-long-range dependence structure, and it provides a useful alternative model for data that exhibit strong dependence.

Similarly, by adding an exponential tempering to the power-law kernel in a LFmSM (2.3), we define the following linear tempered fractional multistable motion. Such process is an extension of both LFmSM and LTFSM mentioned above.

Definition 1 Let $\alpha(x) \in [a,b] \subset (0,2]$ be a continuous function on $\mathbf{R}$. Given an independently scattered symmetric multistable random measure $dM_\alpha(x)$ on $\mathbf{R}$, the multistable stochastic integral

$$X_{H,\alpha(x),\lambda}(t) := \int_{-\infty}^{\infty} \Big[e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha(x)}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{\alpha(x)}}\Big] dM_\alpha(x) \qquad (2.5)$$

with $0 < H < 1, \lambda > 0, (x)_+ = \max\{x, 0\}$, and $0^\gamma = 0, \gamma \in \mathbf{R}$, will be called a *linear tempered fractional multistable motion* (LTFmSM).

Remark 1 With the exponential tempering, we can also define *multistable Yaglom noise*

$$Y_{H,\alpha(x),\lambda}(t) = \int_{-\infty}^{\infty} \Big[e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha(x)}}\Big] dM_\alpha(x), \qquad \lambda > 0.$$

In particular, when $\alpha(x) \equiv 1/H \in (0,2]$, multistable Yaglom noise is known as Ornstein-Uhlenbeck process, see Example 3.6.3 of Samorodnitsky and Taqqu [11]. When $\alpha(x) \equiv \alpha$ for some constant α, multistable Yaglom noise is called stable Yaglom noise, see Meerschaert and Sabzikar [9]. It is obvious that fractional multistable Yaglom noise is a multistable stochastic integral. It is also easy to see that

$$X_{H,\alpha(x),\lambda}(t) = Y_{H,\alpha(x),\lambda}(t) - Y_{H,\alpha(x),\lambda}(0), \qquad \lambda > 0.$$

Denote by

$$G_{H,\alpha(x),\lambda}(t,x) = e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha(x)}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{\alpha(x)}}, \qquad \lambda > 0. \qquad (2.6)$$

It is easy to check that the function $G_{H,\alpha(x),\lambda}(t,x)$ belong to $\mathcal{F}_\alpha$, so that LTFmSM is well defined. Moreover, by the definition of multistable integral (cf. Falconer and Liu [7]), the characteristic function of $X_{H,\alpha(x),\lambda}(t)$ is given as follows:

$$\mathbf{E}\Big[e^{i\sum_{k=1}^{d}\theta_k X_{H,\alpha(x),\lambda}(t_k)}\Big] = \exp\bigg\{-\int_{-\infty}^{\infty}\Big|\sum_{k=1}^{d}\theta_k G_{H,\alpha(x),\lambda}(t_k,x)\Big|^{\alpha(x)} dx\bigg\}. \qquad (2.7)$$

Similarly, when the Hurst parameter H of (2.4) varies with time t, we have another extension of LTFSM.

Definition 2 Let $H_t \in [a,b]$ be a continuous function on $\mathbf{R}$. Given an independent scattered SαS stable random measure $dZ_\alpha(x)$ on $\mathbf{R}$ with control measure dx, the stable stochastic integral

$$X_{H_t,\alpha,\lambda}(t) := \int_{-\infty}^{\infty} \Big[e^{-\lambda(t-x)_+}(t-x)_+^{H_t-\frac{1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H_t-\frac{1}{\alpha}}\Big] dZ_\alpha(x) \qquad (2.8)$$

with $0 < \alpha \le 2$, $\lambda > 0, (x)_+ = \max\{x, 0\}$, and $0^\gamma = 0, \gamma \in \mathbf{R}$, will be called a *linear tempered multifractional stable motion* (LTmFSM).

Denote

$$G_{H_t,\alpha,\lambda}(t,x) = e^{-\lambda(t-x)_+}(t-x)_+^{H_t-\frac{1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H_t-\frac{1}{\alpha}}, \qquad \lambda \ge 0.$$

By the definition of stable integral (cf. Samorodnitsky and Taqqu [11]), the characteristic function of $X_{H_t,\alpha,\lambda}(t)$ is given as follows:

$$\mathbf{E}\Big[e^{i\sum_{k=1}^{n}\theta_k X_{H_t,\alpha,\lambda}(t_k)}\Big] = \exp\bigg\{-\int_{-\infty}^{\infty}\Big|\sum_{k=1}^{n}\theta_k G_{H_t,\alpha,\lambda}(t_k,x)\Big|^{\alpha} dx\bigg\}.$$

The characteristic function of $X_{H_t,\alpha,\lambda}(t)$ is given as follows:

$$\mathbf{E}\Big[e^{i\sum_{k=1}^{n}\theta_k X_{H_t,\alpha,\lambda}(t_k)}\Big] = \exp\bigg\{-\int_{-\infty}^{\infty}\Big|\sum_{k=1}^{n}\theta_k G_{H_t,\alpha,\lambda}(t_k,x)\Big|^{\alpha}dx\bigg\}.$$

3 Dependence structure of LTFmSM and LTmFSM

In this section, we study the behaviour of increments of LTFmSM and LTmFSM, usually termed the "noise" of these processes.

Denote by

$$Y(t) = X(t+1) - X(t) \quad \text{for integers } -\infty < t < \infty$$

the noise of the processes X. Astrauskas et al. [1] studied the dependence structure of linear fractional stable motion using the following nonparametric measure of dependence (see also Meerschaert and Sabzikar [9]). Define

$$R_{t_1}(t) = R(\theta_1,\theta_2,t_1,t_1+t) := \mathbf{E}\Big[e^{i(\theta_1 Y(t_1)+\theta_2 Y(t_1+t))}\Big] - \mathbf{E}\Big[e^{i\theta_1 Y(t_1)}\Big]\mathbf{E}\Big[e^{i\theta_2 Y(t_1+t)}\Big]$$

for $t_1, t, \theta_1, \theta_2 \in \mathbf{R}$. If we also define

$$\begin{aligned} I(\theta_1,\theta_2,t_1,t_1+t) &= \log\Big(\mathbf{E}\Big[e^{i\theta_1 Y(t_1)}\Big]\Big) + \log\Big(\mathbf{E}\Big[e^{i\theta_2 Y(t_1+t)}\Big]\Big) \\ &\quad - \log\Big(\mathbf{E}\Big[e^{i(\theta_1 Y(t_1)+\theta_2 Y(t_1+t))}\Big]\Big), \end{aligned}$$

then we have

$$R_{t_1}(t) = K(\theta_1,\theta_2,t_1,t_1+t)\Big(e^{-I(\theta_1,\theta_2,t_1,t_1+t)} - 1\Big), \tag{3.9}$$

where

$$K(\theta_1,\theta_2,t_1,t_1+t) = \mathbf{E}\Big[e^{i\theta_1 Y(t_1)}\Big]\mathbf{E}\Big[e^{i\theta_2 Y(t_1+t)}\Big].$$

In particular, for stationary processes, $R_{t_1}(t)$ does not depend on t_1, see Meerschaert and Sabzikar [9]. In this case, we denote $R_{t_1}(t)$ by $R(t)$ for simplicity. Note however that the increments of the two processes that we define in this work are not stationary in general.

We first recall the dependence structure of LTFSM. Given two real-valued functions $f(t), g(t)$ on $\mathbf{R}$, we will write

$$f(t) \preceq g(t)$$

if $|f(t)/g(t)| \le C_1$ for all $t > 0$ sufficiently large and some $0 < C_1 < \infty$. In particular, if $f(t) \preceq g(t)$ and $g(t) \preceq f(t)$, we will write

$$f(t) \asymp g(t).$$

Thus $f(t) \asymp g(t)$ is equivalent to $C_1 \le |f(t)/g(t)| \le C_2$ for all $t > 0$ sufficiently large and some $0 < C_1 < C_2 < \infty$. With these notations, Meerschaert and Sabzikar [9] recently proved that if $\lambda > 0$ and $0 < \alpha \le 1$, then TFSN has the following property

$$R(t) \asymp e^{-\lambda\alpha t} t^{H\alpha - 1}$$

for $\theta_1\theta_2 \ne 0$. Meerschaert and Sabzikar [9] also proved that if $\lambda > 0, 1 < \alpha < 2$ and $\frac{1}{\alpha} < H$, then TFSN has the following property

$$R(t) \asymp e^{-\lambda t} t^{H - \frac{1}{\alpha}}$$

for $\theta_1\theta_2 \ne 0$.

3.1 Dependence structure of LTFmSM

The following two theorems show that LTFmSM and LTFSM share the similar dependence structure.

Definition 3 Given an LTFmSM defined by (2.5), we define the tempered fractional multistable noise (TFmSN)

$$Y_{H,\alpha(x),\lambda}(t) := X_{H,\alpha(x),\lambda}(t+1) - X_{H,\alpha(x),\lambda}(t) \tag{3.10}$$

for integers $-\infty < t < \infty$.

In particular, if $\alpha(x) \equiv \alpha$ for a constant $\alpha \in (0,2]$, then the TFmSN reduces to the tempered fractional stable noise, see Meerschaert and Sabzikar [9].

Proposition 1 *Let $\alpha(x) \in [a,b] \subset (0,1)$ be a continuous function on $\mathbf{R}$. Let $Y_{H,\alpha(x),\lambda}(t)$ be the tempered fractional multistable noise (3.10). Recall $R_{t_1}(t)$ defined by (3.9) with $Y(t) = Y_{H,\alpha(x),\lambda}(t)$. Assume $\lambda > 0$. Then*

$$e^{-\lambda b t} t^{Ha-1} \preceq R_{t_1}(t) \preceq e^{-\lambda a t} t^{Hb-1} \tag{3.11}$$

for any $t_1 \in \mathbf{R}$ and $\theta_1\theta_2 \neq 0$.

Proof. By the definition (2.5), TFmSN has the following representation

$$Y_{H,\alpha(x),\lambda}(t) = \int_{-\infty}^{\infty} \Big[e^{-\lambda(t+1-x)_+} (t+1-x)_+^{H-\frac{1}{\alpha(x)}} - e^{-\lambda(t-x)_+} (t-x)_+^{H-\frac{1}{\alpha(x)}} \Big] dM_\alpha(x).$$

Define $g_t(x) = e^{-\lambda(t-x)_+} (t-x)_+^{H-\frac{1}{\alpha(x)}}$ for $t \in \mathbf{R}$ and write

$$\begin{aligned} I(\theta_1,\theta_2,t_1,t_1+t) &= \int_{-\infty}^{\infty} \Big| \theta_1[g_{t_1+1}(x) - g_{t_1}(x)] + \theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} dx \\ &\quad - \int_{-\infty}^{\infty} \Big| \theta_1[g_{t_1+1}(x) - g_{t_1}(x)] \Big|^{\alpha(x)} dx \\ &\quad - \int_{-\infty}^{\infty} \Big| \theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} dx \\ &= I_1(t) + I_2(t), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} I_1(t) = \int_{-\infty}^{t_1} \bigg(&\Big| \theta_1[g_{t_1+1}(x) - g_{t_1}(x)] + \theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} \\ &- \Big| \theta_1[g_{t_1+1}(x) - g_{t_1}(x)] \Big|^{\alpha(x)} - \Big| \theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} \bigg) dx \end{aligned}$$

and

$$\begin{aligned} I_2(t) = \int_{t_1}^{t_1+1} \bigg(&\Big| \theta_1 g_{t_1+1}(x) + \theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} \\ &- \Big| \theta_1 g_{t_1+1}(x) \Big|^{\alpha(x)} - \Big| \theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} \bigg) dx. \end{aligned}$$

Using the following inequalities

$$0 \leq |x_1|^\alpha + |x_2|^\alpha - |x_1 + x_2|^\alpha \leq 2\,|x_2|^\alpha \tag{3.13}$$

for all $x_1, x_2 \in \mathbf{R}$ and $0 < \alpha \leq 1$, we obtain

$$I_1(t) \leq 0 \qquad \text{and} \qquad I_2(t) \leq 0. \tag{3.14}$$

First, we give an estimation for $I_1(t)$. By (3.13), it is easy to see that for $t \geq 1$,

$$\begin{aligned}|I_1(t)| &\leq 2 \int_{-\infty}^{t_1} \Big|\theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)]\Big|^{\alpha(x)} dx \\ &\leq 2\Big(|\theta_2|^a + |\theta_2|^b\Big) e^{-\lambda a t} t^{Hb-1} \int_{-\infty}^{t_1} \Big|[g_{t_1+t+1}(x) - g_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha(x)} - H}\Big|^{\alpha(x)} dx.\end{aligned}$$

Notice that $H\alpha(x) \leq 1$. For $x \leq t_1$ and $t > 1$, we deduce that

$$\begin{aligned}&\Big|[g_{t_1+t+1}(x) - g_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha(x)} - H}\Big|^{\alpha(x)} \\ &= \Bigg|e^{-\lambda(t_1-x)}\Bigg(e^{-\lambda}\Big(1 + \frac{1+t_1-x}{t}\Big)^{H - \frac{1}{\alpha(x)}} - \Big(1 + \frac{t_1-x}{t}\Big)^{H - \frac{1}{\alpha(x)}}\Bigg)\Bigg|^{\alpha(x)} \\ &\leq e^{-\lambda a(t_1-x)}(1+e^{-\lambda})^b\Bigg(\Big(1 + \frac{1+t_1-x}{t}\Big)^{H\alpha(x)-1} + \Big(1 + \frac{t_1-x}{t}\Big)^{H\alpha(x)-1}\Bigg) \\ &\leq 2e^{-\lambda a(t_1-x)}(1+e^{-\lambda})^b \\ &=: F_\lambda(x).\end{aligned} \tag{3.15}$$

Thus

$$\begin{aligned}|I_1(t)|\, e^{\lambda a t} t^{1-Hb} &\leq 2(|\theta_2|^a + |\theta_2|^b) \int_{-\infty}^{t_1} F_\lambda(x) dx \\ &\leq C_1(|\theta_2|^a + |\theta_2|^b),\end{aligned}$$

where $C_1 > 0$ depends only on a, b and λ. Hence

$$|I_1(t)| \leq C_1(|\theta_2|^a + |\theta_2|^b) e^{-\lambda a t} t^{Hb-1}. \tag{3.16}$$

Next for $I_2(t)$, we have the following estimation. Using inequality (3.13) again, we obtain

$$|I_2(t)| \leq 2 \int_{t_1}^{t_1+1} \Big|\theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)]\Big|^{\alpha(x)} dx. \tag{3.17}$$

Applying the mean value theorem to see that for $t \geq 2$ and any $x \in (t_1, t_1+1)$, we have

$$\begin{aligned}&\Big|g_{t_1+t+1}(x) - g_{t_1+t}(x)\Big| \\ &\leq \Big| - \lambda e^{-\lambda(u-x)}(u-x)^{H - \frac{1}{\alpha(x)}} + (H - \frac{1}{\alpha(x)}) e^{-\lambda(u-x)}(u-x)^{H - \frac{1}{\alpha(x)} - 1}\Big| \\ &\leq e^{-\lambda(t-1)}\Big(\lambda(t-1)^{H - \frac{1}{\alpha(x)}} + (\frac{1}{\alpha(x)} - H)(t-1)^{H - \frac{1}{\alpha(x)} - 1}\Big) \\ &\leq e^{-\lambda(t-1)}(\frac{1}{\alpha(x)} - H + \lambda)(t-1)^{H - \frac{1}{\alpha(x)}},\end{aligned}$$

where $u \in (t_1+t, t_1+t+1)$. Returning to (3.17), we get

$$|I_2(t)| \leq 2(|\theta_2|^a + |\theta_2|^b) \int_{t_1}^{t_1+1} \Big| e^{-\lambda(t-1)}(\frac{1}{\alpha(x)} - H + \lambda)|t-1|^{H-\frac{1}{\alpha(x)}} \Big|^{\alpha(x)} dx$$
$$\leq C_2(|\theta_2|^a + |\theta_2|^b) e^{-\lambda a t} t^{Hb-1} \tag{3.18}$$

for large t, where $C_2 > 0$ depends only on a, b, H and λ. Combining the inequalities (3.12), (3.14),(3.16) and (3.18) together, we obtain

$$0 \leq -I(\theta_1, \theta_2, t_1, t_1+t) \leq C_3 e^{-\lambda a t} t^{Hb-1} \quad \text{as } t \to \infty, \tag{3.19}$$

where C_3 does not depend on t. Using the following equality

$$|x_1|^\alpha + |x_2|^\alpha - |x_1+x_2|^\alpha = |x_2|^\alpha - \frac{\alpha}{|x_1+\theta x_2|^{1-\alpha}} |x_2|$$

for all $x_1, x_2 \neq 0$ with $|x_2|$ small enough, $0 < \alpha < 1$ and some $|\theta| \leq 1$, we obtain for any $x_1 \neq 0$ and $0 < \alpha < 1$,

$$|x_1|^\alpha + |x_2|^\alpha - |x_1+x_2|^\alpha \sim |x_2|^\alpha \tag{3.20}$$

as $x_2 \to 0$. It is easy to see that for $t_1 \leq x \leq t_1+1$ and $t > 2$,

$$\lim_{t\to\infty} [g_{t_1+t+1}(x) - g_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha(x)} - H}$$
$$= \lim_{t\to\infty} e^{-\lambda(t_1-x)} \Big(e^{-\lambda} \Big(1 + \frac{1+t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}} - \Big(1 + \frac{t_1-x}{t}\Big)_+^{H-\frac{1}{\alpha(x)}} \Big)$$
$$= e^{-\lambda(t_1-x)} \big(e^{-\lambda} - 1\big) \tag{3.21}$$

and

$$\Big| [g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} \leq \Big| e^{-\lambda(t_1-x)} \big(2 + e^{-\lambda}\big) \Big|^{\alpha(x)} e^{-a\lambda t} t^{1-Ha}.$$

Thus $[g_{t_1+t+1}(x) - g_{t_1+t}(x)]$, $t_1 \leq x \leq t_1+1$, converges uniformly to 0 as $t \to \infty$. Applying the dominated convergence theorem yields for $\theta_1\theta_2 \neq 0$, we have

$$|I_2(t)| \succeq \int_{t_1}^{t_1+1} \Big| \theta_2 [g_{t_1+t+1}(x) - g_{t_1+t}(x)] \Big|^{\alpha(x)} dx \tag{3.22}$$

and

$$\liminf_{t\to\infty} |I_2(t)|\, e^{\lambda b t} t^{1-Ha} \geq \lim_{t\to\infty} \int_{t_1}^{t_1+1} \Big| \theta_2 [g_{t_1+t+1}(x) - g_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha(x)} - H} \Big|^{\alpha(x)} dx$$
$$= \int_{t_1}^{t_1+1} \Big| \theta_2 \big[e^{-\lambda(t_1-x)} \big(1 - e^{-\lambda}\big) \big] \Big|^{\alpha(x)} dx. \tag{3.23}$$

Then (3.12), (3.14) and (3.23) implies that for all large t,

$$-I(\theta_1, \theta_2, t_1, t_1+t) \geq -I_2(t) = |I_2(t)| \geq \frac{1}{2} C_2\, e^{-\lambda b t} t^{Ha-1}, \tag{3.24}$$

where $C_2 = \int_{t_1}^{t_1+1} \big| \theta_2 \big[e^{-\lambda(t_1-x)} \big(1 - e^{-\lambda}\big) \big] \big|^{\alpha(x)} dx$ does not depend on t. Combining (3.19) and (3.24) together, we have

$$e^{-\lambda b t} t^{Ha-1} \preceq I(\theta_1, \theta_2, t_1, t_1+t) \preceq e^{-\lambda a t} t^{Hb-1} \tag{3.25}$$

for $\theta_1\theta_2 \neq 0$. It is easy to see that

$$\begin{aligned}
&K(\theta_1,\theta_2,t_1,t_1+t)\\
&= \exp\Big\{-\int_{-\infty}^{1}\Big|\theta_1[e^{-\lambda(1-u)_+}(1-u)_+^{H-\frac{1}{\alpha(t_1+u)}} - e^{-\lambda(-u)_+}(-u)_+^{H-\frac{1}{\alpha(t_1+u)}}]\Big|^{\alpha(t_1+u)}du\Big\}\\
&\times \exp\Big\{-\int_{-\infty}^{1}\Big|\theta_2[e^{-\lambda(1-u)_+}(1-u)_+^{H-\frac{1}{\alpha(t_1+t+u)}} - e^{-\lambda(-u)_+}(-u)_+^{H-\frac{1}{\alpha(t_1+t+u)}}]\Big|^{\alpha(t_1+t+u)}du\Big\}\\
&\geq \exp\Big\{-2\big(|\theta_1|^a+|\theta_2|^b\big)\int_{-\infty}^{1}M_\lambda(u)du\Big\},
\end{aligned}$$

where

$$M_\lambda(u) = e^{-\lambda a(1-u)_+}\Big((1-u)_+^{Ha-1}+(1-u)_+^{Hb-1}\Big) + e^{-\lambda a(-u)_+}\Big((-u)_+^{Ha-1}+(-u)_+^{Hb-1}\Big)$$

is integrable on $(-\infty,1]$ with respect to u, and that

$$K(\theta_1,\theta_2,t_1,t_1+t) \leq 1.$$

Since $I(\theta_1,\theta_2,t_1,t_1+t)\to 0$ as $t\to\infty$, it follows from (3.9) that $R_{t_1}(t) \sim -K(\theta_1,\theta_2,t_1,t_1+t)I(\theta_1,\theta_2,t_1,t_1+t)$. Hence (3.11) follows by (3.25). □

Proposition 2 *Let $\alpha(x)\in[a,b]\subset(1,2]$ be a continuous function on $\mathbf{R}$. Let $Y_{H,\alpha(x),\lambda}(t)$ be the tempered fractional multistable noise (3.10). Recall $R_{t_1}(t)$ defined by (3.9) with $Y(t) = Y_{H,\alpha(x),\lambda}(t)$. Assume $\lambda>0$ and $1/a<H<1$. Then*

$$e^{-\lambda t}t^{H-\frac{1}{a}} \preceq R_{t_1}(t) \preceq e^{-\lambda t}t^{H-\frac{1}{b}} \tag{3.26}$$

for any $t_1\in\mathbf{R}$ and $\theta_1\theta_2\neq 0$.

Proof. Recall $I_1(t)$ and $I_2(t)$ defined by (3.12). Notice that

$$|x_1+x_2|^\alpha - |x_1|^\alpha - |x_2|^\alpha \sim \alpha\, x_1\, |x_2|^{\alpha-1} \tag{3.27}$$

for all $x_2\neq 0, x_1\to 0$ and $1<\alpha\leq 2$. First, we give an estimation for $I_1(t)$. It is easy to see that for $x\leq t_1+1$, $1/a<H$ and large t,

$$\begin{aligned}
&\Big|[g_{t_1+t+1}(x)-g_{t_1+t}(x)]e^{\lambda t}t^{\frac{1}{\alpha(x)}-H}\Big|\\
&= \Big|e^{-\lambda(t_1-x)}\Big(e^{-\lambda}\Big(1+\frac{1+t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}} - \Big(1+\frac{t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}}\Big)\Big|\\
&\leq e^{-\lambda(t_1-x)}(1+e^{-\lambda})\Big(\Big(1+\frac{1+t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}} + \Big(1+\frac{t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}}\Big)\\
&\leq 2e^{-\lambda(t_1-x)}(1+e^{-\lambda})(2+t_1-x)^{H-\frac{1}{\alpha(x)}}
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
&[g_{t_1+t+1}(x)-g_{t_1+t}(x)]e^{\lambda t}t^{\frac{1}{\alpha(x)}-H}\\
&= e^{-\lambda(t_1-x)}\Big(\Big(e^{-\lambda/(H-\frac{1}{\alpha(x)})}(1+\frac{1}{t}) + e^{-\lambda/(H-\frac{1}{\alpha(x)})}\frac{t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}} - \Big(1+\frac{t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}}\Big)\\
&\leq e^{-\lambda(t_1-x)}\Big(\Big(1+e^{-\lambda/(H-\frac{1}{\alpha(x)})}\frac{t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}} - \Big(1+\frac{t_1-x}{t}\Big)^{H-\frac{1}{\alpha(x)}}\Big)\\
&\leq 0.
\end{aligned} \tag{3.29}$$

Then (3.28) and (3.29) together implies that for $x \le t_1 + 1$, $1/a < H$ and large t,

$$\begin{aligned} 0 &\ge g_{t_1+t+1}(x) - g_{t_1+t}(x) \\ &\ge -2(1+e^{-\lambda})e^{-\lambda t}t^{H-\frac{1}{b}}e^{-\lambda(t_1-x)}(2+t_1-x)^{H-\frac{1}{b}} \\ &\ge -2(1+e^{-\lambda})e^{2\lambda+\frac{1}{b}-H}\Big(\frac{H-\frac{1}{b}}{b}\Big)^{H-\frac{1}{b}}\ e^{-\lambda t}t^{H-\frac{1}{b}}. \end{aligned}$$

Thus $[g_{t_1+t+1}(x) - g_{t_1+t}(x)]$ is negative and converges to 0 uniformly for $x \in (-\infty, t_1]$ as $t \to \infty$. By (3.27), we obtain for large t,

$$\begin{aligned} &|I_1(t)| \\ &\le 2\int_{-\infty}^{t_1} \alpha(x)\Big|\theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)]\Big|\Big|\theta_1[g_{t_1+1}(x) - g_{t_1}(x)]\Big|^{\alpha(x)-1}dx \\ &\le 4|\theta_2|\max\Big\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\Big\}\int_{-\infty}^{t_1}\Big|g_{t_1+t+1}(x) - g_{t_1+t}(x)\Big|\Big|g_{t_1+1}(x) - g_{t_1}(x)\Big|^{\alpha(x)-1}dx. \end{aligned}$$

Therefore, for large t and $x \le t_1$,

$$\begin{aligned} &|I_1(t)| \\ &\le 4|\theta_2|\max\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\} \\ &\quad\times\int_{-\infty}^{t_1} e^{-\lambda t}t^{H-\frac{1}{\alpha(x)}}(1+e^{-\lambda})\Big(2+t_1-x\Big)^{H-\frac{1}{\alpha(x)}}e^{-\lambda(t_1-x)}\Big|g_{t_1+1}(x) - g_{t_1}(x)\Big|^{\alpha(x)-1}dx \\ &\le 4|\theta_2|\max\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\}e^{-\lambda t}t^{H-\frac{1}{b}} \\ &\quad\times\int_{-\infty}^{t_1}(1+e^{-\lambda})\Big(2+t_1-x\Big)^{H-\frac{1}{\alpha(x)}}e^{-\lambda(t_1-x)}\Big|g_{t_1+1}(x) - g_{t_1}(x)\Big|^{\alpha(x)-1}dx. \end{aligned} \tag{3.30}$$

Recall $g_t(x) = e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha(x)}}$, and that

$$\Big|g_{t_1+1}(x) - g_{t_1}(x)\Big|^{\alpha(x)-1} \le \Big|g_{t_1+1}(x)\Big|^{\alpha(x)-1} + \Big|g_{t_1}(x)\Big|^{\alpha(x)-1}$$

(cf. (3.13) for the last inequality). Since $\alpha(x) - 1 \le b - 1 < 1$ and $H > \frac{1}{a} \ge \frac{1}{\alpha(x)}$, from (3.30), we obtain

$$|I_1(t)| \le C_1|\theta_2|\max\Big\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\Big\}e^{-\lambda t}t^{H-\frac{1}{b}}, \tag{3.31}$$

where C_1 does not depend on t. Next, we give an estimation for $I_2(t)$. Using (3.27) again, we obtain for large t,

$$\begin{aligned} |I_2(t)| = &\int_{t_1}^{t_1+1}\Bigg|\Big|\theta_1 g_{t_1+1}(x) + \theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)]\Big|^{\alpha(x)} \\ &\qquad -\Big|\theta_1 g_{t_1+1}(x)\Big|^{\alpha(x)} - \Big|\theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)]\Big|^{\alpha(x)}\Bigg|dx \\ &\le 2\int_{t_1}^{t_1+1}\alpha(x)\Big|\theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)]\Big|\Big|\theta_1 g_{t_1+1}(x)\Big|^{\alpha(x)-1}dx \\ &\le 4|\theta_2|\max\Big\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\Big\}\int_{t_1}^{t_1+1}\Big|g_{t_1+t+1}(x) - g_{t_1+t}(x)\Big|\Big|g_{t_1+1}(x)\Big|^{\alpha(x)-1}dx. \end{aligned}$$

By (3.28), it follows that for large t,

$$
\begin{aligned}
|I_2(t)| &\le 4|\theta_2| \max\Big\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\Big\} e^{-\lambda t} t^{H-\frac{1}{b}} \\
&\quad \times \int_{t_1}^{t_1+1} e^{-\lambda(t_1-x)}(1+e^{-\lambda})(2+t_1-x)^{H-\frac{1}{\alpha(x)}} \Big|g_{t_1+1}(x)\Big|^{\alpha(x)-1} dx \\
&\le C_2|\theta_2| \max\Big\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\Big\}\ e^{-\lambda t} t^{H-\frac{1}{b}}, \qquad (3.32)
\end{aligned}
$$

where C_2 does not depend on t. Therefore, from (3.31) and (3.32), for large t,

$$
|I(\theta_1, \theta_2, t_1, t_1+t)| \le C_3 e^{-\lambda t} t^{H-\frac{1}{b}}. \qquad (3.33)
$$

where C_3 does not depend on t.

By (3.27), we have

$$
|I_2(t)| \succeq \int_{t_1}^{t_1+1} \alpha(x)\Big|\theta_2[g_{t_1+t+1}(x) - g_{t_1+t}(x)]\Big|\Big|\theta_1 g_{t_1+1}(x)\Big|^{\alpha(x)-1} dx.
$$

Applying (3.21) and the dominated convergence theorem yields

$$
\begin{aligned}
&\liminf_{t\to\infty} |I_2(t)| e^{\lambda t} t^{\frac{1}{a}-H} \\
&\ge \lim_{t\to\infty} |\theta_2| \min\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\} \\
&\quad \times \int_{t_1}^{t_1+1} \Big|[g_{t_1+t+1}(x) - g_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha(x)}-H}\Big|\Big|g_{t_1+1}(x)\Big|^{\alpha(x)-1} dx \\
&= |\theta_2| \min\{|\theta_1|^{a-1}, |\theta_1|^{b-1}\} \int_{t_1}^{t_1+1} e^{-\lambda(t_1-x)}\big(1-e^{-\lambda}\big)\Big|g_{t_1+1}(x)\Big|^{\alpha(x)-1} dx.
\end{aligned}
$$

Thus for $\theta_1\theta_2 \ne 0$,

$$
|I_2(t)| \succeq e^{-\lambda t} t^{H-\frac{1}{a}}. \qquad (3.34)
$$

Notice that (3.27) and (3.29) implies that

$$
I_1(t) I_2(t) \ge 0 \qquad (3.35)
$$

for large t. Combining (3.33) and (3.34) together, we have for $\theta_1\theta_2 \ne 0$,

$$
e^{-\lambda t} t^{H-\frac{1}{a}} \preceq |I_2(t)| \preceq |I(\theta_1, \theta_2, t_1, t_1+t)| \preceq e^{-\lambda t} t^{H-\frac{1}{b}}. \qquad (3.36)
$$

Since $I(\theta_1, \theta_2, t_1, t_1+t) \to 0$ as $t \to \infty$, it follows from (3.9) that $R_{t_1}(t) \sim -K(\theta_1, \theta_2, t_1, t_1+t) I(\theta_1, \theta_2, t_1, t_1+t)$; hence (3.26) holds. □

3.2 Dependence structure of LTmFSM

In this section, we consider the increment of LTmFSM. The following two theorems extend the dependence structure of LTFSM to the case of LTmFSM.

Definition 4 Given an LTmFSM defined by (2.8), we define the tempered multifractional stable noise (TmFSN)

$$Y_{H_t,\alpha,\lambda}(t) := X_{H_{t+1},\alpha,\lambda}(t+1) - X_{H_t,\alpha,\lambda}(t) \tag{3.37}$$

for integers $-\infty < t < \infty$.

In particular, if $H_t \equiv H$ for a constant $H \in (0,1)$, then the TmFSN reduces to the tempered fractional stable noise. The next theorem shows that LTmFSM has a dependence structure more general than that of LTFSM.

Proposition 3 *Let $H_t \in [a,b]$ be a continuous function on $\mathbf{R}$. Let $Y_{H_t,\alpha,\lambda}(t)$ be a tempered multifractional stable noise (3.37) for some $0 < \alpha < 1$. Recall $R_{t_1}(t)$ defined by (3.9) with $Y(t) = Y_{H_t,\alpha,\lambda}(t)$. Assume $\lambda > 0$. Then*

$$R_{t_1}(t) \asymp e^{-\lambda\alpha t} t^{\alpha H_t - 1} \tag{3.38}$$

for $\theta_1\theta_2 \neq 0$.

Proof. By the definition (2.8), TmFSN has the following representation

$$Y_{H_t,\alpha,\lambda}(t) = \int_{-\infty}^{\infty} \Big[e^{-\lambda(t+1-x)_+}(t+1-x)_+^{H_{t+1}-\frac{1}{\alpha}} - e^{-\lambda(t-x)_+}(t-x)_+^{H_t-\frac{1}{\alpha}} \Big] dZ_\alpha(x).$$

Define $h_t(x) = (t-x)_+^{H_t-\frac{1}{\alpha}} e^{-\lambda(t-x)_+}$ for $t \in \mathbf{R}$ and write

$$\begin{aligned} I(\theta_1,\theta_2,t_1,t_1+t) &= \int_{-\infty}^{\infty} \Big| \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] + \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \Big|^\alpha dx \\ &\quad - \int_{-\infty}^{\infty} \Big| \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] \Big|^\alpha dx \\ &\quad - \int_{-\infty}^{\infty} \Big| \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \Big|^\alpha dx \\ &= I_3(t) + I_4(t), \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} I_3(t) = \int_{-\infty}^{t_1} \Big(&\Big| \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] + \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \Big|^\alpha \\ &- \Big| \theta_1[h_{t_1+1}(x) - h_{t_1}(x)] \Big|^\alpha - \Big| \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \Big|^\alpha \Big) dx \end{aligned}$$

and

$$\begin{aligned} I_4(t) = \int_{t_1}^{t_1+1} \Big(&\Big| \theta_1 h_{t_1+1}(x) + \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \Big|^\alpha \\ &- \Big| \theta_1 h_{t_1+1}(x) \Big|^\alpha - \Big| \theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)] \Big|^\alpha \Big) dx. \end{aligned}$$

Using (3.13) again, we obtain

$$I_3(t) \le 0 \qquad \text{and} \qquad I_4(t) \le 0. \tag{3.40}$$

First, we give an estimation for $I_3(t)$. For large t,

$$\begin{aligned}|I_3(t)| &\le 2\int_{-\infty}^{t_1} \Big|\theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)]\Big|^{\alpha} dx \\ &\le 2|\theta_2|^{\alpha} e^{-\lambda\alpha t} t^{\alpha H_t - 1} \int_{-\infty}^{t_1} \Big|[h_{t_1+t+1}(x) - h_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha} - H_t}\Big|^{\alpha} dx.\end{aligned}$$

Recall $H_t \in [a, b]$. It is easy to see that for $x \le t_1$ and $t > 1$,

$$\begin{aligned}&\Big|[h_{t_1+t+1}(x) - h_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha} - H_t}\Big|^{\alpha} \\ &= \bigg|e^{-\lambda(t_1 - x)}\Big(e^{-\lambda}\Big(1 + \frac{1 + t_1 - x}{t}\Big)^{H_t - \frac{1}{\alpha}} - \Big(1 + \frac{t_1 - x}{t}\Big)^{H_t - \frac{1}{\alpha}}\Big)\bigg|^{\alpha} \\ &\le e^{-\lambda\alpha(t_1 - x)}(1 + e^{-\lambda})^{\alpha}\bigg(\Big(1 + \frac{1 + t_1 - x}{t}\Big)^{H_t\alpha - 1} + \Big(1 + \frac{t_1 - x}{t}\Big)^{H_t\alpha - 1}\bigg) \\ &\le e^{-\lambda\alpha(t_1 - x)}(1 + e^{-\lambda})^{\alpha} \max\Big\{2,\ \big(2 + t_1 - x\big)^{b\alpha - 1} + \big(1 + t_1 - x\big)^{b\alpha - 1}\Big\} \\ &:= F_{\lambda}(x).\end{aligned}$$

Thus

$$\begin{aligned}|I_3(t)| e^{\lambda\alpha t} t^{1 - \alpha H_t} &\le 2|\theta_2|^{\alpha} \int_{-\infty}^{t_1} F_{\lambda}(x) dx \\ &\le C_1\, |\theta_2|^{\alpha},\end{aligned}$$

where $C_1 > 0$ depends only on α, b and λ. Hence

$$|I_3(t)| \le C_1\, |\theta_2|^{\alpha} e^{-\lambda\alpha t} t^{\alpha H_t - 1}. \tag{3.41}$$

Next for $I_4(t)$, we have the following estimation. Using inequality (3.13) again, we obtain

$$\Big|I_4(t)\Big| \le 2\int_{t_1}^{t_1+1} \Big|\theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)]\Big|^{\alpha} dx. \tag{3.42}$$

Applying the mean value theorem to see that for $t \ge 2$ and any $x \in (t_1, t_1 + 1)$, we have

$$\begin{aligned}&\Big|h_{t_1+t+1}(x) - h_{t_1+t}(x)\Big| \\ &\le \Big| -\lambda e^{-\lambda(u - x)}(u - x)^{H_t - \frac{1}{\alpha}} + (H_t - \frac{1}{\alpha}) e^{-\lambda(u - x)}(u - x)^{H_t - \frac{1}{\alpha} - 1}\Big| \\ &\le e^{-\lambda(t - 1)}\Big(\lambda(t - 1)^{H_t - \frac{1}{\alpha}} + (\frac{1}{\alpha} - H_t)(t - 1)^{H_t - \frac{1}{\alpha} - 1}\Big) \\ &\le e^{-\lambda(t - 1)}(\frac{1}{\alpha} - H_t + \lambda)(t - 1)^{H_t - \frac{1}{\alpha}},\end{aligned} \tag{3.43}$$

where $u \in (t_1 + t, t_1 + t + 1)$. Returning to (3.42), we get

$$\begin{aligned}\Big|I_4(t)\Big| &\le 2|\theta_2|^{\alpha} \int_{t_1}^{t_1+1} \Big|e^{-\lambda(t - 1)}(\frac{1}{\alpha} - H_t + \lambda)|t - 1|^{H_t - \frac{1}{\alpha}}\Big|^{\alpha} dx \\ &\le C_2 |\theta_2|^{\alpha} e^{-\lambda\alpha t} t^{\alpha H_t - 1}\end{aligned} \tag{3.44}$$

for large t, where $C_2 > 0$ depends only on α, b and λ. Combining the inequalities (3.39), (3.40), (3.41) and (3.44) together, we obtain

$$0 \le -I(\theta_1, \theta_2, t_1, t_1 + t) \le C_3 |\theta_2|^\alpha e^{-\lambda\alpha t} t^{\alpha H_t - 1} \tag{3.45}$$

for large t, where C_3 does not depend on t. By (3.20), it holds for $t \to \infty$,

$$|I_4(t)| \succeq \int_{t_1}^{t_1+1} \Big| \theta_2 [h_{t_1+t+1}(x) - h_{t_1+t}(x)] \Big|^\alpha dx.$$

Similar to (3.21), it is easy to see that for $t_1 \le x \le t_1 + 1$ and $t > 2$,

$$\lim_{t\to\infty} \Big| [h_{t_1+t+1}(x) - h_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha} - H_t} \Big|^\alpha = \big| e^{-\lambda(t_1 - x)} \big(1 - e^{-\lambda}\big) \big|^\alpha. \tag{3.46}$$

Applying the dominated convergence theorem yields

$$\begin{aligned} \liminf_{t\to\infty} |I_4(t)|\, e^{\lambda\alpha t} t^{1 - H_t \alpha} &\ge \lim_{t\to\infty} \int_{t_1}^{t_1+1} \Big| \theta_2 [h_{t_1+t+1}(x) - h_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha} - H_t} \Big|^\alpha dx \\ &= \int_{t_1}^{t_1+1} \big| \theta_2 e^{-\lambda(t_1 - x)} \big(1 - e^{-\lambda}\big) \big|^\alpha dx. \end{aligned} \tag{3.47}$$

Then (3.39), (3.40) and (3.47) implies that for large t,

$$-I(\theta_1, \theta_2, t_1, t_1 + t) \ge -I_4(t) = |I_4(t)| \ge \frac{1}{2} C_3\, e^{-\lambda\alpha t} t^{\alpha H_t - 1}, \tag{3.48}$$

where $C_3 = \int_{t_1}^{t_1+1} \big| \theta_2 e^{-\lambda(t_1 - x)} \big(1 - e^{-\lambda}\big) \big|^\alpha dx$ does not depend on t. Combining (3.45) and (3.48) together, we have

$$I(\theta_1, \theta_2, t_1, t_1 + t) \asymp e^{-\lambda\alpha t} t^{\alpha H_t - 1} \tag{3.49}$$

for $\theta_1\theta_2 \ne 0$. It is easy to see that

$$\begin{aligned} &K(\theta_1, \theta_2, t_1, t_1 + t) \\ &= \exp\Big\{ - \int_{-\infty}^1 \Big| \theta_1 [e^{-\lambda(1-u)_+} (1 - u)_+^{H_{t_1+1} - \frac{1}{\alpha}} - e^{-\lambda(-u)_+} (-u)_+^{H_{t_1} - \frac{1}{\alpha}}] \Big|^\alpha du \Big\} \\ &\quad \times \exp\Big\{ - \int_{-\infty}^1 \Big| \theta_2 [e^{-\lambda(1-u)_+} (1 - u)_+^{H_{t_1+t+1} - \frac{1}{\alpha}} - e^{-\lambda(-u)_+} (-u)_+^{H_{t_1+t} - \frac{1}{\alpha}}] \Big|^\alpha du \Big\} \\ &\ge \exp\Big\{ - 2\Big(|\theta_1|^\alpha + |\theta_2|^\alpha \Big) \int_{-\infty}^1 T(u) du \Big\}, \end{aligned}$$

where

$$T(u) := e^{-\lambda\alpha(1-u)_+} \Big((1 - u)_+^{\alpha a - 1} + (1 - u)_+^{\alpha b - 1} \Big) + e^{-\lambda\alpha(-u)_+} \Big((-u)_+^{\alpha a - 1} + (-u)_+^{\alpha b - 1} \Big)$$

is integrable on $(-\infty, 1]$ with respect to u, and that $|K(\theta_1, \theta_2, t_1, t_1 + t)| \le 1$. Since $I(\theta_1, \theta_2, t_1, t_1 + t) \to 0$ as $t \to \infty$, it follows that $R_{t_1}(t) \sim -K(\theta_1, \theta_2, t_1, t_1 + t) I(\theta_1, \theta_2, t_1, t_1 + t)$; hence (3.38) follows by (3.49). □

Proposition 4 *Let $H_t \in [a,b]$ be a continuous function on* $\mathbf{R}$. *Let $Y_{H_t,\alpha,\lambda}(t)$ be a tempered multifractional stable noise (3.37). Recall $R_{t_1}(t)$ defined by (3.9) with $Y(t) = Y_{H_t,\alpha,\lambda}(t)$. Assume $\lambda > 0$, $1 < \alpha \le 2$ and $1/\alpha < H_t$. Then*

$$R_{t_1}(t) \asymp e^{-\lambda t} t^{H_t - \frac{1}{\alpha}} \tag{3.50}$$

for $\theta_1\theta_2 \neq 0$.

Proof. Recall $I_3(t)$ and $I_4(t)$ defined by (3.39). By an argument similar to (3.35), we have for large t,

$$I_3(t) I_4(t) \ge 0. \tag{3.51}$$

First, we give an estimation for $I_3(t)$. Using the inequality (3.27), we obtain

$$\begin{aligned} |I_3(t)| &\le 2 \int_{-\infty}^{t_1} \alpha \Big|\theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)]\Big| \Big|\theta_1[h_{t_1+1}(x) - h_{t_1}(x)]\Big|^{\alpha-1} dx \\ &\le 4|\theta_2||\theta_1|^{\alpha-1} \int_{-\infty}^{t_1} \Big|h_{t_1+t+1}(x) - h_{t_1+t}(x)\Big| \Big|h_{t_1+1}(x) - h_{t_1}(x)\Big|^{\alpha-1} dx. \end{aligned}$$

By an argument similar to (3.28), it is easy to see that for large t and $x \le t_1$,

$$\Big|[h_{t_1+t+1}(x) - h_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha} - H_t}\Big| \le 2e^{-\lambda(t_1-x)}(1+e^{-\lambda})(2+t_1-x)^{H_t - \frac{1}{\alpha}}. \tag{3.52}$$

Therefore, for large t and $x \le t_1$,

$$\begin{aligned} |I_3(t)| &\le 8|\theta_2||\theta_1|^{\alpha-1} \\ &\quad \times \int_{-\infty}^{t_1} e^{-\lambda t} t^{H_t - \frac{1}{\alpha}} (1+e^{-\lambda}) \big(2+t_1-x\big)^{H_t - \frac{1}{\alpha}} e^{-\lambda(t_1-x)} \Big|h_{t_1+1}(x) - h_{t_1}(x)\Big|^{\alpha-1} dx \\ &\le 8|\theta_2||\theta_1|^{\alpha-1} e^{-\lambda t} t^{H_t - \frac{1}{\alpha}} \\ &\quad \times \int_{-\infty}^{t_1} (1+e^{-\lambda}) \Big(2+t_1-x\Big)^{b - \frac{1}{\alpha}} e^{-\lambda(t_1-x)} \Big|h_{t_1+1}(x) - h_{t_1}(x)\Big|^{\alpha-1} dx. \end{aligned} \tag{3.53}$$

From (3.53), we obtain

$$|I_3(t)| \le C_1 |\theta_2||\theta_1|^{\alpha-1} e^{-\lambda t} t^{H_t - \frac{1}{\alpha}}, \tag{3.54}$$

where C_1 does not depend on t. Similarly, we have for large t,

$$|I_4(t)| \le C_2 |\theta_2||\theta_1|^{\alpha-1} e^{-\lambda t} t^{H_t - \frac{1}{\alpha}}, \tag{3.55}$$

where C_2 does not depend on t. Therefore, from (3.54) and (3.55), for large t,

$$|I(\theta_1, \theta_2, t_1, t_1+t)| \le C_3 |\theta_2||\theta_1|^{\alpha-1} e^{-\lambda t} t^{H_t - \frac{1}{\alpha}}. \tag{3.56}$$

where C_3 does not depend on t. By (3.27) we have for large t,

$$|I_4(t)| \ge \frac{1}{2} \int_{t_1}^{t_1+1} \alpha \Big|\theta_2[h_{t_1+t+1}(x) - h_{t_1+t}(x)]\Big| \Big|\theta_1 h_{t_1+1}(x)\Big|^{\alpha-1} dx.$$

Applying (3.46) with $\alpha = 1$ and the dominated convergence theorem yields

$$\begin{aligned}
&\liminf_{t\to\infty} |I_4(t)| e^{\lambda t} t^{\frac{1}{\alpha}-H_t} \\
&\geq \lim_{t\to\infty} \frac{1}{2}\alpha|\theta_2||\theta_1|^{\alpha-1} \int_{t_1}^{t_1+1} \Big|[h_{t_1+t+1}(x) - h_{t_1+t}(x)] e^{\lambda t} t^{\frac{1}{\alpha}-H_t}\Big| \Big|h_{t_1+1}(x)\Big|^{\alpha-1} dx \\
&= \frac{1}{2}\alpha|\theta_2||\theta_1|^{\alpha-1} \int_{t_1}^{t_1+1} e^{-\lambda(t_1-x)}\big(1-e^{-\lambda}\big)\Big|h_{t_1+1}(x)\Big|^{\alpha-1} dx.
\end{aligned}$$

Thus

$$|I_4(t)| \succeq e^{-\lambda t} t^{H_t-\frac{1}{\alpha}}. \tag{3.57}$$

for $\theta_1\theta_2 \neq 0$. Combining (3.51), (3.56) and (3.57) together, we have

$$e^{-\lambda t} t^{H_t-\frac{1}{\alpha}} \preceq |I_4(t)| \preceq |I(\theta_1,\theta_2,t_1,t_1+t)| \preceq e^{-\lambda t} t^{H_t-\frac{1}{\alpha}} \tag{3.58}$$

for $\theta_1\theta_2 \neq 0$. Since $I(\theta_1,\theta_2,t_1,t_1+t) \to 0$ as $t \to \infty$, it follows from (3.9) that $R_{t_1}(t) \sim -K(\theta_1,\theta_2,t_1,t_1+t)I(\theta_1,\theta_2,t_1,t_1+t)$; hence (3.50) holds. □

Remark 2 One says that a symmetric α-stable process $X(t)$ exhibits long-range dependence if for any $t_1 \in \mathbf{R}$,

$$\sum_{n=0}^{\infty} \Big|R_{t_1}(n)\Big| = \infty, \tag{3.59}$$

where $R_{t_1}(t)$ is defined by (3.9). It is obvious that LTFmSM and LTmFSM are not long-range dependent, but they exhibit semi-long-range dependence, that is, for $\lambda > 0$ sufficiently small, the sum (3.59) is large, and it tends to infinity as $\lambda \to 0$. Therefore, LTFmSM and LTmFSM provide two useful alternative models for data that exhibit strong dependence.

4 Scaling property and tail probabilities

The following result shows that LTmFSM (2.8) has a nice scaling property, involving both the time scale and the tempering. Denote by $\overset{fdd}{=}$ equality in the sense of finite dimensional distributions.

Proposition 5 *For any scale factor $c > 0$, it holds*

$$\Big\{X_{H_{ct},\alpha,\lambda}(c\,t)\Big\}_{t\in\mathbf{R}} \overset{fdd}{=} \Big\{c^{H_{c\,t}} X_{H_{ct},\alpha,c\lambda}(t)\Big\}_{t\in\mathbf{R}}. \tag{4.60}$$

Proof. It is easy to see that

$$G_{H_{ct},\alpha,\lambda}(c\,t, c\,x) = c^{H_{ct}-\frac{1}{\alpha}} G_{H_{ct},\alpha,c\lambda}(t,x).$$

Notice that $dZ_\alpha(c\,x)$ has control measure $c^{\frac{1}{\alpha}}dx$. Given $t_1 < t_2 < ... < t_n$, a change of variable $x = c\,x'$ then yields

$$\begin{aligned}
(X_{H_{c\,t_i},\alpha,\lambda}(c\,t_i) : i = 1,...,n) &= \Big(\int_{-\infty}^{\infty} G_{H_{c\,t_i},\alpha,\lambda}(c\,t_i, x) dZ_\alpha(x) : i = 1,...,n\Big) \\
&= \Big(\int_{-\infty}^{\infty} G_{H_{ct_i},\alpha,\lambda}(c\,t_i, cx') dZ_\alpha(c\,x') : i = 1,...,n\Big)
\end{aligned}$$

$$\stackrel{d}{=} \Big(\int_{-\infty}^{\infty} c^{H_{ct_i}-\frac{1}{\alpha}} G_{H_{ct_i},\alpha,c\lambda}(t_i,x')\, c^{\frac{1}{\alpha}} dZ_\alpha(x') : i=1,...,n\Big)$$
$$= \Big(c^{H_{ct_i}} \int_{-\infty}^{\infty} G_{H_{ct_i},\alpha,c\lambda}(t_i,x') dZ_\alpha(x') : i=1,...,n\Big)$$
$$= \Big(c^{H_{ct_i}} X_{H_{ct_i},\alpha,c\lambda}(t_i) : i=1,...,n\Big),$$

where $\stackrel{d}{=}$ indicates equality in distribution. So that (4.60) holds. □

We say that a stochastic process $X(t)$, $t \in I$, is *stochastic Hölder continuous* of exponent $\beta \in (0,\infty)$ if it holds

$$\limsup_{t,v \in I,\ |t-v|\to 0} \mathbf{P}\big(|X(t)-X(v)| \geq C|t-v|^{\beta}\big) = 0$$

for a positive constant C. It is obvious that if $X(u)$ is stochastic Hölder continuous of exponent β_1, then $X(u)$ is stochastic Hölder continuous of exponent $\beta_2 \in (0,\beta_1]$.

The following proposition shows that LTFmSM is stochastic Hölder continuous. Denote $a \wedge b = \min\{a,b\}$.

Proposition 6 *There is a number C, depending only on a, b, λ and H, such that for all $t, v \in \mathbf{R}$ and any $y > 0$,*

$$\mathbf{P}\Big(\Big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\Big| \geq y\Big) \leq \frac{C}{y^a \wedge y^b}\Big(|t-v|^{Ha} + |t-v|^{Hb}\Big) \tag{4.61}$$

In particular, (4.61) implies that for any $\beta \in (0, Ha/b)$ and all t, v satisfying $|t-v| \leq 1$,

$$\mathbf{P}\Big(\Big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\Big| \geq |t-v|^{\beta}\Big) \leq C|t-v|^{Ha-\beta b},$$

which implies that $X_{H,\alpha(x),\lambda}(t)$ is stochastic Hölder continuous of exponent $\beta \in (0, Ha/b)$.

Proof. By Proposition 2.3 of Falconer and Liu [7], it follows that for any $y > 0$,

$$\begin{aligned} &\mathbf{P}\Big(\Big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\Big| \geq y\Big) \\ &\leq C_1 \int_{-\infty}^{\infty} \Big|\frac{G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)}{y}\Big|^{\alpha(x)} dx \\ &\leq \frac{C_1}{y^a \wedge y^b} \int_{-\infty}^{\infty} \Big|G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)} dx, \end{aligned} \tag{4.62}$$

where $G_{H,\alpha(x),\lambda}(t,x)$ is defined by (2.6). Without loss of generality, we assume that $t \geq v$. Then

$$\int_{-\infty}^{\infty} \Big|G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)} dx = I_1 + I_2, \tag{4.63}$$

where

$$I_1 = \int_{-\infty}^{v} \Big|G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)} dx,$$
$$I_2 = \int_{v}^{t} e^{-\lambda\alpha(x)(t-x)}(t-x)^{H\alpha(x)-1} dx.$$

Using the inequality $|x+y|^\alpha \le 2^\alpha(|x|^\alpha + |y|^\alpha)$ for all $x, y \in \mathbf{R}$ and any $\alpha > 0$, we have

$$I_1 \le 4(I_{11} + I_{12}),$$

where

$$I_{11} = \int_{-\infty}^{v} \Big| e^{-\lambda(t-x)}(t-x)^{H-\frac{1}{\alpha(x)}} - e^{-\lambda(t-x)}(v-x)^{H-\frac{1}{\alpha(x)}} \Big|^{\alpha(x)} dx,$$
$$I_{12} = \int_{-\infty}^{v} \Big| e^{-\lambda(t-x)}(v-x)^{H-\frac{1}{\alpha(x)}} - e^{-\lambda(v-x)}(v-x)^{H-\frac{1}{\alpha(x)}} \Big|^{\alpha(x)} dx.$$

Let $h = t - v$. We deduce the following estimation of I_{11} :

$$\begin{aligned}
I_{11} &\le \int_{-\infty}^{v} \Big| (t-x)^{H-\frac{1}{\alpha(x)}} - (v-x)^{H-\frac{1}{\alpha(x)}} \Big|^{\alpha(x)} dx \\
&\le \int_{-\infty}^{v} \Big| (h+v-x)^{H-\frac{1}{\alpha(x)}} - (v-x)^{H-\frac{1}{\alpha(x)}} \Big|^{\alpha(x)} dx \\
&= \int_{-\infty}^{v} \Big| \Big(1+\frac{v-x}{h}\Big)^{H-\frac{1}{\alpha(x)}} - \Big(\frac{v-x}{h}\Big)^{H-\frac{1}{\alpha(x)}} \Big|^{\alpha(x)} h^{H\alpha(x)-1} dx \\
&= \int_{0}^{\infty} \Big| (1+u)^{H-\frac{1}{\alpha(v-hu)}} - u^{H-\frac{1}{\alpha(v-hu)}} \Big|^{\alpha(v-hu)} h^{H\alpha(v-hu)} du \\
&\le \int_{0}^{\infty} \Big| (1+u)^{H-\frac{1}{\alpha(v-hu)}} - u^{H-\frac{1}{\alpha(v-hu)}} \Big|^{\alpha(v-hu)} du \, \Big(h^{Ha} + h^{Hb}\Big) \\
&\le C_{11}\Big(|t-v|^{Ha} + |t-v|^{Hb}\Big).
\end{aligned}$$

Next, we estimate I_{12}. Notice that $|e^{-x} - e^{-y}| \le |x-y|$ for $x, y > 0$. Substitute $u = v - x$ to see that for $\lambda > 0$,

$$\begin{aligned}
I_{12} &= \int_{-\infty}^{v} \frac{1}{(\lambda\alpha(x))^{H\alpha(x)-1}} \big(\lambda\alpha(x)(v-x)\big)^{H\alpha(x)-1} e^{-\lambda\alpha(x)(v-x)} \Big| e^{-\lambda(t-v)} - 1 \Big|^{\alpha(x)} dx \\
&\le C_{12} \int_{-\infty}^{v} \Big(\lambda\alpha(x)(v-x)\Big)^{H\alpha(x)-1} e^{-\lambda\alpha(x)(v-x)} \min\Big\{(t-v)^{\alpha(x)}, 1\Big\} dx \\
&\le C_{12} \min\Big\{|t-v|^a, 1\Big\} \int_{0}^{\infty} (\lambda\alpha(v-u)u)^{H\alpha(v-u)-1} e^{-\lambda\alpha(v-u)u} du \\
&\le C_{12} \min\Big\{|t-v|^a, 1\Big\} \int_{0}^{\infty} \max_{\alpha\in[a,b]} \Big\{(\lambda\alpha u)^{H\alpha-1} e^{-\lambda\alpha u}\Big\} du \\
&\le C_{13} \min\Big\{|t-v|^a, 1\Big\}. \qquad (4.64)
\end{aligned}$$

It is obvious that if $\lambda = 0$, then $I_{12} = 0$, and thus (4.64) holds obviously for all $\lambda \ge 0$. By simple calculations, we get

$$\begin{aligned}
I_2 &\le \int_{v}^{t} (t-x)^{H\alpha(x)-1} dx \\
&\le \begin{cases} \int_v^t (t-x)^{Ha-1} dx & \text{if } t-v \le 1 \\ \int_v^{t-1} (t-x)^{Hb-1} dx + \int_{t-1}^{t} (t-x)^{Ha-1} dx & \text{if } t-v > 1 \end{cases} \\
&\le C_4\Big(|t-v|^{Ha} + |t-v|^{Hb}\Big).
\end{aligned}$$

Returning to (4.63), we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big|G_{H,\alpha(x),\lambda}(t,x)-G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)}dx\\&\leq C_5\Big(|t-v|^{Ha}+|t-v|^{Hb}+\min\Big\{|t-v|^{a},1\Big\}\Big)\\&\leq C_6\Big(|t-v|^{Ha}+|t-v|^{Hb}\Big).\end{aligned}\tag{4.65}$$

Hence, for $y>0$,

$$\mathbf{P}\Big(\Big|X_{H,\alpha(x),\lambda}(t)-X_{H,\alpha(x),\lambda}(v)\Big|\geq y\Big)\leq\frac{C_7}{y^a\wedge y^b}\Big(|t-v|^{Ha}+|t-v|^{Hb}\Big).$$

This completes the proof of Proposition 6. □

The following proposition shows that LTmFSM is also stochastic Hölder continuous.

Proposition 7 *Let $\lambda>0$. There is a number C depending only on a,b, α and λ, such that for all $z>0$,*

$$\mathbf{P}\Big(\Big|X_{H_t,\alpha,\lambda}(t)-X_{H_s,\alpha,\lambda}(s)\Big|\geq z\Big)\leq\frac{C}{z^\alpha}\Big(|t-s|^{\alpha H_t}+\big|H_t-H_s\big|^{\alpha}\Big)\tag{4.66}$$

for all $t,s\in\mathbf{R}$ satisfying $t\geq s$. In particular, if H_t is $\gamma-$Hölder continuous, that is

$$\big|H_t-H_s\big|\leq C|t-s|^{\gamma}\qquad\text{for } t,s \text{ satisfying } |t-s|\leq 1,$$

then (4.66) implies that for any $\beta\in(0,\min\{a,\gamma\})$ and all $t,s\in\mathbf{R}$ satisfying $|t-s|\leq 1$,

$$\mathbf{P}\Big(\Big|X_{H_t,\alpha,\lambda}(t)-X_{H_s,\alpha,\lambda}(s)\Big|\geq|t-s|^{\beta}\Big)\leq C\left(|t-s|^{\alpha(a-\beta)}+|t-s|^{\alpha(\gamma-\beta)}\right),$$

which implies that $X_{H_t,\alpha,\lambda}(t)$ is stochastic Hölder continuous of exponent $\beta\in(0,\min\{a,\gamma\})$.

Proof. By Proposition 1.2.15 of Samorodnitsky and Taqqu [11], it follows that for $z>0$,

$$\begin{aligned}&\mathbf{P}\Big(\Big|X_{H_t,\alpha,\lambda}(t)-X_{H_s,\alpha,\lambda}(s)\Big|\geq z\Big)\\&\leq C_0\frac{1}{z^\alpha}\int_{-\infty}^{\infty}\Big|G_{H_t,\alpha,\lambda}(t,x)-G_{H_s,\alpha,\lambda}(s,x)\Big|^{\alpha}dx.\end{aligned}\tag{4.67}$$

Using the inequality for any $\alpha>0$,

$$|x+y+z|^\alpha\leq 3^\alpha(|x|^\alpha+|y|^\alpha+|z|^\alpha),\qquad x,y,z\in\mathbf{R},$$

we have

$$\int_{-\infty}^{\infty}\Big|G_{H_t,\alpha,\lambda}(t,x)-G_{H_s,\alpha,\lambda}(s,x)\Big|^{\alpha}dx\leq 3^\alpha(I_1+I_2+I_3),\tag{4.68}$$

where

$$\begin{aligned}I_1&=\int_{-\infty}^{\infty}\Big|e^{-\lambda(t-x)_+}(t-x)_+^{H_t-\frac{1}{\alpha}}-e^{-\lambda(s-x)_+}(s-x)_+^{H_t-\frac{1}{\alpha}}\Big|^{\alpha}dx,\\I_2&=\int_{-\infty}^{\infty}\Big|e^{-\lambda(s-x)_+}(s-x)_+^{H_t-\frac{1}{\alpha}}-e^{-\lambda(s-x)_+}(s-x)_+^{H_s-\frac{1}{\alpha}}\Big|^{\alpha}dx,\\I_3&=\int_{-\infty}^{\infty}\Big|e^{-\lambda(-x)_+}(-x)_+^{H_t-\frac{1}{\alpha}}-e^{-\lambda(-x)_+}(-x)_+^{H_s-\frac{1}{\alpha}}\Big|^{\alpha}dx.\end{aligned}$$

It is easy to see that

$$I_1 \leq 2^\alpha (I_{11} + I_{12}), \tag{4.69}$$

where

$$I_{11} = \int_{-\infty}^{\infty} \Big| e^{-\lambda(t-x)_+}(t-x)_+^{H_t - \frac{1}{\alpha}} - e^{-\lambda(t-x)_+}(s-x)_+^{H_t-\frac{1}{\alpha}} \Big|^\alpha dx,$$

$$I_{12} = \int_{-\infty}^{\infty} \Big| e^{-\lambda(t-x)_+}(s-x)_+^{H_t - \frac{1}{\alpha}} - e^{-\lambda(s-x)_+}(s-x)_+^{H_t-\frac{1}{\alpha}} \Big|^\alpha dx.$$

Let $h = t - s > 0$. Notice that $|\big(1+u\big)^{H_t - \frac{1}{\alpha}} - u^{H_t - \frac{1}{\alpha}}| \leq 2\beta u^{H_t - \frac{1}{\alpha} - 1}, u \to \infty$. Then we deduce the following estimation of I_{11} :

$$\begin{aligned}
I_{11} &= \int_{-\infty}^{t} e^{-\lambda\alpha(t-x)} \Big| (t-x)^{H_t - \frac{1}{\alpha}} - (s-x)_+^{H_t - \frac{1}{\alpha}} \Big|^\alpha dx \\
&= \int_{-\infty}^{t} e^{-\lambda\alpha(t-x)} \Big| \Big(1 + \frac{s-x}{h}\Big)^{H_t - \frac{1}{\alpha}} - \Big(\frac{s-x}{h}\Big)_+^{H_t - \frac{1}{\alpha}} \Big|^\alpha h^{H_t\alpha - 1} dx \\
&= \int_{-1}^{\infty} e^{-\lambda\alpha h(1+u)} \Big| \big(1+u\big)^{H_t - \frac{1}{\alpha}} - u_+^{H_t - \frac{1}{\alpha}} \Big|^\alpha h^{H_t\alpha} du \\
&\leq \int_{-1}^{\infty} e^{-\lambda\alpha h(1+u)} \Big| \big(1+u\big)^{H_t - \frac{1}{\alpha}} - u_+^{H_t - \frac{1}{\alpha}} \Big|^\alpha du \; h^{H_t\alpha} \\
&\leq C_{11} h^{H_t\alpha} = C_{11}|t-s|^{H_t\alpha}.
\end{aligned} \tag{4.70}$$

Next, consider the item I_{12}. Substitute $u = s - x$ and then $w = \lambda$ to see that for $\lambda > 0$,

$$\begin{aligned}
I_{12} &= \int_{-\infty}^{s} (s-x)^{\alpha H_t - 1} e^{-\lambda\alpha(s-x)} \Big| e^{-\lambda(t-s)} - 1 \Big|^\alpha dx \\
&\leq \int_{-\infty}^{s} (s-x)^{\alpha H_t - 1} e^{-\lambda\alpha(s-x)} dx \; \min\Big\{(t-s)^\alpha, 1\Big\} \\
&= \int_{0}^{\infty} u^{\alpha H_t - 1} e^{-\lambda\alpha u} du \; \min\Big\{(t-s)^\alpha, 1\Big\} \\
&\leq C_{12} \min\Big\{|t-s|^\alpha, 1\Big\},
\end{aligned} \tag{4.71}$$

where the second line of the last inequalities follows by the inequality $|e^{-x} - e^{-y}| \leq |x - y|$ for all $x, y \geq 0$. It is obvious that if $\lambda = 0$, then $I_{12} = 0$. Thus (4.71) also holds for $\lambda = 0$. Combining (4.69), (4.70) and (4.71) together, we get

$$I_1 \leq C_1 \Big(|t-s|^{\alpha H_t} + \min\Big\{|t-s|^\alpha, 1\Big\} \Big). \tag{4.72}$$

In the sequel, we give the estimations of I_2 and I_3. Without loss of generality, we assume that $H_t \geq H_s$. By some simple calculations, we get

$$\begin{aligned}
I_2 &= \int_{-\infty}^{s} e^{-\lambda\alpha(s-x)} \Big| (s-x)^{H_t - \frac{1}{\alpha}} - (s-x)^{H_s - \frac{1}{\alpha}} \Big|^\alpha dx \\
&= \int_{0}^{\infty} e^{-\lambda\alpha u} u^{-1} \Big| u^{H_t} - u^{H_s} \Big|^\alpha du \\
&= \int_{0}^{\infty} e^{-\lambda\alpha u} \Big| H_t - H_s \Big|^\alpha u^{\alpha H_\theta - 1} |\log u|^\alpha du \\
&\leq C_2 \big| H_t - H_s \big|^\alpha,
\end{aligned} \tag{4.73}$$

where $H_\theta \in [H_s, H_t]$. Similarly, we have

$$I_3 \le C_3 \big|H_t - H_s\big|^\alpha. \tag{4.74}$$

Combining the inequalities (4.68), (4.72), (4.73) and (4.74) together, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Big|G_{H_t,\alpha,\lambda}(t,x) - G_{H_s,\alpha,\lambda}(s,x)\Big|^\alpha dx \\
&\le C_4\Big(|t-s|^{\alpha H_t} + \min\big\{|t-s|^\alpha, 1\big\} + \big|H_t - H_s\big|^\alpha\Big) \\
&\le C_5\Big(|t-s|^{\alpha H_t} + \big|H_t - H_s\big|^\alpha\Big).
\end{aligned} \tag{4.75}$$

Returning to (4.67), we get for $z > 0$,

$$\mathbf{P}\Big(\Big|X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s)\Big| \ge z\Big) \le \frac{C_6}{z^\alpha}\Big(|t-s|^{\alpha H_t} + \big|H_t - H_s\big|^\alpha\Big).$$

This completes the proof of Proposition 7. □

5 Absolute moments

We estimate the absolute (incremental) moments of the LTFmSM.

Proposition 8 *If $0 < p < a$, then there exists a number C_1, depending only on a, b, λ and H, such that for all $t, v \in \mathbf{R}$ and $|t - v| \ge 1$,*

$$\mathbf{E}\Big[\big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\big|^p\Big] \le C_1\Big(1 + \frac{p}{a-p}\Big)|t-v|^{Hb}.$$

Proof. When $|t - v| \ge 1$, using Proposition 6, we deduce that

$$\begin{aligned}
&\mathbf{E}\Big[\big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\big|^p\Big] \\
&= p\int_0^\infty y^{p-1}\mathbf{P}\Big(\Big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\Big| \ge y\Big)dy \\
&\le p\Big(\int_0^1 y^{p-1}dy + C_1\int_1^\infty y^{p-1-a}dy\Big)|t-v|^{Hb} \\
&\le C_2\Big(1 + \frac{p}{a-p}\Big)|t-v|^{Hb}.
\end{aligned}$$

This completes the proof of Proposition 8. □

The next proposition gives an estimate for the absolute (incremental) moment of the LTmFSM.

Proposition 9 *If $0 < p < \alpha$, then there is a number C depending only on p, a, b and λ, such that for all $t, s \in \mathbf{R}$ satisfying $t \ge s$,*

$$\mathbf{E}\Big[\big|X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s)\big|^p\Big] \le \Big(1 + \frac{2C_1}{p-\alpha}\Big)\Big(|t-s|^{pH_t} + \big|H_t - H_s\big|^p\Big).$$

Proof. Using Proposition 7, we have for any $\varepsilon > 0$,

$$
\begin{aligned}
&\mathbf{E}\Big[\big|X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s)\big|^p\Big] \\
&= p\int_0^\infty y^{p-1}\mathbf{P}\Big(\Big|X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s)\Big| \geq y\Big)dy \\
&\leq p\int_0^\varepsilon y^{p-1}dy + C_1\int_\varepsilon^\infty y^{p-1-\alpha}dy\Big(|t-s|^{\alpha H_t} + \big|H_t - H_s\big|^\alpha\Big) \\
&= \varepsilon^p + \frac{C_1}{\alpha - p}\varepsilon^{p-\alpha}\Big(|t-s|^{\alpha H_t} + \big|H_t - H_s\big|^\alpha\Big).
\end{aligned}
$$

Taking $\varepsilon = \max\{|t-s|^{H_t}, \big|H_t - H_s\big|\}$, we get

$$
\mathbf{E}\Big[\big|X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s)\big|^p\Big] \leq \Big(1 + \frac{2C_1}{\alpha - p}\Big)\Big(|t-s|^{pH_t} + \big|H_t - H_s\big|^p\Big),
$$

which gives the desired inequality. □

For LFmSM, Le Guével and Lévy Véhel [8] have investigated the asymptotic behaviour of $\mathbf{E}\big[|X(t+r) - X(t)|^\eta\big], r \to 0$, for some positive constant $\eta > 0$. The following proposition gives a result similar to the one of Le Guével and Lévy Véhel for LTFmSM.

Proposition 10 *For each $t \in \mathbf{R}$ satisfying $H\alpha(t) \neq 1$ and all $\gamma \in (0, a)$, it holds*

$$
\lim_{r \to 0+} \frac{\mathbf{E}\big[|X_{H,\alpha(x),\lambda}(t+r) - X_{H,\alpha(x),\lambda}(t)|^\gamma\big]}{r^{\gamma H}} = F(\gamma, t),
$$

where

$$
F(\gamma, t) = \Bigg(\int_{-\infty}^{\infty}\Big[(1-x)_+^{H - \frac{1}{\alpha(t)}} - (-x)_+^{H - \frac{1}{\alpha(t)}}\Big]^{\alpha(t)}dx\Bigg)^{\gamma/\alpha(t)} \frac{2^{\gamma-1}\Gamma\Big(1 - \frac{\gamma}{\alpha(t)}\Big)}{\gamma\int_0^\infty u^{-\gamma-1}\sin^2(u)du}
$$

and $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is the gamma function.

Proof. Notice that for all $\gamma \in (0, a)$ and all $u \in [0, 1)$,

$$
\begin{aligned}
&\mathbf{E}\Big[\Big|\frac{X_{H,\alpha(x),\lambda}(t+r) - X_{H,\alpha(x),\lambda}(t)}{r^H}\Big|^\gamma\Big] \\
&= \gamma\int_0^\infty z^{\gamma-1}\,\mathbf{P}\Big(\Big|\frac{X_{H,\alpha(x),\lambda}(t+r) - X_{H,\alpha(x),\lambda}(t)}{r^H}\Big| \geq z\Big)dz.
\end{aligned}
$$

Notice that $X_{H,\alpha(x),\lambda}(t)$ is $H-$localisable to X defined by (8.83) (cf. Proposition 19 whose proof does not involve Proposition 10). Thus

$$
\mathbf{P}\Bigg(\Big|\frac{X_{H,\alpha(x),\lambda}(t+r) - X_{H,\alpha(x),\lambda}(t)}{r^H}\Big| \geq z\Bigg) \to \mathbf{P}\Bigg(\big|X(1)\big| \geq z\Bigg), \qquad r \to 0.
$$

By Proposition 6, for z large enough,

$$
\mathbf{P}\Big(\Big|\frac{X_{H,\alpha(x),\lambda}(t+r) - X_{H,\alpha(x),\lambda}(t)}{r^H}\Big| \geq z\Big) \leq C\frac{1}{z^a}.
$$

Hence, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{r\to 0+} \mathbf{E}\Big[\Big|\frac{X_{H,\alpha(x),\lambda}(t+r)-X_{H,\alpha(x),\lambda}(t)}{r^H}\Big|^{\gamma}\Big] \\
&=\gamma\int_0^{\infty} z^{\gamma-1}\,\mathbf{P}\Big(\big|X(1)\big|\ge z\Big)dz \\
&=\mathbf{E}\big[\big|X(1)\big|^{\gamma}\big] \\
&=\Big(\int_{-\infty}^{\infty}\Big[(1-x)_+^{H-\frac{1}{\alpha(t)}}-(-x)_+^{H-\frac{1}{\alpha(t)}}\Big]^{\alpha(t)}dx\Big)^{\gamma/\alpha(t)}\frac{2^{\gamma-1}\Gamma\Big(1-\frac{\gamma}{\alpha(t)}\Big)}{\gamma\int_0^{\infty}u^{-\gamma-1}\sin^2(u)du},
\end{aligned}$$

where X is $\alpha(t)$-stable. We refer to Property 1.2.17 of Samorodnitsky and Taqqu [11] for the last line of the last equality. □

6 Sample path properties

When $Ha > 1$ with $a > 1$, the following proposition implies that every LTFmSM process has an a.s. Hölder continuous version.

Proposition 11 *If $Ha > 1$ with $a > 1$, then for any $0 < \beta < H - 1/a$, $X_{H,\alpha(x),\lambda}(t)$ has a continuous version such that its paths are almost surely $\beta-$Hölder continuous on each bounded interval.*

Proof. Recall that

$$X_{H,\alpha(x),\lambda}(t)=\int_{-\infty}^{\infty}G_{H,\alpha(x),\lambda}(t,x)dM_{\alpha}(x).$$

By (4.65), we have for $|t-v|\le 1$,

$$\int_{-\infty}^{\infty}\Big|G_{H,\alpha(x),\lambda}(t,x)-G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)}dx\le C_1|t-v|^{Ha}. \tag{6.76}$$

By Proposition 3.1 of Falconer and Liu [7], $X_{H,\alpha(x),\lambda}(t)$ has a continuous version such that its paths are almost surely $\beta-$Hölder continuous on each bounded interval, where $0<\beta<(Ha-1)/a$. □

Recall that a stochastic process $X(t), t\in T$, on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ is called separable if there is a countable set $T^*\subset T$ and an even $\Omega_0\in\mathcal{F}$ with $\mathbf{P}(\Omega_0)=0$, such that for any closed set $F\subset\mathbf{R}$ we have

$$\{\omega: X(t)\in F,\quad \forall t\in T^*\}\setminus\{\omega: X(t)\in F,\quad \forall t\in T\}\subset\Omega_0.$$

See Chapter 9 of Samorodnitsky and Taqqu [11] for more details.

When $H_t\alpha<1$ and $\lambda>0$, the following proposition shows that every separable version of LTmFSM process has unbounded paths.

Proposition 12 *If $H_t\alpha<1$ and $\lambda>0$, then for any separable version of the LTmFSM process, we have for any interval $(c,\, d)$,*

$$\mathbf{P}\Big(\Big\{\omega:\sup_{t\in(c,\,d)}\big|X_{H_t,\alpha,\lambda}(t)\big|=\infty\Big\}\Big)=1.$$

Proof. We may assume that $(c,\, d)$ is bound. Consider the countable set $T^* := \mathbf{Q} \cap [c,\, d]$, where $\mathbf{Q}$ denotes the set of rational numbers. Since T^* is dense in $[c,\, d]$, there exists a sequence of numbers $\{t_n\}_{n\in\mathbf{N}} \in T^*$, such that for any $x \in [c,\, d]$, $t_n \to x$ as $n \to \infty$. Therefore, it holds

$$f^*(T^*;x) := \sup_{t\in T^*} \Big| G_{H_t,\alpha,\lambda}(t,x) \Big| \geq \sup_{t_n\in T^*} \Big| G_{H_t,\alpha,\lambda}(t_n,x) \Big| =: f_n^*(T^*;x) = \infty, \quad n \to \infty.$$

Thus $\int_c^d f^*(T^*;x)dx = \infty$, and this contradicts Condition (10.2.14) of Theorem 10.2.3 in Samorodnitsky and Taqqu [11]. Therefore, the stochastic process $\{X_{H_t,\alpha,\lambda}\}$ does not have a version with bounded paths on the interval $(c,\, d)$, and this completes the proof. □

For LTmFSM process with $H_t\alpha > 1$, we have the following proposition.

Proposition 13 *Assume that H_t is $\gamma-$Hölder continuous, $\gamma > 1/\alpha$, that is*

$$\big| H_t - H_s \big| \leq C|t-s|^\gamma \tag{6.77}$$

for $t, s \in \mathbf{R}$ satisfying $|t-s| \leq 1$. If $\alpha \min\{a,\gamma\} > 1$, then for any $0 < \beta < \min\{a,\gamma\} - 1/\alpha$, $X_{H_t,\alpha,\lambda}(t)$ has a continuous version, such that its paths are almost surely $\beta-$Hölder continuous on each compact set.

Proof. By Proposition 9 and (6.77), we have for any $0 < p < \alpha$ and all t, s satisfying $|t-s| \leq 1$,

$$\begin{aligned}\mathbf{E}\Big[\big|X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s)\big|^p\Big] &\leq C_1\Big(|t-s|^{pa} + \big|H_t - H_s\big|^p\Big) \\ &\leq C_2\Big(|t-s|^{pa} + \big|t-s\big|^{p\gamma}\Big).\end{aligned}$$

The Kolmogorov continuity theorem implies that $X_{H_t,\alpha,\lambda}(t)$ has a continuous version, such that its paths are almost surely $\beta-$Hölder continuous on each compact set, $0 < \beta < (p\min\{a,\gamma\} - 1)/p$. Let $p \to \alpha$. We completes the proof of Proposition 13. □

Remark 3 For LmFSM, Ayache and Hamonier [2] have obtained the uniform pointwise Hölder exponent of $X_{H_t,\alpha,\lambda}(t)$. By Theorem 8.1 of Ayache and Hamonier [2], it is easy to see that when $a \geq \gamma$, the β in Proposition 13 cannot exceed $\gamma - 1/\alpha$.

Denote by

$$\widetilde{\mathcal{H}}_t(\omega) = \sup\left\{\gamma : \lim_{r\to 0} \frac{|X_{H,\alpha(x),\lambda}(t+r,\omega) - X_{H,\alpha(x),\lambda}(t,\omega)|}{|r|^\gamma} = 0\right\}$$

the pontwise Hölder exponent of the LTFmSM $X_{H,\alpha(x),\lambda}(\cdot)$ at t.

Proposition 14 *If $Ha > 1$ with $a > 1$, then $\widetilde{\mathcal{H}}_t(\omega) \geq H - 1/a$ almost surely.*

Proof. It follows by Proposition 11. □

Let

$$\widehat{\mathcal{H}}_t(\omega) = \sup\left\{\gamma : \lim_{r\to 0} \frac{|X_{H_t,\alpha,\lambda}(t+r,\omega) - X_{H_t,\alpha,\lambda}(t,\omega)|}{|r|^\gamma} = 0\right\}$$

be the pontwise Hölder exponent of the LTmFSM $X_{H_t,\alpha,\lambda}(\cdot)$ at t.

Proposition 15 *Assume that H_t is $\gamma-$Hölder continuous, $\gamma > 1/\alpha$. If $\alpha\min\{H_{t_0},\gamma\} > 1$ for some $t_0 \in \mathbf{R}$, then $\widehat{\mathcal{H}}_{t_0}(\omega) \geq \min\{H_{t_0},\gamma\} - 1/\alpha$ almost surely.*

Proof. Since H_t is continuous, we have for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $s \in [t_0 - \delta,\ t_0 + \delta]$, it holds $H_s \in [H_{t_0} - \varepsilon,\ H_{t_0} + \varepsilon]$. If $\alpha\min\{H_{t_0},\gamma\} > 1$, by an argument similar to the proof of Proposition 13, then for any $0 < \beta < \min\{H_{t_0} - \varepsilon, \gamma\} - 1/\alpha$, $X_{H_s,\alpha,\lambda}(s)$ has a continuous version, such that its paths are almost surely $\beta-$Hölder continuous on $s \in [t_0 - \delta,\ t_0 + \delta]$. Thus if $\alpha\min\{H_{t_0},\gamma\} > 1$, then $\widehat{\mathcal{H}}_{t_0}(\omega) \geq \min\{H_{t_0} - \varepsilon, \gamma\} - 1/\alpha$ almost surely. The claim follows by the fact that ε can be arbitrary small. □

7 Hölder continuity of quasi norm

Denote by

$$\Big|\Big|X_{H,\alpha(x),\lambda}(t)\Big|\Big|_\alpha := \left\{ y>0: \quad \int_{-\infty}^{\infty} \Big|\frac{G_{H,\alpha(x),\lambda}(t,x)}{y}\Big|^{\alpha(x)} dx = 1 \right\}$$

for $t \in \mathbf{R}$. Then $\big|\big|\cdot\big|\big|_\alpha$ is a quasi norm. In particular, if $\alpha(x) \equiv p \geq 1$ for a constant p, then $||X_{H,p,\lambda}(t)||_p$ is the $L^p(\mathbf{R})$ norm of $G_{H,p,\lambda}(t,x)$. Moreover, when $\alpha(x) \equiv \alpha$ for a constant $\alpha \in (0,2]$, then it holds

$$\Big|\Big|X_{H,\alpha,\lambda}(t)\Big|\Big|_\alpha = \Big(-\log \mathbf{E}\big[e^{iX_{H,\alpha,\lambda}(t)}\big]\Big)^{1/\alpha} = \left(\int_{-\infty}^{\infty}\Big|G_{H,\alpha,\lambda}(t,x)\Big|^\alpha dx\right)^{1/\alpha},$$

see Meerschaert and Sabzikar [9].

The next proposition implies that the quasi norm of LTFmSM process is Hölder continuous in time t.

Proposition 16 *There are two positive numbers c and C, depending only on a, b, λ and H, such that*

$$c\,|t-v|^{Hb/a} \leq \Big|\Big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\Big|\Big|_\alpha \leq C\,|t-v|^{Ha/b}$$

for all $t, v \in \mathbf{R}$ satisfying $|t-v| \leq 1$.

Proof. Denote by $\rho = \big|\big|X_{H,\alpha(x),\lambda}(t) - X_{H,\alpha(x),\lambda}(v)\big|\big|_\alpha$. Assume that $t > v$, and write

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Big|G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)} dx \\
&\geq \int_v^t e^{-\lambda\alpha(x)(t-x)}(t-x)^{H\alpha(x)-1}dx \\
&\geq e^{-\lambda b(t-v)}\int_v^t (t-x)^{H\alpha(x)-1}dx \\
&\geq e^{-\lambda b(t-v)}\int_v^t (t-x)^{Hb-1}dx \\
&\geq e^{-\lambda b}\frac{1}{Hb}(t-v)^{Hb}
\end{aligned}$$

uniformly for all $t, v \in \mathbf{R}$ satisfying $|t-v| \leq 1$. Therefore, we have

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\Big|\frac{G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)}{\rho}\Big|^{\alpha(x)} dx \\
&\geq \int_{-\infty}^{\infty}\Big|G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)} dx \,\min\Big\{\frac{1}{\rho^a}, \frac{1}{\rho^b}\Big\} \\
&\geq e^{-\lambda b}\frac{1}{Hb}(t-v)^{Hb}\,\min\Big\{\frac{1}{\rho^a}, \frac{1}{\rho^b}\Big\}.
\end{aligned}$$

The last inequality implies the lower bound of ρ. By (4.65), we have

$$\int_{-\infty}^{\infty}\Big|G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)\Big|^{\alpha(x)} dx \leq C_1|t-v|^{Ha}$$

uniformly for all $t, v \in \mathbf{R}$ satisfying $|t-v| \leq 1$. Then

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \Big| \frac{G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x)}{\rho} \Big|^{\alpha(x)} dx \\ &\leq \int_{-\infty}^{\infty} \Big| G_{H,\alpha(x),\lambda}(t,x) - G_{H,\alpha(x),\lambda}(v,x) \Big|^{\alpha(x)} dx \ \max\Big\{ \frac{1}{\rho^a}, \frac{1}{\rho^b} \Big\} \\ &\leq C_1 |t-v|^{Ha} \ \max\Big\{ \frac{1}{\rho^a}, \frac{1}{\rho^b} \Big\}, \end{aligned} \tag{7.78}$$

whenever $|t-v| \leq 1$. Inequality (7.78) implies the upper bound of ρ. □

When $a = b$ and $1/a < H < 1$, Proposition 16 reduces to Lemma 4.2 of Meerschaert and Sabzikar [9]. Hence Proposition 16 can be regarded as a generalization of this lemma.

The next proposition implies that the quasi norm of LTmFSM process is Hölder continuous in time t.

Proposition 17 *There exist two positive numbers c and C, depending only on a, b, λ and α, such that*

$$c\,|t-s|^{H_t} \leq \Big|\Big| X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s) \Big|\Big|_{\alpha} \leq C\,\Big(|t-s|^{H_t} + \big|H_t - H_s\big| \Big) \tag{7.79}$$

for all t, s satisfying $0 \leq s \leq t \leq s+1$.

Proof. From the poof of Proposition 7, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \Big| G_{H_t,\alpha,\lambda}(t,x) - G_{H_t,\alpha,\lambda}(v,x) \Big|^{\alpha} dx &\leq C_1 \Big(|t-s|^{\alpha H_t} + \big|H_t - H_s\big|^{\alpha} \Big) \\ &\leq 2C_1 \max\Big\{ |t-s|^{\alpha H_t}, \quad \big|H_t - H_s\big|^{\alpha} \Big\}. \end{aligned}$$

Hence,

$$\Big|\Big| X_{H_t,\alpha,\lambda}(t) - X_{H_t,\alpha,\lambda}(v) \Big|\Big|_{\alpha} \leq (2C_1)^{1/\alpha} \Big(|t-s|^{H_t} + \big|H_t - H_s\big| \Big),$$

which gives the desired upper bound in (7.79).

Next, consider the lower bound of $\big|\big| X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s) \big|\big|_{\alpha}$. Write

$$\begin{aligned} \int_{-\infty}^{\infty} \Big| G_{H_t,\alpha,\lambda}(t,x) - G_{H_s,\alpha,\lambda}(s,x) \Big|^{\alpha} dx \ &\geq \int_s^t \Big| e^{-\lambda(t-x)} (t-x)^{H_t - \frac{1}{\alpha}} \Big|^{\alpha} dx \\ &\geq e^{-\lambda\alpha(t-s)} \int_s^t (t-x)^{H_t\alpha - 1} dx \\ &\geq e^{-\lambda\alpha(t-s)} \int_s^t (t-x)^{H_t\alpha - 1} dx \\ &\geq e^{-\lambda\alpha} \frac{1}{H_t \alpha} (t-s)^{H_t \alpha} \\ &\geq e^{-\lambda\alpha} \frac{1}{b\alpha} (t-s)^{H_t \alpha} \end{aligned}$$

uniformly for all t, s satisfying $s \leq t \leq s+1$. Therefore, we have

$$\Big|\Big| X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s) \Big|\Big|_{\alpha} \geq \ e^{-\lambda} \Big(\frac{1}{b\alpha} \Big)^{1/\alpha} |t-s|^{H_t},$$

which gives the desired lower bound in (7.79). □

For $\alpha \in (0, 1]$, the next proposition shows that the upper bound of (7.79) is also exact.

Proposition 18 *Assume $\alpha \in (0,1]$ and $t_0 > 0$. Then there is a positive number c, depending only on a, b, λ, t_0 and α, such that*

$$\Big|\Big|X_{H_t,\alpha,\lambda}(t) - X_{H_s,\alpha,\lambda}(s)\Big|\Big|_\alpha \geq c\left(|t-s|^{H_t} + \big|H_t - H_s\big|\right) \tag{7.80}$$

for all t, s satisfying $t_0 \leq s \leq t \leq s+1$.

Proof. If $|t-s|^{H_t} \geq c_1\big|H_t - H_s\big|$ for some $c_1 > 0$, depending only on a, b, λ, t_0 and α, then (7.79) implies (7.80). Otherwise, we have

$$\big|H_t - H_s\big|/|t-s|^{H_t} \to \infty \tag{7.81}$$

as $|t-s| \to 0$. Applying the inequality

$$\Big||x|^\alpha - |y|^\alpha\Big| \leq |x-y|^\alpha, \qquad x, y \in \mathbf{R} \text{ and } \alpha \in (0,1],$$

we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Big|G_{H_t,\alpha,\lambda}(t,x) - G_{H_s,\alpha,\lambda}(s,x)\Big|^\alpha dx \\
&\geq \int_0^s \Big|e^{-\lambda(t-x)}(t-x)^{H_t-\frac{1}{\alpha}} - e^{-\lambda(s-x)}(s-x)^{H_s-\frac{1}{\alpha}}\Big|^\alpha dx \\
&= \int_0^s \Big|e^{-\lambda(t-x)}(t-x)^{H_t-\frac{1}{\alpha}} - e^{-\lambda(s-x)}(s-x)^{H_t-\frac{1}{\alpha}} \\
&\qquad + e^{-\lambda(s-x)}(s-x)^{H_t-\frac{1}{\alpha}} - e^{-\lambda(s-x)}(s-x)^{H_s-\frac{1}{\alpha}}\Big|^\alpha dx \\
&\geq \int_0^s \Big|e^{-\lambda(s-x)}(s-x)^{H_t-\frac{1}{\alpha}} - e^{-\lambda(s-x)}(s-x)^{H_s-\frac{1}{\alpha}}\Big|^\alpha dx \\
&\qquad - \int_0^s \Big|e^{-\lambda(t-x)}(t-x)^{H_t-\frac{1}{\alpha}} - e^{-\lambda(s-x)}(s-x)^{H_t-\frac{1}{\alpha}}\Big|^\alpha dx.
\end{aligned}$$

By the mean value theorem and (4.72), the last inequality implies that for $\alpha \in (0,1]$ and all t, s satisfying $0 \leq s \leq t \leq s+1$,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Big|G_{H_t,\alpha,\lambda}(t,x) - G_{H_s,\alpha,\lambda}(s,x)\Big|^\alpha dx \\
&\geq \int_0^s e^{-\lambda\alpha(s-x)}(s-x)^{\alpha H_\theta - 1}|\log(s-x)|^\alpha dx\,\Big|H_t - H_s\Big|^\alpha - C_{11}|t-s|^{\alpha H_t} \\
&\geq c_{00}\Big|H_t - H_s\Big|^\alpha - C_{11}|t-s|^{\alpha H_t},
\end{aligned}$$

where $c_{00}, C_{11} > 0$ depending only on a, b, λ, t_0 and α. By (7.81), it follows that for $\alpha \in (0,1]$,

$$\int_{-\infty}^{\infty}\Big|G_{H_t,\alpha,\lambda}(t,x) - G_{H_s,\alpha,\lambda}(s,x)\Big|^\alpha dx \geq c\Big|H_t - H_s\Big|^\alpha,$$

where $c > 0$ depending only on a, b, λ, t_0 and α. Therefore (7.80) holds. □

8 Localisability and strong localisability

Recall that a stochastic process $X(t), t \in \mathbf{R}$, is said to be $h-$localisable at u (cf. Falconer [3,4]), with $h > 0$, if there exists a non-trivial process X'_u, called the tangent process of X at u, such that

$$\lim_{r \searrow 0} \frac{X(u+rv) - X(u)}{r^h} \overset{fdd}{=} X'_u(v), \tag{8.82}$$

where $\overset{fdd}{=}$ stands for convergence in finite-dimensional distributions.

The following proposition shows that LTFmSM is $H-$localisable.

Proposition 19 *Assume that $\alpha(x)$ is continuous on $\mathbf{R}$. When $1/a < H < 1$, the LTFmSM process $X_{H,\alpha(x),\lambda}(t)$ is $H-$localisable at u with local form*

$$X(t) := \int_{-\infty}^{\infty} \Big[(t-x)_+^{H-\frac{1}{\alpha(u)}} - (-x)_+^{H-\frac{1}{\alpha(u)}} \Big] dZ_{\alpha(u)}(x), \tag{8.83}$$

where $dZ_{\alpha(u)}(x)$ is a symmetric $\alpha(u)-$stable random measure.

Proof. Given $u_1 < u_2 < ... < u_d$, denote

$$S_r(u_k) = \frac{X_{H,\alpha,\lambda}(u+ru_k) - X_{H,\alpha,\lambda}(u)}{r^H}$$

for $r > 0$ and $k = 1, ..., d$. Then

$$\begin{aligned} &\mathbf{E}\Big[e^{i\sum_{k=1}^d \theta_k S_r(u_k)}\Big] \\ &= \exp\Bigg\{ - \int_{-\infty}^{\infty} \Big| \sum_{k=1}^d \theta_k r^{-H} \Big(G_{H,\alpha(x),\lambda}(u+ru_k, x) - G_{H,\alpha(x),\lambda}(u,x) \Big) \Big|^{\alpha(x)} dx \Bigg\}. \end{aligned}$$

Let $x = u + rz$. It follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} \Big| \sum_{k=1}^d \theta_k r^{-H} \Big(G_{H,\alpha(x),\lambda}(u+ru_k, x) - G_{H,\alpha(x),\lambda}(u,x) \Big) \Big|^{\alpha(x)} dx \\ &= \int_{-\infty}^{\infty} \Big| \sum_{k=1}^d \theta_k \Big(e^{-\lambda r(u_k - z)_+} (u_k - z)_+^{H-\frac{1}{\alpha(u+rz)}} - e^{-\lambda r(-z)_+} (-z)_+^{H-\frac{1}{\alpha(u+rz)}} \Big) \Big|^{\alpha(u+rz)} dz. \end{aligned}$$

Recall that $\alpha(x) \in [a,b]$ is a continuous function on $\mathbf{R}$. Thus

$$\begin{aligned} &\lim_{r\to 0} \Big| \sum_{k=1}^d \theta_k \Big(e^{-\lambda r(u_k - z)_+} (u_k - z)_+^{H-\frac{1}{\alpha(u+rz)}} - e^{-\lambda r(-z)_+} (-z)_+^{H-\frac{1}{\alpha(u+rz)}} \Big) \Big|^{\alpha(u+rz)} \\ &= \Big| \sum_{k=1}^d \theta_k \Big((u_k - z)_+^{H-\frac{1}{\alpha(u)}} - (-z)_+^{H-\frac{1}{\alpha(u)}} \Big) \Big|^{\alpha(u)}. \end{aligned}$$

It is obvious that for $z < \min\{u_1, 0\} - 1$, $0 < r < 1$ and $1/a < H < 1$,

$$\begin{aligned}
&\Big|\sum_{k=1}^{d}\theta_k\Big(e^{-\lambda r(u_k-z)_+}(u_k-z)_+^{H-\frac{1}{\alpha(u+rz)}} - e^{-\lambda r(-z)_+}(-z)_+^{H-\frac{1}{\alpha(u+rz)}}\Big)\Big|^{\alpha(u+rz)}\\
&\leq \sum_{k=1}^{d}|\theta_k|^{\alpha(u+rz)}\Big|e^{-\lambda r u_k}(u_k-z)^{H-\frac{1}{\alpha(u+rz)}} - (-z)^{H-\frac{1}{\alpha(u+rz)}}\Big|^{\alpha(u+rz)}\\
&\leq \sum_{k=1}^{d}|\theta_k|^{\alpha(u+rz)}e^{\lambda r|u_k|}\Big|(u_k-z)^{H-\frac{1}{\alpha(u+rz)}} - (-z)^{H-\frac{1}{\alpha(u+rz)}}\Big|^{\alpha(u+rz)}\\
&\leq \sum_{k=1}^{d}\Big(|\theta_k|^a+|\theta_k|^b\Big)e^{\lambda|u_k|}\sup_{\alpha\in[a,b]}\Big|(u_k-z)^{H-1/\alpha}-(-z)^{H-1/\alpha}\Big|^{\alpha}\\
&\leq \sum_{k=1}^{d}\Big(|\theta_k|^a+|\theta_k|^b\Big)e^{\lambda|u_k|}\Big|H-\frac{1}{a}\Big|^{a}\Big(\min\{u_1,0\}-z\Big)^{Ha-1-a}(|u_k|^a+|u_k|^b),
\end{aligned}$$

and that for $z \geq \min\{u_1, 0\} - 1$ and $0 < r < 1$,

$$\begin{aligned}
&\Big|\sum_{k=1}^{d}\theta_k\Big(e^{-\lambda r(u_k-z)_+}(u_k-z)_+^{H-\frac{1}{\alpha(u+rz)}} - e^{-\lambda r(-z)_+}(-z)_+^{H-\frac{1}{\alpha(u+rz)}}\Big)\Big|^{\alpha(u+rz)}\\
&\leq \sum_{k=1}^{d}|\theta_k|^{\alpha(u+rz)}\Big(e^{-\lambda r(u_k-z)_+\alpha(u+rz)}(u_k-z)_+^{H\alpha(u+rz)-1} + e^{-\lambda r(-z)_+\alpha(u+rz)}(-z)_+^{H\alpha(u+rz)-1}\Big)\\
&\leq \sum_{k=1}^{d}|\theta_k|^{\alpha(u+rz)}\Big((u_k-z)_+^{H\alpha(u+rz)-1} + (-z)_+^{H\alpha(u+rz)-1}\Big)\\
&\leq \sum_{k=1}^{d}\Big(|\theta_k|^a+|\theta_k|^b\Big)\Big((u_k-z)_+^{Ha-1}+(u_k-z)_+^{Hb-1}+(-z)_+^{Ha-1}+(-z)_+^{Hb-1}\Big).
\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}
&\lim_{r\to 0}\mathbf{E}\Big[e^{i\sum_{k=1}^{d}\theta_k S_r(u_k)}\Big]\\
&= \exp\Big\{-\int_{-\infty}^{\infty}\Big|\sum_{k=1}^{d}\theta_k\Big((u_k-z)_+^{H-\frac{1}{\alpha(u)}}-(-z)_+^{H-\frac{1}{\alpha(u)}}\Big)\Big|^{\alpha(u)}dz\Big\}\\
&= \mathbf{E}\Big[e^{i\sum_{k=1}^{d}\theta_k X(u_k)}\Big],
\end{aligned}$$

where $X(\cdot)$ is defined by (8.83). By Lévy's continuous theorem, we have

$$\lim_{r\to 0} S_r(u_k) \overset{fdd}{=} X(u_k).$$

Thus $X_{H,\alpha(x),\lambda}(t), t \in \mathbf{R}$, is H-localisable at u to $X(\cdot)$ defined by (8.83). □

Recall that $X(t), t \in \mathbf{R}$, is said to be h-strongly localisable at u to $X'_u(v)$ with $h > 0$ (cf. Falconer and Liu [7]), if the convergence in (8.82) occurs in distribution with respect to the metric of uniform convergence on bounded intervals, and X and X'_t have versions in $C(R)$ (the space of continuous function on $\mathbf{R}$).

The next proposition shows that when $1/a < H < 1$, the LTFmSM is H-strongly localisable.

Proposition 20 *Assume that $\alpha(x)$ is continuous on* $\mathbf{R}$. *When $1/a < H < 1$, the process $X_{H,\alpha(x),\lambda}(t)$ is $H-$strongly localisable at u to the LFSM defined by (8.83).*

Proof. By Theorem 3.2 of Falconer and Liu [7], it is sufficient to prove that for each bounded interval J, there is a positive r_0 such that for any $r \in (0, r_0)$,

$$\int_{-\infty}^{\infty} \Big| \frac{G_{H,\alpha(x),\lambda}(u+rt,x) - G_{H,\alpha(x),\lambda}(u+rv,x)}{r^H} \Big|^{\alpha(x)} dx \leq C\, |t-v|^{aH}, \qquad t, v \in J,$$

where C is a constant. Indeed, by (4.65), for any $0 < r \leq \min\{1/|t-v|, 1\}$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \Big| \frac{G_{H,\alpha(x),\lambda}(u+rt,x) - G_{H,\alpha(x),\lambda}(u+rv,x)}{r^H} \Big|^{\alpha(x)} dx \\ &\leq \frac{1}{r^{Ha}} \int_{-\infty}^{\infty} \Big| G_{H,\alpha(x),\lambda}(u+rt,x) - G_{H,\alpha(x),\lambda}(u+rv,x) \Big|^{\alpha(x)} dx \\ &\leq \frac{1}{r^{Ha}} C\, |rt - rv|^{Ha} = C\, |t-v|^{aH}. \end{aligned}$$

This completes the proof of Proposition 20. □

When $\lambda = 0$ and $1/a < H < 1 + 1/b - 1/a$, Falconer and Liu proved that $X_{H,\alpha(x),0}(t), t \in \mathbf{R}$, is $H-$strongly localisable, see Proposition 4.3 of [7]. Now Proposition 20 extends the result of Falconer and Liu.

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