New Regularity of Kolmogorov Equation and Application on Approximation of Semi-linear SPDEs with Hölder Continuous Drifts

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Abstract

In this paper, some new results on the the regularity of Kolmogorov equations associated to the infinite dimensional OU-process are obtained. As an application, the average L^2-error on $[0, T]$ of exponential integrator scheme for a range of semi-linear stochastic partial differential equations is derived, where the drift term is assumed to be Hölder continuous with respect to the Sobolev norm $\|\cdot\|_\beta$ for some appropriate $\beta > 0$. In addition, under a stronger condition on the drift, the strong convergence estimate is obtained, which covers the result of the SDEs with Hölder continuous drift.

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1 Introduction

Recently, the regularity of the Kolmogorov equation with singular coefficients is applied to study the pathwise uniqueness of stochastic (partial) differential equations (S(P)DEs) with singular drifts, which are, for instance, Hölder continuous, Dini continuous or integrable etc. The main idea is to construct Zvonkin's transform ([33]) which is a homeomorphism map depending on the solution of the Kolmogorov equation to transform the original S(P)DEs to a new one, where the singular drift is killed with the aid of the Kolmogorov equation and the pathwise uniqueness can be obtained. There are many results on this topic, see

[4, 7, 8, 9, 10, 13, 20, 28, 29, 30, 31, 32], and references therein. Encouraged by this idea, some researchers have adopted Zvonkin's transform to study the strong convergence rate of the Euler Maruyama (EM) method for SDEs with singular drift, for instance, [14, 24, 25, 26] and the SDEs with Jumps [17]. However, so far, there are no results on the numerical method for the semi-linear SPDEs with singular drift. The main difficulty lies in the following. After Zvonkin's transform, compared to the exact solution, the SPDE for the numerical solution contains two additional items produced by the temporal discretization, one of which depends on the trace of the second-ordered gradient operator of the solution to the associated Kolmogorov equation, see (4.3) and (3.8) below for more details. In $\mathbb{R}^d$, the trace of a linear operator can be controlled by the operator norm, while this is generally not true in infinite dimension case, for instance, the identical operator. Thus, in the SPDEs, the previous results in [28] on the regularity of the Kolmogorov equation, i.e. the estimate for the operator norm of the second-ordered gradient operator of the solution is not available. In other words, to obtain the convergence rate of the numerical method for the semi-linear SPDEs with singular drift, some new regularity, i.e. the trace of the second-ordered gradient operator of the solution to the associated Kolmogorov equation is required. The trace strictly depends on the spectrum of A and the regularity of the drift term b, which will be showed in Theorem 2.7 below by the gradient estimate of the semigroup. This is a new result in related fields.

Through out this paper, let $(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}}, \| \cdot \|_{\mathbb{H}})$ be a real separable Hilbert space. Denote by $\mathscr{L}(\mathbb{H})$ (resp. $\mathscr{L}_2(\mathbb{H})$) the space of all bounded linear operators (resp. Hilbert-Schmidt operators) on $\mathbb{H}$. Let $\| \cdot \|$ (resp. $\| \cdot \|_{\mathscr{L}_2}$) stand for the operator norm (resp. the Hilbert-Schmidt norm). Let $(W_t)_{t \geq 0}$ be an $\mathbb{H}$-valued cylindrical Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$, i.e., $W_t = \sum_{k=1}^{\infty} \beta_t^{(k)} e_k$, where $(\beta^{(k)})_{k \geq 1}$ is a sequence of independent real-valued Brownian motions on the probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ and $(e_k)_{k \geq 1}$ is an orthonormal basis of $\mathbb{H}$. Fix $T > 0$ and set $\|f\|_{T,\infty} := \sup_{t \in [0,T], x \in \mathbb{H}} \|f(t,x)\|$ for an operator-valued map f on $[0,T] \times \mathbb{H}$. Let $\mathscr{B}_b(\mathbb{H}; \mathbb{H})$ be the collection of all bounded measurable functions from $\mathbb{H}$ to $\mathbb{H}$.

Let $(-A, \mathscr{D}(-A))$ be a positive definite self-adjoint operator on $\mathbb{H}$ satisfying $(-A)e_i = \lambda_i e_i, i \geq 1$, where $(0 <)\lambda_1 \leq \lambda_2 \leq \cdots$ are all eigenvalues with counting multiplicities. For any $r \in \mathbb{R}$, Let $\mathbb{H}_r = \{x \in \mathbb{H}, |(-A)^r x| < \infty\}$ equipped the Sobolev norm $\|x\|_r := |(-A)^r x|$, $x \in \mathbb{H}_r$. Then $(\mathbb{H}_r, \| \cdot \|_r)$ is a Banach space and $\mathbb{H}_0 = \mathbb{H}$.

Convention: The letter c or C with or without subscripts will denote an unimportant constant, whose values may change in different places. Moreover, we use the shorthand notation $a \lesssim b$ to mean $a \leq c\, b$. If the constant c depends on a parameter p, we shall also write c_p and $a \lesssim_p b$.

The remainder of this paper is organized as follows: In Section 2, we investigate the new regularity of the Kolmogrov equation associated to the OU-process. In Section 4, as an application of Section 2, we give a result on the average L^2-error on $[0,T]$ of exponential integrator scheme for a range of semi-linear SPDEs with Hölder continuous drift. Moreover, under a stronger condition on the drift, the strong convergence rate is obtained.

2 New Regularity of Kolmogorov Equation Associated to OU-process

In this section, we consider the Kolmogorov Equation with singular coefficients:

$$u_t^\lambda(x) = \int_t^T \mathrm{e}^{-\lambda(s-t)} P_{s-t}^0(\nabla_{b_s} u_s^\lambda + b_s)(x)\mathrm{d}s, \quad x \in \mathbb{H}, \quad t \in [0, T], \tag{2.1}$$

where $\lambda > 0$, $b : [0, T] \times \mathbb{H} \to \mathbb{H}$, $(P_t^0)_{t\geq 0}$ is the Markov semigroup associated to the following O-U process:

$$\mathrm{d}Z_t^x = AZ_t^x \mathrm{d}t + \mathrm{d}W_t, \quad t \geq 0, \quad Z_0^x = x. \tag{2.2}$$

To characterize the regularity of the solution to (2.1), we need some assumptions:

(A1) There exists a constant $\alpha \in (0, 1)$ such that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1-\alpha}} < \infty. \tag{2.3}$$

(A2) b is bounded, i.e.,

$$\|b\|_{T,\infty} < \infty, \tag{2.4}$$

and there exist $c > 0$, $\varepsilon \in (0, 1)$ and $\beta \geq 0$ such that

$$\|b_t(x) - b_t(y)\|_{\mathbb{H}} \leq c\|x - y\|_{-\beta}^{\varepsilon}, \quad t \in [0, T], \quad x, y \in \mathbb{H}. \tag{2.5}$$

Remark 2.1. Under **(A1)**, it is well known that (2.2) has an up to modifications unique mild solution $(Z_t^x)_{t\geq 0}$ (see, e.g., [6]) with the associated Markov semigroup $(P_t^0)_{t\geq 0}$.

Remark 2.2. The condition (2.5) means that the continuity of b_t has weaker dependence on the higher dimensional components. More precisely, for any $i \geq 1$,

$$\|b_t(x) - b_t(x + \langle y - x, e_i\rangle e_i)\|_{\mathbb{H}} \leq \frac{c}{\lambda_i^{\beta}} |\langle y - x, e_i\rangle|^{\varepsilon}, \quad t \in [0, T], \quad x, y \in \mathbb{H}. \tag{2.6}$$

On the other hand, (2.5) implies

$$\|b_t(x) - b_t(y)\|_{\mathbb{H}} \leq c_0\|x - y\|_{\mathbb{H}}^{\varepsilon}, \quad t \in [0, T], \quad x, y \in \mathbb{H} \tag{2.7}$$

for some $c_0 > 0$. Let b_t^0 satisfy (2.7), then $b_t = b_t^0 \circ (-A)^{-\beta}$ satisfy (2.5).

Before moving on, we introduce the results on the regularity of (2.1) in [28, Lemma 2.3].

Lemma 2.3. Assume **(A1)**, (2.4). Then there exists a constant $\lambda_T > 0$ such that for any $\lambda \geq \lambda_T$, (2.1) has a unique solution $u^\lambda \in C([0,T]; C_b^1(\mathbb{H};\mathbb{H}))$ satisfying

$$\lim_{\lambda\to\infty} \{\|u^\lambda\|_{T,\infty} + \|\nabla u^\lambda\|_{T,\infty}\} = 0. \tag{2.8}$$

If moreover (2.7) holds, then for any $\lambda \geq \lambda_T$, $\|\nabla^2 u^\lambda\|_{T,\infty} < \infty$ and

$$\lim_{\lambda\to\infty} \|\nabla^2 u^\lambda\|_{T,\infty} = 0. \tag{2.9}$$

Remark 2.4. We should remark that [28, Lemma 2.3] gives the above result under **(A1)**, (2.4) and

$$\|b_t(x) - b_t(y)\|_{\mathbb{H}} \leq \phi(\|x-y\|_{\mathbb{H}}), \quad t \in [0,T], \quad x, y \in \mathbb{H}, \tag{2.10}$$

where $\phi \in \mathscr{D}$ with

$$\mathscr{D} = \left\{\phi : [0,\infty) \to [0,\infty) \text{ is increasing}, \phi^2 \text{ is concave}, \int_0^1 \frac{\phi(s)}{s}\mathrm{d}s < \infty\right\}.$$

In fact, when $\varepsilon \in (0, 1/2]$, take $\phi(x) = x^\varepsilon$, then $\phi \in \mathscr{D}$ and (2.7) implies (2.10). As for the case $\varepsilon \in (1/2, 1)$, it is easy to obtain Lemma 2.3 by repeating the proof of [28, Lemma 2.3].

The main results in this section are the following:

Theorem 2.5. Assume **(A1)**, (2.4). For any $\lambda \geq \lambda_T$, let u^λ be the unique solution to (2.1). Then the following assertions hold.

(1) For any $\kappa \in [0, 1/2)$,

$$\lim_{\lambda\to\infty} \|\nabla u^\lambda (-A)^\kappa\|_{T,\infty} = 0. \tag{2.11}$$

(2) For any $\theta \in [0, \alpha)$,

$$\sum_{i=1}^\infty \lambda_i^\theta \|\nabla_{e_i} u^\lambda\|_{T,\infty}^2 < \infty. \tag{2.12}$$

(3) If in addition, there exists $\gamma > 0$ such that

$$\|(-A)^\gamma b\|_{T,\infty} < \infty, \tag{2.13}$$

then there exists $\lambda_T' \geq \lambda_T$ such that, for any $\lambda \geq \lambda_T'$, $\|(-A)^\gamma \nabla u^\lambda\|_{T,\infty} < \infty$ and

$$\lim_{\lambda\to\infty} \|(-A)^\gamma \nabla u^\lambda\|_{T,\infty} = 0. \tag{2.14}$$

Proof. (1) Note that the following Bismut formula

$$\nabla_\eta P_t^0 f(x) = \mathbb{E}\Big(\frac{f(Z_t^x)}{t}\int_0^t \langle \nabla_\eta Z_s^x, \mathrm{d}W_s\rangle\Big), \quad t>0, x,\eta\in\mathbb{H}, f\in\mathscr{B}_b(\mathbb{H};\mathbb{H}) \tag{2.15}$$

holds; see, e.g., [28, (2.8)]. By Hölder's inequality and Itô's isometry, together with $\nabla_\eta Z_t^x = \mathrm{e}^{tA}\eta$, we deduce that

$$\begin{aligned} \|\nabla_{(-A)^\kappa\eta} P_t^0 f(x)\|_{\mathbb{H}}^2 &\le \frac{\mathbb{E}\|f(Z_t^x)\|_{\mathbb{H}}^2}{t^2}\int_0^t \|\mathrm{e}^{sA}(-A)^\kappa\eta\|_{\mathbb{H}}^2\mathrm{d}s \\ &\lesssim \frac{P_t^0\|f(x)\|_{\mathbb{H}}^2}{t^2}\sum_{j=1}^\infty \frac{(1-\mathrm{e}^{-2\lambda_j t})\langle \eta, e_j\rangle^2}{\lambda_j^{1-2\kappa}}, \qquad \kappa\in[0,1/2), \end{aligned} \tag{2.16}$$

which, combining $u^\lambda \in C([0,T]; C_b^1(\mathbb{H};\mathbb{H}))$ with $\|b\|_{T,\infty}<\infty$ and $\lambda_j^{2\kappa-1}(1-\mathrm{e}^{-2\lambda_j t}) \lesssim t^{1-2\kappa}$, yields that

$$\begin{aligned} \|\nabla_{(-A)^\kappa\eta} u_t^\lambda\|_{\mathbb{H}} &\le \int_t^T \mathrm{e}^{-\lambda(s-t)}\|\nabla_{(-A)^\kappa\eta} P_{s-t}^0(\nabla_{b_s}u_s^\lambda + b_s)\|_{\mathbb{H}}\mathrm{d}s \\ &\lesssim \|\eta\|_{\mathbb{H}}\int_0^T \mathrm{e}^{-\lambda s}s^{-(\kappa+\frac{1}{2})}\mathrm{d}s \\ &\lesssim \|\eta\|_{\mathbb{H}}\lambda^{\kappa-\frac{1}{2}}. \end{aligned} \tag{2.17}$$

Thereby, (2.11) follows immediately.

(2) Next, in (2.16), taking $\kappa=0$ and $\eta=e_i$, we have

$$\|\nabla_{e_i} P_t^0 f(x)\|_{\mathbb{H}}^2 \lesssim \frac{(1-\mathrm{e}^{-2\lambda_i t})P_t^0\|f(x)\|_{\mathbb{H}}^2}{\lambda_i t^2}. \tag{2.18}$$

This, again together with $u^\lambda \in C([0,T]; C_b^1(\mathbb{H};\mathbb{H}))$ and $\|b\|_{T,\infty}<\infty$, leads to

$$\begin{aligned} \|\nabla_{e_i} u_t^\lambda\|_{\mathbb{H}} &\le \int_t^T \mathrm{e}^{-\lambda(s-t)}\|\nabla_{e_i} P_{s-t}^0(\nabla_{b_s}u_s^\lambda + b_s)\|_{\mathbb{H}}\mathrm{d}s \\ &\lesssim \int_0^T \frac{\mathrm{e}^{-\lambda s}(1-\mathrm{e}^{-2\lambda_i s})^{\frac{1}{2}}}{\lambda_i^{\frac{1}{2}}s}\mathrm{d}s \\ &\lesssim \frac{1}{\lambda_i^{\frac{1-\theta}{2}}}\int_0^T \mathrm{e}^{-\lambda s}s^{\frac{\theta}{2}-1}\mathrm{d}s \\ &\lesssim \frac{1}{\lambda_i^{\frac{1-\theta}{2}}}, \qquad \theta\in(0,1), \end{aligned} \tag{2.19}$$

where we have used the elementary inequality

$$|\mathrm{e}^{-x}-\mathrm{e}^{-y}| \le c_\theta|x-y|^\theta, \qquad x,y\ge 0, \quad \theta\in[0,1] \tag{2.20}$$

for some constant $c_\theta > 0$. Hence, we deduce from (2.3) and (2.19) that

$$\sum_{i=1}^{\infty} \lambda_i^{\alpha-\theta} \|\nabla_{e_i} u^\lambda\|_{T,\infty}^2 \le c_0 \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1-\alpha}} \le c, \qquad \theta \in (0, \alpha]$$

for some constants $c_0, c > 0$. As a result, (2.12) holds.

(3) Let $\mathscr{H} = C([0,T]; C_b^1(\mathbb{H}; \mathbb{H}_\gamma))$, which is a Banach space under the norm

$$\|u\|_{\mathscr{H}} := \|(-A)^\gamma u\|_{T,\infty} + \|(-A)^\gamma \nabla u\|_{T,\infty}, \quad u \in \mathscr{H}.$$

For any $u \in \mathscr{H}$, define

$$(\Gamma u)_s(x) = \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{t-s}^0 (\nabla_{b_t} u_t + b_t)(x) \mathrm{d}t, \quad s \in [0,T].$$

Then we have $\Gamma\mathscr{H} \subset \mathscr{H}$. In fact, for any $u \in \mathscr{H}$, by (2.4) and (2.13), it holds that

$$\begin{aligned}
\|(-A)^\gamma \Gamma u\|_{T,\infty} &= \sup_{s\in[0,T], x\in\mathbb{H}} \left\| \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{t-s}^0 \left((-A)^\gamma \nabla_{b_t} u_t + (-A)^\gamma b_t\right)(x) \mathrm{d}t \right\|_{\mathbb{H}} \\
&\le \sup_{s\in[0,T]} \int_s^T \mathrm{e}^{-\lambda(t-s)} (\|b\|_{T,\infty} \|(-A)^\gamma \nabla u\|_{T,\infty} + \|(-A)^\gamma b\|_{T,\infty}) \mathrm{d}t \\
&\le (\|b\|_{T,\infty} \|(-A)^\gamma \nabla u\|_{T,\infty} + \|(-A)^\gamma b\|_{T,\infty}) \int_0^T \mathrm{e}^{-\lambda t} \mathrm{d}t \\
&\le \frac{\|b\|_{T,\infty} \|(-A)^\gamma \nabla u\|_{T,\infty} + \|(-A)^\gamma b\|_{T,\infty}}{\lambda} < \infty.
\end{aligned}$$

Again by (2.4), (2.13) and (2.16) with $\kappa = 0$, we have

$$\begin{aligned}
\|(-A)^\gamma \nabla \Gamma u\|_{T,\infty} &= \sup_{s\in[0,T], x\in\mathbb{H}, |\eta|\le 1} \left\| \int_s^T \mathrm{e}^{-\lambda(t-s)} \nabla_\eta P_{t-s}^0 ((-A)^\gamma \nabla_{b_t} u_t + (-A)^\gamma b_t)(x) \mathrm{d}t \right\|_{\mathbb{H}} \\
&\le C \sup_{s\in[0,T]} \int_s^T \frac{\mathrm{e}^{-\lambda(t-s)}}{\sqrt{t-s}} (\|b\|_{T,\infty} \|(-A)^\gamma \nabla u\|_{T,\infty} + \|(-A)^\gamma b\|_{T,\infty}) \mathrm{d}t \\
&\le C(\|b\|_{T,\infty} \|(-A)^\gamma \nabla u\|_{T,\infty} + \|(-A)^\gamma b\|_{T,\infty}) \int_0^T \frac{\mathrm{e}^{-\lambda t}}{\sqrt{t}} \mathrm{d}t \\
&\le C \frac{\|b\|_{T,\infty} \|(-A)^\gamma \nabla u\|_{T,\infty} + \|(-A)^\gamma b\|_{T,\infty}}{\sqrt{\lambda}} < \infty.
\end{aligned}$$

So, $\Gamma\mathscr{H} \subset \mathscr{H}$. Next, by the fixed-point theorem, it suffices to show that for large enough $\lambda > 0$, Γ is contractive on $\mathscr{H}$. To do this, for any u, $\tilde{u} \in \mathscr{H}$, similarly to the estimates of $\|(-A)^\gamma \Gamma u\|_{T,\infty}$ and $\|(-A)^\gamma \nabla \Gamma u\|_{T,\infty}$, we obtain that

$$\begin{aligned}
&\|(-A)^\gamma \Gamma u - (-A)^\gamma \Gamma \tilde{u}\|_{T,\infty} \le \frac{\|b\|_{T,\infty}}{\lambda} \|(-A)^\gamma \nabla u - (-A)^\gamma \nabla \tilde{u}\|_{T,\infty}, \\
&\|(-A)^\gamma \nabla \Gamma u - (-A)^\gamma \nabla \Gamma \tilde{u}\|_{T,\infty} \le C \frac{\|b\|_{T,\infty}}{\sqrt{\lambda}} \|(-A)^\gamma \nabla u - (-A)^\gamma \nabla \tilde{u}\|_{T,\infty}.
\end{aligned} \tag{2.21}$$

So we can find $\lambda_T' \geq \lambda_T$ such that Γ is contractive on $\mathscr{H}$ with $\lambda \geq \lambda_T'$, by fixed-point theorem, (2.1) has a unique solution $u^\lambda \in C([0,T]; C_b^1(\mathbb{H};\mathbb{H}_\gamma))$. Finally, substituting $\Gamma u^\lambda = u^\lambda$ into (2.21) and letting $u = u^\lambda$, $\tilde{u} = 0$, we obtain (2.14). □

Remark 2.6. In the multiplicative noise case, i.e.

$$\mathrm{d}Z_t^x = AZ_t^x \mathrm{d}t + \sigma_t(Z_t^x)\mathrm{d}W_t$$

there is also Bismut derivative formula if the diffusion coefficient is non-degenerate, see [28, (2.8)]. Differently, in this case, $\nabla_\eta Z_t^x$ is no longer equal to $\mathrm{e}^{tA}\eta$. Instead, $\nabla_\eta Z_t^x$ satisfies an SPDE:

$$\mathrm{d}\nabla_\eta Z_t^x = A\nabla_\eta Z_t^x \mathrm{d}t + \nabla_{\nabla_\eta Z_t^x}\sigma_t(Z_t^x)\mathrm{d}W_t.$$

Thus it is very hard to obtain (2.17). We will leave the multiplicative noise case in the future research.

The next theorem gives an estimate for the trace of $\nabla^2 u^\lambda$ we explained in the introduction, which plays a crucial role in analyzing error of numerical schemes.

Theorem 2.7. Let **(A1)** and **(A2)** hold and assume further $\nu := \frac{\varepsilon + 2\beta\wedge\alpha\varepsilon^2}{2} + \alpha - 1 > 0$. Then, for any $\lambda \geq \lambda_T$,

$$\sum_{i=1}^{\infty} \lambda_i^\nu \|\nabla_{e_i}\nabla_{e_i} u^\lambda\|_{T,\infty} < \infty. \tag{2.22}$$

Proof. From (2.15) and $\nabla_\eta Z_t^x = \mathrm{e}^{tA}\eta$, we have

$$\frac{1}{2}(\nabla_{e_i}\nabla_\eta P_t^0 f)(x) = \frac{\mathrm{e}^{-\frac{\lambda_i t}{2}}}{t}\mathbb{E}\Big((\nabla_{e_i} P_{t/2}^0 f)(Z_{t/2}^x)\int_0^{t/2}\langle \mathrm{e}^{sA}\eta, \mathrm{d}W_s\rangle\Big). \tag{2.23}$$

By Hölder's inequality and Itô's isometry, it then follows from (2.18), (2.20) with $\theta = 1$, contractive property of e^{tA} and semigroup property of P_t^0 that

$$\begin{aligned}\|(\nabla_{e_i}\nabla_\eta P_t^0 f)(x)\|_{\mathbb{H}}^2 &\lesssim \frac{\mathrm{e}^{-\lambda_i t}\|\eta\|_{\mathbb{H}}^2}{t}\mathbb{E}\|(\nabla_{e_i}P_{t/2}^0 f)(Z_{t/2}^x)\|_{\mathbb{H}}^2 \\ &\lesssim \frac{\mathrm{e}^{-\lambda_i t}\|\eta\|_{\mathbb{H}}^2}{t^2} P_t^0\|f(x)\|_{\mathbb{H}}^2.\end{aligned} \tag{2.24}$$

Furthermore, Itô's isometry, (2.3) as well as (2.20) with $\theta = \alpha$ yield that

$$\mathbb{E}\|Z_t^x - \mathrm{e}^{tA}x\|_{\mathbb{H}}^2 = \int_0^t \|\mathrm{e}^{(t-s)A}\|_{\mathrm{HS}}^2 \mathrm{d}s = \sum_{i=1}^{\infty}\frac{1-\mathrm{e}^{-2\lambda_i t}}{2\lambda_i} \lesssim t^\alpha. \tag{2.25}$$

For a mapping $f : \mathbb{H} \to \mathbb{H}$ such that $\|f(x) - f(y)\|_{\mathbb{H}} \lesssim \|x - y\|_{\mathbb{H}}^\varepsilon$, let $\tilde{f}_t(y) = f(y) - f(\mathrm{e}^{tA}x)$. Taking (2.24) and (2.25) into account and employing Jensen's inequality, we derive that

$$\begin{aligned}\|(\nabla_{e_i}\nabla_\eta P_t^0 f)(x)\|_{\mathbb{H}}^2 &= \|(\nabla_{e_i}\nabla_\eta P_t^0 \tilde{f})(x)\|_{\mathbb{H}}^2 \\ &\lesssim \frac{\mathrm{e}^{-\lambda_i t}\|\eta\|_{\mathbb{H}}^2\mathbb{E}\|Z_t^x - \mathrm{e}^{tA}x\|_{\mathbb{H}}^{2\varepsilon}}{t^2} \\ &\lesssim \frac{\mathrm{e}^{-\lambda_i t}\|\eta\|_{\mathbb{H}}^2}{t^{2-\alpha\varepsilon}}.\end{aligned} \tag{2.26}$$

For notational simplicity, set

$$\hat{f}_t^\lambda(x) := (\nabla_{b_t} u_t^\lambda + b_t)(x). \tag{2.27}$$

It is easy to see from (2.7) and (2.11) that

$$\|\hat{f}_t^\lambda(x) - \hat{f}_t^\lambda(y)\|_{\mathbb{H}} \lesssim \|x-y\|_{\mathbb{H}}^{\varepsilon}, \quad x, y \in \mathbb{H}.$$

Thus, combining (2.1) with (2.26) yields that

$$\begin{aligned} \|\nabla_{e_i}\nabla_\eta u_s^\lambda\|_{\mathbb{H}} &\le \int_s^T \mathrm{e}^{-\lambda(t-s)}\|\nabla_{e_i}\nabla_\eta P_{t-s}^0 \hat{f}_t^\lambda\|_{\mathbb{H}}\mathrm{d}t \\ &\lesssim \|\eta\|_{\mathbb{H}}\int_0^T \frac{\mathrm{e}^{-\lambda_i t/2}}{t^{1-\alpha\varepsilon/2}}\mathrm{d}t \\ &\lesssim \frac{\|\eta\|_{\mathbb{H}}}{\lambda_i^{\alpha\varepsilon/2}}. \end{aligned} \tag{2.28}$$

For a mapping $f:\mathbb{H}\to\mathbb{H}$ satisfying

$$\|f(x)-f(y)\|_{\mathbb{H}} \le \|x-y\|_{-\beta_0}^{\varepsilon_0}, \quad x, y \in \mathbb{H} \tag{2.29}$$

for some $\beta_0>0$ and $\varepsilon_0\in(0,1)$. Fixing $x\in\mathbb{H}$, let

$$\tilde{f}_t^i(y) := f(y) - f(y + \langle \mathrm{e}^{tA}x - y, e_i\rangle e_i), \quad y \in \mathbb{H}.$$

Now, (2.29), Jensen's inequality and Itô's isometry imply that

$$P_t^0\|\tilde{f}_t^i\|_{\mathbb{H}}^2(x) \le \lambda_i^{-2\beta_0}\mathbb{E}|\Lambda_t^i|^{2\varepsilon_0} \lesssim \frac{(1-\mathrm{e}^{-2\lambda_i t})^{\varepsilon_0}}{\lambda_i^{\varepsilon_0+2\beta_0}}, \tag{2.30}$$

where

$$\Lambda_t^i := \int_0^t \langle \mathrm{e}^{(t-s)A}\mathrm{d}W_s, e_i\rangle = \int_0^t \mathrm{e}^{-\lambda_i(t-s)}\mathrm{d}\beta_s^{(i)}.$$

In terms of (2.24) with $\eta = e_i$ and (2.30), together with the notion of $\tilde{f}_t^i$, it follows from (2.20) that

$$\begin{aligned} \|(\nabla_{e_i}\nabla_{e_i}P_t^0 f)(x)\|_{\mathbb{H}}^2 &= \|(\nabla_{e_i}\nabla_{e_i}P_t^0 \tilde{f}_t^i)(x)\|_{\mathbb{H}}^2 \\ &\lesssim \frac{\mathrm{e}^{-\lambda_i t}P_t^0\|\tilde{f}_t^i\|_{\mathbb{H}}^2(x)}{t^2} \\ &\lesssim \frac{\mathrm{e}^{-\lambda_i t}}{t^{2-\varepsilon_0\theta}\lambda_i^{\varepsilon_0(1-\theta)+2\beta_0}}, \quad \theta\in(0,1]. \end{aligned} \tag{2.31}$$

Next, thanks to (2.6), $\|b\|_{T,\infty}<\infty$ and (2.28), we obtain

$$\|\hat{f}_t^\lambda(x) - \hat{f}_t^\lambda(x + \langle y-x, e_i\rangle e_i)\|_{\mathbb{H}} \lesssim \lambda_i^{-(\beta\wedge\frac{\alpha\varepsilon^2}{2})}|x_i - y_i|^\varepsilon, \tag{2.32}$$

where $\hat{f}_t^\lambda : \mathbb{H} \to \mathbb{H}$ is defined in (2.27), Hence, in light of (2.31) with $\varepsilon_0 = \varepsilon$ and $\beta_0 = \beta \wedge \frac{\alpha\varepsilon^2}{2}$ and (2.32), one infers that

$$\begin{aligned}\|\nabla_{e_i}\nabla_{e_i} u_s^\lambda\|_{\mathbb{H}} &\lesssim \frac{1}{\lambda_i^{\frac{\varepsilon(1-\theta)+2\beta\wedge\alpha\varepsilon^2}{2}}} \int_0^T \frac{\mathrm{e}^{-\frac{\lambda_i t}{2}}}{t^{1-\frac{\varepsilon\theta}{2}}}\mathrm{d}t \\ &\lesssim \frac{1}{\lambda_i^{\frac{\varepsilon+2\beta\wedge\alpha\varepsilon^2}{2}}}.\end{aligned}$$

This, along with (2.3), implies (2.22). □

3 Approximation of Semi-linear SPDEs with Hölder Continuous Drifts

The numerical approximation of SPDEs has been a very active field of research. Due to the infinite dimensional nature of state space, in order to be able to simulate a numerical approximation on a computer, both temporal discretization and spatial discretization are implemented. The temporal discretization is achieved generally by Euler type approximations, Milstein type approximations, and splitting-up method (see, e.g., [2, 5, 12, 16, 21, 27]), and the spatial discretization is in general done by finite element, finite difference and spectral Galerkin methods (see, e.g., [19]). In contrast to substantial literature on approximations of semi-linear SPDEs with regular coefficients, the counterpart with irregular terms (e.g., Hölder continuous drifts) is scarce. Whereas, our goal in this section is to make an attempt to discuss strong convergence of an exponential integrator (EI) scheme, coupled with a Galerkin scheme for the spatial discretization (see (3.3) and (3.4) below), for a class of semi-linear SPDEs with Hölder continuous drifts. With regard to convergence of EI scheme for SPDEs with smooth drift coefficients, we refer to [18, 22, 23] for further details, to name a few. Also, there is a number of literature on approximation of SPDEs with non-globally Lipschitz continuous nonlinearities; see, for instance, [15].

Consider the following semi-linear SPDE on $\mathbb{H}$

$$\mathrm{d}X_t = \{AX_t + b_t(X_t)\}\mathrm{d}t + \mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = x \in \mathbb{H}, \tag{3.1}$$

where A, b, W are introduced in Section 2.

Thus, according to [28, Theorem 1.1], under **(A1)** and **(A2)**, (3.1) has a unique mild solution, i.e., there exists a unique continuous adapted process $(X_t)_{t\geq 0}$ such that $\mathbb{P}$-a.s.,

$$X_t = \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{(t-s)A} b_s(X_s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A}\mathrm{d}W_s, \quad t > 0.$$

For any $n \in \mathbb{N}$, let $\pi_n : \mathbb{H} \mapsto \mathbb{H}_n := \mathrm{span}\{e_1, \cdots, e_n\}$ be the orthogonal projection, $A_n = \pi_n A$, $b_t^{(n)} = \pi_n b_t$ and $W_t^{(n)} = \pi_n W_t$. With the notation above, we consider the following finite-dimensional approximation associated with (3.1) on $\mathbb{H}_n \simeq \mathbb{R}^n$

$$\mathrm{d}X_t^{(n)} = \{A_n X_t^n + b_t^{(n)}(X_t^{(n)})\}\mathrm{d}t + \mathrm{d}W_t^{(n)}, \quad t > 0, \quad X_0^{(n)} = x_n := \pi_n x, \tag{3.2}$$

which is the Galerkin projection of (3.1) onto $\mathbb{H}_n$. Since $A_n x = Ax$ for any $x \in \mathbb{H}_n$ and b_n is Hölder continuous in terms of (2.6), by virtue of [28, Theorem 1.1], (3.2) has a unique strong solution.

Now we define a numerical scheme to approximate $X_t^{(n)}$ in time, which is called discrete-time EI scheme: for a stepsize $\delta \in (0,1)$ and each integer $k \geq 0$,

$$\bar{Y}_{(k+1)\delta}^{(n),\delta} = \mathrm{e}^{\delta A_n}\{\bar{Y}_{k\delta}^{(n),\delta} + b_{k\delta}^{(n)}(\bar{Y}_{k\delta}^{(n),\delta})\delta + \Delta W_k^{(n)}\}, \quad \bar{Y}_0^{(n),\delta} = x_n, \tag{3.3}$$

which is also named as Lord-Rougemont scheme (see, e.g., [14, (3.2)]), where $\Delta W_k^{(n)} := W_{(k+1)\delta}^{(n)} - W_{k\delta}^{(n)}$, and continuous-time EI scheme

$$Y_t^{(n),\delta} = \mathrm{e}^{tA_n} x_n + \int_0^t \mathrm{e}^{(t-s_\delta)A_n} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s_\delta)A_n}\mathrm{d}W_s^{(n)}, \quad t \geq 0, \tag{3.4}$$

where $t_\delta := \lfloor t/\delta \rfloor \delta$ with $\lfloor t/\delta \rfloor$ being the integer part of t/δ. It is easy to see that $Y_{k\delta}^{(n),\delta} = \bar{Y}_{k\delta}^{(n),\delta}$ for any $k \geq 0$.

To obtain the convergence rate of the EI scheme, we need the continuity of b_t with respect to t, i.e.

(A3) For $c > 0$ and $\varepsilon \in (0,1)$ in (2.5),

$$\|b_s(x) - b_t(x)\|_{\mathbb{H}} \leq c|s-t|^\varepsilon, \quad s,t \in [0,T], x \in \mathbb{H}. \tag{3.5}$$

The main result of this section is stated as follows.

Theorem 3.1. Assume **(A1)-(A3)**. Suppose $\frac{\varepsilon + 2\beta\wedge\alpha\varepsilon^2}{2} + \alpha - 1 =: \nu \in (0, \frac{1}{2})$ and $x \in \mathscr{D}(A)$. Then, there exists some $C = C(\nu, T) > 0$ such that the following assertions hold.

(1) The average L^2-error on $[0,T]$ satisfies

$$\frac{1}{T}\int_0^T \mathbb{E}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 \mathrm{d}t \leq C\Big\{\delta^{2\nu} + \frac{1}{\lambda_n^{2\nu}}\Big\}. \tag{3.6}$$

(2) If in addition, there exists an $\gamma \in (\frac{1}{2}, 1]$ such that $\|(-A)^\gamma b\|_{T,\infty} < \infty$, then one has the strong convergence estimate:

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 \leq C\Big\{\delta^{2\nu} + \frac{1}{\lambda_n^{2\nu}}\Big\}. \tag{3.7}$$

Remark 3.2. If $1 - \alpha < \frac{\varepsilon}{2}$, then we can take $\beta = 0$ and (2.5) reduces to (2.7). Otherwise, to ensure $\nu > 0$, we have to take $\beta > 0$. See the example in Section 4 for more details.

Remark 3.3. For each n, the EI scheme (3.3) is an explicit one in finite dimensional, this means we can compute $\bar{Y}_{k\delta}^{(n),\delta}, k = 1, 2, \dots$ explicitly given $\bar{Y}_0^{(n),\delta}$. Moreover, Theorem 3.1 (2) covers the result of the finite dimensional case, see [14]. In fact, when $\mathbb{H}$ is finite dimensional, $(-A)^\gamma$ is a bounded linear operator, and the second term on the right side of (3.7) disappears. In this case, we can take $\alpha = 1$, $\beta = 0$ and then $\nu = \frac{\varepsilon}{2}$. Thus

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 \leq C\delta^\varepsilon.$$

Remark 3.4. To obtain the strong convergence estimate, as in the proof of Theorem 3.1 (2), we need to deal with

$$\mathbb{E}\Big\| \int_0^t \mathrm{e}^{(t-s)A}(-A)\{u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta})\}\mathrm{d}s\Big\|_{\mathbb{H}}^2.$$

Noting that $\int_0^t \|\mathrm{e}^{A(t-s)}(-A)\|\mathrm{d}s = \infty$, if $\|(-A)^\gamma \nabla u\|_{T,\infty} < \infty$ for some $\gamma \in (1/2, 1]$, then this term can be treated as in (4.14) below. On the other hand, by Theorem 2.5 (3), $\|(-A)^\gamma b\|_{T,\infty}$ implies $\|(-A)^\gamma \nabla u\|_{T,\infty} < \infty$. For more details, see the proof of the Theorem 3.1. In fact, the above trick is used to prove the pathwise uniqueness of the neutral functional SPDE, see [11], where the condition [11, (H3)] is something like $\|(-A)^\delta \nabla u\|_{T,\infty} < \infty$ for some $\delta > 0$.

Remark 3.5. To avoid complicated computation, in the present setup we work only on the case that the drift is uniformly bounded. Nevertheless, employing the standard cut-off approach (see, e.g., [1]), we of course can extend our framework to the setting that the drift coefficient is unbounded.

The following lemma provides us with a regular representation of the continuous-time EI scheme (3.4).

Lemma 3.6. For any $t \in [0, T]$ and $\lambda \geq \lambda_T$, it holds that

$$\begin{aligned}
&Y_t^{(n),\delta} + u_t^\lambda(Y_t^{(n),\delta}) \\
&= \mathrm{e}^{tA}\{x_n + u_0^\lambda(x_n)\} + \int_0^t \mathrm{e}^{(t-s)A}(\lambda\mathbf{I} - A)u_s^\lambda(Y_s^{(n),\delta})\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{(t-s)A}\{\mathrm{e}^{(s-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})\}\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{(t-s)A}\nabla u_s^\lambda(Y_s^{(n),\delta})\left(\mathrm{e}^{(s-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})\right)\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^n \int_0^t \mathrm{e}^{(t-s)A}(\nabla^2_{\mathrm{e}^{(s-s_\delta)A}e_i}u_s^\lambda - \nabla^2_{e_i}u_s^\lambda)(Y_s^{(n),\delta})\mathrm{d}s \\
&\quad - \frac{1}{2}\sum_{i=n+1}^\infty \int_0^t \mathrm{e}^{(t-s)A}(\nabla^2_{e_i}u_s^\lambda)(Y_s^{(n),\delta})\mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{(t-s_\delta)A}\mathrm{d}W_s^{(n)} + \int_0^t \mathrm{e}^{(t-s)A}(\nabla_{\mathrm{e}^{(s-s_\delta)A}\mathrm{d}W_s^{(n)}}u_s^\lambda)(Y_s^{(n),\delta}),
\end{aligned} \tag{3.8}$$

in which $\mathbf{I}$ is the identity operator on $\mathbb{H}$.

Proof. Since $A_n x = Ax$ for any $x \in \mathbb{H}_n$, (3.4) can be reformulated as

$$\mathrm{d}Y_t^{(n),\delta} = \{AY_t^{(n),\delta} + \mathrm{e}^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta})\}\mathrm{d}t + \mathrm{e}^{(t-t_\delta)A}\mathrm{d}W_t^{(n)}, \quad t > 0, \quad Y_0^{(n),\delta} = x_n.$$

For any $\lambda \geq \lambda_T$, let u_t^λ be the solution to (2.1), then by Theorem 2.7, for any $x \in \cup_{n=1}^\infty \mathbb{H}_n$,

$$\Big(\partial_t u_t^\lambda + \nabla_{b_t} u_t^\lambda + b_t + \frac{1}{2}\sum_{i=1}^\infty \nabla_{e_i}^2 u_t^\lambda + \nabla_{A\cdot} u_t^\lambda\Big)(x) = \lambda u_t^\lambda(x), \quad t \in [0,T], \quad u_T^\lambda = 0, \tag{3.9}$$

where $\nabla_{e_i}^2 := \nabla_{e_i}\nabla_{e_i}$, the second order gradient operator along the direction e_i. Applying Itô's formula, for $\lambda \geq \lambda_T$ we deduce from (3.9) that

$$\begin{aligned}
&\mathrm{d}\{Y_t^{(n),\delta} + u_t^\lambda(Y_t^{(n),\delta})\} \\
&= \Big\{AY_t^{(n),\delta} + \mathrm{e}^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta}) + (\partial_t u_t^\lambda)(Y_t^{(n),\delta}) + (\nabla_{A\cdot} u_t^\lambda)(Y_t^{(n),\delta}) \\
&\quad + \Big(\nabla_{\mathrm{e}^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta})} u_t^\lambda\Big)(Y_t^{(n),\delta}) + \frac{1}{2}\sum_{k=1}^n \Big(\nabla^2_{\mathrm{e}^{(t-t_\delta)A} e_k} u_t^\lambda\Big)(Y_t^{(n),\delta})\Big\}\mathrm{d}t \\
&\quad + \mathrm{e}^{(t-t_\delta)A}\mathrm{d}W_t^{(n)} + \Big(\nabla_{\mathrm{e}^{(t-t_\delta)A}\mathrm{d}W_t^{(n)}} u_t^\lambda\Big)(Y_t^{(n),\delta}) \\
&= \Big\{AY_t^{(n),\delta} + \lambda u_t^\lambda(Y_t^{(n),\delta}) + \mathrm{e}^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta}) - b_t(Y_t^{(n),\delta}) \\
&\quad + \nabla u_t^\lambda(Y_t^{(n),\delta})\Big(\mathrm{e}^{(t-t_\delta)A} b_{t_\delta}^{(n)}(Y_{t_\delta}^{(n),\delta}) - b_t(Y_t^{(n),\delta})\Big) + \frac{1}{2}\sum_{k=1}^n \Big(\nabla^2_{\mathrm{e}^{(t-t_\delta)A} e_k} u_t^\lambda\Big)(Y_t^{(n),\delta}) \\
&\quad - \frac{1}{2}\sum_{k=1}^\infty (\nabla_{e_k}^2 u_t^\lambda)(Y_t^{(n),\delta})\Big\}\mathrm{d}t + \mathrm{e}^{(t-t_\delta)A}\mathrm{d}W_t^{(n)} + \Big(\nabla_{\mathrm{e}^{(t-t_\delta)A}\mathrm{d}W_t^{(n)}} u_t^\lambda\Big)(Y_t^{(n),\delta}).
\end{aligned}$$

This, in addition to integration by parts, further implies that

$$\begin{aligned}
&\int_0^t A\mathrm{e}^{(t-s)A}\{Y_s^{(n),\delta} + u_s^\lambda(Y_s^{(n),\delta})\}\mathrm{d}s \\
&= -\mathrm{e}^{(t-s)A}\{Y_s^{(n),\delta} + u_s^\lambda(Y_s^{(n),\delta})\}\Big|_0^t + \int_0^t \mathrm{e}^{(t-s)A}\mathrm{d}\{Y_s^{(n),\delta} + u_s^\lambda(Y_s^{(n),\delta})\} \\
&= -\{Y_t^{(n),\delta} + u_t^\lambda(Y_t^{(n),\delta})\} + \mathrm{e}^{tA}\{x_n + u_0^\lambda(x_n)\} + \int_0^t A\mathrm{e}^{(t-r)A} Y_r^{(n),\delta}\mathrm{d}r \\
&\quad + \lambda\int_0^t \mathrm{e}^{(t-r)A} u_r^\lambda(Y_r^{(n),\delta})\mathrm{d}r + \int_0^t \mathrm{e}^{(t-r)A}\{\mathrm{e}^{(r-r_\delta)A} b_{r_\delta}^{(n)}(Y_{r_\delta}^{(n),\delta}) - b_r(Y_r^{(n),\delta})\}\mathrm{d}r \\
&\quad + \int_0^t \mathrm{e}^{(t-r)A}\nabla u_r^\lambda(Y_r^{(n),\delta})\left(\mathrm{e}^{(r-r_\delta)A} b_{r_\delta}^{(n)}(Y_{r_\delta}^{(n),\delta}) - b_r(Y_r^{(n),\delta})\right)\mathrm{d}r \\
&\quad + \frac{1}{2}\sum_{k=1}^n \int_0^t \mathrm{e}^{(t-r)A}\Big(\nabla^2_{\mathrm{e}^{(r-r_\delta)A} e_k} u_r^\lambda\Big)(Y_r^{(n),\delta})\mathrm{d}r - \frac{1}{2}\sum_{k=1}^\infty \int_0^t \mathrm{e}^{(t-r)A}(\nabla_{e_k}^2 u_r^\lambda)(Y_r^{(n),\delta})\mathrm{d}r \\
&\quad + \int_0^t \mathrm{e}^{(t-r_\delta)A}\mathrm{d}W_r^{(n)} + \int_0^t \mathrm{e}^{(t-r)A}\Big(\nabla_{\mathrm{e}^{(r-r_\delta)A}\mathrm{d}W_r^{(n)}} u_r^\lambda\Big)(Y_r^{(n),\delta}).
\end{aligned}$$

As a consequence, the desired assertion (3.8) is now available. □

The following lemma concerns the continuity in the mean L^2-norm sense for the displacement of $(Y_t^{(n),\delta})_{t\in[0,T]}$.

Lemma 3.7. Let **(A1)** and **(A2)** hold and assume that the inial value $x \in \mathscr{D}(A)$. Then,

$$\sup_{t\in[0,T]} \mathbb{E}\|Y_t^{(n),\delta} - Y_{t_\delta}^{(n),\delta}\|_{\mathbb{H}}^2 \lesssim_T \delta^\alpha. \tag{3.10}$$

Proof. To make the content self-contained, we here give a sketch although the corresponding argument of (3.10) is quite standard. By virtue of (3.4), it follows immediately that

$$\begin{aligned} Y_t^{(n),\delta} - Y_{t_\delta}^{(n),\delta} &= (\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})\mathrm{e}^{t_\delta A}x_n \\ &\quad + \int_0^{t_\delta} (\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})\mathrm{e}^{(t_\delta - s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\mathrm{d}s \\ &\quad + \int_0^{t_\delta} (\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})\mathrm{e}^{(t_\delta - s_\delta)A}\mathrm{d}W_s^{(n)} \\ &\quad + \int_{t_\delta}^t \mathrm{e}^{(t-s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\mathrm{d}s + \int_{t_\delta}^t \mathrm{e}^{(t-s_\delta)A}\mathrm{d}W_s^{(n)}. \end{aligned}$$

Recall the elementary inequalities: for any $\eta \in (0,1]$,

$$\|(-A)^\eta \mathrm{e}^{tA}\| \le t^{-\eta} \qquad \text{and} \qquad \|(-A)^{-\eta}(\mathrm{e}^{tA} - \mathbf{I})\| \le t^\eta. \tag{3.11}$$

Next, according to Hölder's inequality and Itô's isometry and by taking contractive property of e^{tA} and $\|b\|_{T,\infty} < \infty$ into account, we derive from (3.11) that

$$\begin{aligned} \mathbb{E}\|Y_t^{(n),\delta} - Y_{t_\delta}^{(n),\delta}\|_{\mathbb{H}}^2 &\lesssim_T \|(\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})\mathrm{e}^{t_\delta A}x_n\|_{\mathbb{H}}^2 \\ &\quad + \int_0^{t_\delta} \mathbb{E}\|(\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})\mathrm{e}^{(t_\delta - s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2\mathrm{d}s \\ &\quad + \int_0^{t_\delta} \|(\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})\mathrm{e}^{(t_\delta - s_\delta)A}\|_{\mathscr{L}_2}^2\mathrm{d}s \\ &\quad + \delta\int_{t_\delta}^t \mathbb{E}\|\mathrm{e}^{(t-s_\delta)A} b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2\mathrm{d}s + \int_{t_\delta}^t \|\mathrm{e}^{(t-s_\delta)A}\|_{\mathscr{L}_2}^2\mathrm{d}s \\ &\lesssim_T \|(\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})(-A)^{-1}\|^2\|Ax\|_{\mathbb{H}}^2 \\ &\quad + \int_0^{t_\delta} \|(\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})(-A)^{-\alpha/2}\|^2\|(-A)^{\alpha/2}\mathrm{e}^{(t_\delta - s_\delta)A}\|^2\mathbb{E}\|b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2\mathrm{d}s \\ &\quad + \int_0^{t_\delta} \|(\mathrm{e}^{(t-t_\delta)A} - \mathbf{I})(-A)^{-\alpha/2}\|^2\|(-A)^{\alpha/2}\mathrm{e}^{(t_\delta - s_\delta)A}\|_{\mathscr{L}_2}^2\mathrm{d}s \\ &\quad + \delta\int_{t_\delta}^t \mathbb{E}\|b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2\mathrm{d}s + \int_{t_\delta}^t \|\mathrm{e}^{(t-s_\delta)A}\|_{\mathscr{L}_2}^2\mathrm{d}s \\ &\lesssim_T \delta^2 + \delta^\alpha\int_0^{t_\delta}(t_\delta - s_\delta)^{-\alpha}\mathrm{d}s + \delta^\alpha\sum_{i=1}^\infty \lambda_i^\alpha\int_0^{t_\delta} \mathrm{e}^{-2\lambda_i(t_\delta - s_\delta)}\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\int_0^{t-t_\delta} \mathrm{e}^{-2\lambda_i s}\mathrm{d}s \\
&\lesssim_T \delta^2+\delta^\alpha\int_0^{t_\delta} s^{-\alpha}\mathrm{d}s+\delta^\alpha\sum_{i=1}^{\infty}\lambda_i^\alpha\int_0^{t_\delta}\mathrm{e}^{-2\lambda_i s}\mathrm{d}s+\sum_{i=1}^{\infty}\int_0^{t-t_\delta}\mathrm{e}^{-2\lambda_i s}\mathrm{d}s \\
&\lesssim_T \delta^\alpha+\delta^\alpha\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{1-\alpha}}+\sum_{i=1}^{\infty}\frac{1-\mathrm{e}^{-2\lambda_i(t-t_\delta)}}{\lambda_i} \\
&\lesssim_T \delta^\alpha+\delta^\alpha\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{1-\alpha}} \\
&\lesssim_T \delta^\alpha,
\end{aligned}
$$

where in the penultimate display we used (2.20) with $\theta=\alpha$ and in the last step utilized (2.3). □

4 Proof of Theorem 3.1

With Theorem 2.5, Lemma 3.6, and Lemma 3.7 in hand, we now in a position to complete the proof of Theorem 3.1. In view of (2.11), we deduce that there exists $\hat{\lambda}_T\ge\lambda_T$ such that for any $\lambda\ge\hat{\lambda}_T$

$$
\|\nabla u^\lambda\|_{T,\infty}^2+\|\nabla u^\lambda(-A)^\kappa\|_{T,\infty}^2+3\int_0^T\|\mathrm{e}^{sA}\|_{\mathscr{L}_2}^2\mathrm{d}s\|\nabla^2u^\lambda\|_{T,\infty}^2\le\frac{1}{38}. \tag{4.1}
$$

In what follows, we shall fix $\lambda\ge\hat{\lambda}_T$ so that (4.1) holds. Since

$$
\begin{aligned}
\|X_t-Y_t^{(n),\delta}\|_{\mathbb{H}}^2&\le 2\|X_t+u_t^\lambda(X_t)-Y_t^{(n),\delta}-u_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2+2\|u_t^\lambda(X_t)-u_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2\\
&\le 2\|X_t+u_t^\lambda(X_t)-Y_t^{(n),\delta}-u_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2+\frac{1}{19}\|X_t-Y_t^{(n),\delta}\|_{\mathbb{H}}^2,
\end{aligned}
$$

we have

$$
\Gamma_t^{(n),\delta}:=\mathbb{E}\|X_t-Y_t^{(n),\delta}\|_{\mathbb{H}}^2\le\frac{19}{9}\mathbb{E}\|X_t+u_t^\lambda(X_t)-Y_t^{(n),\delta}-u_t^\lambda(Y_t^{(n),\delta})\|_{\mathbb{H}}^2. \tag{4.2}
$$

According to [28, Proposition 2.5], one has

$$
\begin{aligned}
X_t+u_t^\lambda(X_t)=\mathrm{e}^{tA}(x+u_0^\lambda(x))&+\int_0^t\{(\lambda\mathbf{I}-A)\mathrm{e}^{(t-s)A}u_s^\lambda(X_s)\}\mathrm{d}s\\
&+\int_0^t\mathrm{e}^{(t-s)A}\{\mathrm{d}W_s+(\nabla_{\mathrm{d}W_s}u_s^\lambda)(X_s)\}.
\end{aligned} \tag{4.3}
$$

In view of (3.8) and (4.3), we find that

$$
\begin{aligned}
\Gamma_t^{(n),\delta} \le 19\Big\{&\|\mathrm{e}^{tA}(x-x_n)\|_{\mathbb{H}}^2+\|\mathrm{e}^{tA}(u_0^\lambda(x)-u_0^\lambda(x_n))\|_{\mathbb{H}}^2\\
&+\mathbb{E}\Big\|\int_0^t \mathrm{e}^{(t-s)A}(\lambda\mathbf{I}-A)\{u_s^\lambda(X_s)-u_s^\lambda(Y_s^{(n),\delta})\}\mathrm{d}s\Big\|_{\mathbb{H}}^2\\
&+\mathbb{E}\Big\|\int_0^t \mathrm{e}^{(t-s)A}\{\mathrm{e}^{(s-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})-b_s(Y_s^{(n),\delta})\}\mathrm{d}s\Big\|_{\mathbb{H}}^2\\
&+\mathbb{E}\Big\|\int_0^t \mathrm{e}^{(t-s)A}\nabla u_s^\lambda(Y_s^{(n),\delta})\left(\mathrm{e}^{(s-s_\delta)A}b_{s_\delta}^{(n)}(Y_{s_\delta}^{(n),\delta})-b_s(Y_s^{(n),\delta})\right)\mathrm{d}s\Big\|_{\mathbb{H}}^2\\
&+\mathbb{E}\Big\|\sum_{i=1}^n\int_0^t \mathrm{e}^{(t-s)A}\Big(\nabla^2_{\mathrm{e}^{(s-s_\delta)A}e_i}u_s^\lambda-\nabla^2_{e_i}u_s^\lambda\Big)(Y_s^{(n),\delta})\mathrm{d}s\Big\|_{\mathbb{H}}^2 \qquad (4.4)\\
&+\mathbb{E}\Big\|\sum_{i=n+1}^\infty\int_0^t \mathrm{e}^{(t-s)A}\Big(\nabla^2_{e_i}u_s^\lambda\Big)(Y_s^{(n),\delta})\mathrm{d}s\Big\|_{\mathbb{H}}^2\\
&+\mathbb{E}\Big\|\int_0^t \mathrm{e}^{(t-s)A}\mathrm{d}W_s-\int_0^t \mathrm{e}^{(t-s_\delta)A}\mathrm{d}W_s^{(n)}\Big\|_{\mathbb{H}}^2\\
&+\mathbb{E}\Big\|\int_0^t \mathrm{e}^{(t-s)A}\Big(\nabla_{\mathrm{d}W_s}u_s^\lambda\Big)(X_s)-\int_0^t \mathrm{e}^{(t-s)A}\Big(\nabla_{\mathrm{e}^{(s-s_\delta)A}\mathrm{d}W_s^{(n)}}u_s^\lambda\Big)(Y_s^{(n),\delta})\Big\|_{\mathbb{H}}^2\Big\}\\
=:&\ 19(\Lambda_t^{(1)}+\Lambda_t^{(2)}+\cdots+\Lambda_t^{(9)}).
\end{aligned}
$$

To achieve the desired assertion (3.6), in the sequel, we aim to estimate the terms $\Lambda_t^{(i)}, i\ge 1$, one-by-one. By the contraction property of e^{tA} and thanks to (4.1) and $x\in\mathscr{D}(A)$,

$$
\Lambda_t^{(1)}+\Lambda_t^{(2)}\lesssim \|x-x_n\|_{\mathbb{H}}^2\lesssim \frac{1}{\lambda_n^2}\sum_{i=n+1}^\infty \lambda_i^2\langle x,e_i\rangle^2\lesssim\frac{\|Ax\|_{\mathbb{H}}^2}{\lambda_n^2}\lesssim\frac{1}{\lambda_n^2}. \qquad (4.5)
$$

Taking Hölder's inequality and (3.11) with $\eta=\alpha\varepsilon/2$ into consideration and taking advantage of contractive property of e^{tA} and $\|b\|_{T,\infty}<\infty$ as well as **(A3)**, we deduce from $\alpha\varepsilon\in(0,1)$

that

$$
\begin{aligned}
\Lambda_t^{(4)} &\lesssim_T \int_0^t \mathbb{E}\|b_{s_\delta}(Y_{s_\delta}^{(n),\delta}) - b_s(Y_{s_\delta}^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s + \int_0^t \mathbb{E}\|b_s(Y_{s_\delta}^{(n),\delta}) - b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\|\mathrm{e}^{(t-s_\delta)A}\{\pi_n - \mathbf{I}\} b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\|\mathrm{e}^{(t-s)A}\{\mathrm{e}^{(s-s_\delta)A} - \mathbf{I}\} b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\lesssim_T \delta^{\alpha\varepsilon} + \sum_{i=n+1}^{\infty} \int_0^t \mathrm{e}^{-2\lambda_i(t-s_\delta)} \mathbb{E}\langle b_s(Y_s^{(n),\delta}), e_i\rangle^2 \mathrm{d}s \\
&\quad + \int_0^t \|\mathrm{e}^{(t-s)A}(-A)^{\alpha\varepsilon/2}\|^2 \|(-A)^{-\alpha\varepsilon/2}\{\mathrm{e}^{(s-s_\delta)A} - \mathbf{I}\}\|^2 \mathbb{E}\|b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\lesssim_T \delta^{\alpha\varepsilon} + \int_0^t \mathrm{e}^{-2\lambda_n(t-s_\delta)} \mathbb{E}\|b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s + \delta^{\alpha\varepsilon} \int_0^t (t-s)^{-\alpha\varepsilon} \mathbb{E}\|b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\lesssim_T \delta^{\alpha\varepsilon} + \frac{1}{\lambda_n}.
\end{aligned} \tag{4.6}
$$

By the aid of Jensen's inequality, in addition to (2.7), (2.11), (3.10), (3.11), (4.1), contractive property of e^{tA} and $\|b\|_{T,\infty} < \infty$ along with **(A3)** yield that

$$
\begin{aligned}
\Lambda_t^{(5)} &\lesssim \int_0^t \mathbb{E}\|b_{s_\delta}(Y_{s_\delta}^{(n),\delta}) - b_{s_\delta}(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\|b_{s_\delta}(Y_s^{(n),\delta}) - b_s(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\|\nabla u_s^\lambda(Y_s^{(n),\delta})(-A)^{\alpha\varepsilon/2}\|^2 \|(-A)^{-\alpha\varepsilon/2}(\mathrm{e}^{(s-s_\delta)A} - \mathbf{I})\|^2 \mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\|\nabla u_s^\lambda(Y_s^{(n),\delta})(-A)^{\nu}\|^2 \|(-A)^{-\nu}(\pi_n - \mathbf{I})\|^2 \mathrm{d}s \\
&\lesssim \delta^{\alpha\varepsilon} + \frac{1}{\lambda_n^{2\nu}},
\end{aligned} \tag{4.7}
$$

where we have also used $\frac{1}{2} > \nu = \frac{\varepsilon + 2\beta\wedge\alpha\varepsilon^2}{2} + \alpha - 1 > 0$. We thus obtain from (2.22) that

$$
\begin{aligned}
\Lambda_t^{(6)} &\lesssim \int_0^t \mathbb{E}\Big(\sum_{i=1}^n (1 - \mathrm{e}^{-\lambda_i(s-s_\delta)})^2 \|(\nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}}\Big)^2 \mathrm{d}s \\
&\lesssim \delta^{2\nu} \int_0^t \mathbb{E}\Big(\sum_{i=1}^n \lambda_i^\nu \|(\nabla_{e_i}^2 u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}}\Big)^2 \mathrm{d}s \\
&\lesssim \delta^{2\nu},
\end{aligned} \tag{4.8}
$$

where we have used $\sup_{x>0}\{(1-\mathrm{e}^{-x})x^{-\nu}\}<\infty$ for $\nu\in(0,1)$. Also, with the help of (2.22),

$$
\begin{aligned}
\Lambda_t^{(7)} &\lesssim \int_0^t \mathbb{E}\Big(\sum_{i=n+1}^{\infty}\|(\nabla^2_{e_i}u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}}\Big)^2\mathrm{d}s\\
&\lesssim \frac{1}{\lambda_n^{2\nu}}\int_0^t \mathbb{E}\Big(\sum_{i=n+1}^{\infty}\lambda_i^{\nu}\|(\nabla^2_{e_i}u_s^\lambda)(Y_s^{(n),\delta})\|_{\mathbb{H}}\Big)^2\mathrm{d}s \qquad (4.9)\\
&\lesssim \frac{1}{\lambda_n^{2\nu}}.
\end{aligned}
$$

Employing Itô's isometry, (2.3) and (2.20) with $\theta=\alpha/2$, we get that

$$
\begin{aligned}
\Lambda_t^{(8)} &\lesssim \sum_{i=n+1}^{\infty}\int_0^t \mathrm{e}^{-2\lambda_i(t-s)}\mathrm{d}s+\sum_{i=1}^{n}\int_0^t \mathrm{e}^{-2\lambda_i(t-s)}(1-\mathrm{e}^{-\lambda_i(s-s_\delta)})^2\mathrm{d}s\\
&\lesssim \sum_{i=n+1}^{\infty}\int_0^t \mathrm{e}^{-2\lambda_i s}\mathrm{d}s+\delta^{\alpha}\sum_{i=1}^{n}\lambda_i^{\alpha}\int_0^t \mathrm{e}^{-2\lambda_i s}\mathrm{d}s \qquad (4.10)\\
&\lesssim \sum_{i=n+1}^{\infty}\frac{1}{\lambda_i}+\delta^{\alpha}\sum_{i=1}^{n}\frac{1}{\lambda_i^{1-\alpha}}\\
&\lesssim \frac{1}{\lambda_n^{\alpha}}+\delta^{\alpha}.
\end{aligned}
$$

Let

$$
\Theta_t=\mathbb{E}\int_0^t \|\mathrm{e}^{(t-r)A}[(\nabla u_r^\lambda)(X_r)-(\nabla u_r^\lambda)(Y_r^{(n),\delta})]\|^2_{\mathscr{L}_2}\mathrm{d}r.
$$

Again, an application of Itô's isometry implies that for any $\theta\in[0,\alpha)$,

$$
\begin{aligned}
\Lambda_t^{(9)} &\le 3\Big\{\Theta_t+\sum_{i=1}^{\infty}\int_0^t(1-\mathrm{e}^{-\lambda_i(r-r_\delta)})^2\mathbb{E}\|(\nabla_{e_i}u_r^\lambda)(Y_r^{(n),\delta})\|^2_{\mathbb{H}}\mathrm{d}r\\
&\quad+\sum_{i=n+1}^{\infty}\int_0^t\mathbb{E}\|(\nabla_{e_i}u_r^\lambda)(Y_r^{(n),\delta})\|^2_{\mathbb{H}}\mathrm{d}r\Big\}\\
&\le 3\Theta_t+c_1\Big\{\delta^{\theta}\sum_{i=1}^{\infty}\lambda_i^{\theta}\int_0^t\mathbb{E}\|(\nabla_{e_i}u_r^\lambda)(Y_r^{(n),\delta})\|^2_{\mathbb{H}}\mathrm{d}r \qquad (4.11)\\
&\quad+\lambda_n^{-\theta}\sum_{i=n+1}^{\infty}\lambda_i^{\theta}\int_0^t\mathbb{E}\|(\nabla_{e_i}u_r^\lambda)(Y_r^{(n),\delta})\|^2_{\mathbb{H}}\mathrm{d}r\Big\}\\
&\le 3\Theta_t+c_2\{\delta^{\theta}+\lambda_n^{-\theta}\},
\end{aligned}
$$

for some constants $c_1,c_2>0$, where we have used (2.12) in the last procedure.

Next, we will use different techniques to estimate $\Lambda_t^{(3)}$ and $\Lambda_t^{(9)}$ under the conditions of assertion (1) and (2) respectively.

(1) By Fubini's theorem, we deduce that

$$3\int_0^t \mathrm{e}^{-2\lambda s}\Theta_s \mathrm{d}s = 3\int_0^t \mathrm{e}^{-2\lambda r}\mathbb{E}\|(\nabla u_r^\lambda)(X_r) - (\nabla u_r^\lambda)(Y_r^{(n),\delta})\|^2 \left(\int_0^{t-r} \mathrm{e}^{-2\lambda s}\|\mathrm{e}^{sA}\|_{\mathscr{L}_2}^2 \mathrm{d}s\right)\mathrm{d}r$$
$$\le 3\left(\int_0^T \|\mathrm{e}^{sA}\|_{\mathscr{L}_2}^2 \mathrm{d}s \|\nabla^2 u^\lambda\|_{T,\infty}^2\right)\int_0^t \mathrm{e}^{-2\lambda s}\Gamma_s^{(n),\delta}\mathrm{d}s.$$

Thus, it follows from (4.11) that

$$\int_0^t \mathrm{e}^{-2\lambda s}\Lambda_s^{(9)}\mathrm{d}s \le 3\left(\int_0^T \|\mathrm{e}^{sA}\|_{\mathscr{L}_2}^2 \mathrm{d}s\|\nabla^2 u^\lambda\|_{T,\infty}^2\right)\int_0^t \mathrm{e}^{-2\lambda s}\Gamma_s^{(n),\delta}\mathrm{d}s + C\{\delta^\theta + \lambda_n^{-\theta}\}. \tag{4.12}$$

Using Hölder's inequality and Fubini's theorem, we deduce from (4.1) that

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-2\lambda s}\Lambda_s^{(3)}\mathrm{d}s\\
&= \sum_{i=1}^\infty \mathbb{E}\int_0^t \mathrm{e}^{-2\lambda s}\Big(\int_0^s \langle \mathrm{e}^{(s-r)A}(\lambda\mathbf{I} - A)\{u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta})\}, e_i\rangle \mathrm{d}r\Big)^2 \mathrm{d}s\\
&= \sum_{i=1}^\infty \mathbb{E}\int_0^t \Big(\int_0^s \mathrm{e}^{-\lambda_i(s-r)-\lambda s}(\lambda+\lambda_i)\langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}), e_i\rangle \mathrm{d}r\Big)^2 \mathrm{d}s\\
&\le \sum_{i=1}^\infty \int_0^t \Big(\int_0^s \mathrm{e}^{-(\lambda_i+\lambda)(s-r)}(\lambda+\lambda_i)\mathrm{d}r\\
&\quad \times \int_0^s \mathrm{e}^{-(\lambda+\lambda_i)(s-r)-2\lambda r}(\lambda+\lambda_i)\mathbb{E}\langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}), e_i\rangle^2 \mathrm{d}r\Big)\mathrm{d}s\\
&\le \sum_{i=1}^\infty \int_0^t\int_0^s \mathrm{e}^{-(\lambda+\lambda_i)(s-r)-2\lambda r}(\lambda+\lambda_i)\mathbb{E}\langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}), e_i\rangle^2 \mathrm{d}r\mathrm{d}s\\
&= \sum_{i=1}^\infty \int_0^t \Big(\int_r^t \mathrm{e}^{-(\lambda+\lambda_i)(s-r)}(\lambda+\lambda_i)\mathrm{d}s\Big)\mathrm{e}^{-2\lambda r}\mathbb{E}\langle u_r^\lambda(X_r) - u_r^\lambda(Y_r^{(n),\delta}), e_i\rangle^2 \mathrm{d}r\\
&\le \sum_{i=1}^\infty \int_0^t \mathrm{e}^{-2\lambda s}\mathbb{E}\langle u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta}), e_i\rangle^2 \mathrm{d}s\\
&= \int_0^t \mathrm{e}^{-2\lambda s}\mathbb{E}\|u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta})\|_{\mathbb{H}}^2 \mathrm{d}s\\
&\le \|\nabla u^\lambda\|_{T,\infty}^2 \int_0^t \mathrm{e}^{-2\lambda s}\Gamma_s^{(n),\delta}\mathrm{d}s,
\end{aligned}$$

which, combining with (4.4), (4.1) and (4.12), further leads to

$$\int_0^t \mathrm{e}^{-2\lambda s}\Gamma_s^{(n),\delta}\mathrm{d}s \le \frac{1}{2}\int_0^t \mathrm{e}^{-2\lambda s}\Gamma_s^{(n),\delta}\mathrm{d}s + C\{\delta^\theta + \lambda_n^{-\theta}\} + 19\sum_{i=1, i\neq 3}^{8}\int_0^t \mathrm{e}^{-2\lambda s}\Lambda_s^{(i)}\mathrm{d}s.$$

We therefore obtain that

$$\int_0^t \mathrm{e}^{-2\lambda s}\Gamma_s^{(n),\delta}\mathrm{d}s \le 38\sum_{i=1,i\neq 3}^{8}\int_0^t \mathrm{e}^{-2\lambda s}\Lambda_s^{(i)}\mathrm{d}s + C\{\delta^\theta + \lambda_n^{-\theta}\}. \tag{4.13}$$

Finally, taking $\theta = \alpha\varepsilon$, combining (4.5)-(4.10) with (4.13), noticing $2\nu \in (0, \alpha\varepsilon)$, we arrive at

$$\int_0^t \mathrm{e}^{-2\lambda s}\Gamma_s^{(n),\delta}\mathrm{d}s \lesssim_T \delta^{2\nu} + \frac{1}{\lambda_n^{2\nu}}.$$

Thus, (3.6) holds.

(2) If $\|(-A)^\gamma b\|_{T,\infty} < \infty$, then by Theorem 2.5 (3), taking $\lambda = \max(\hat{\lambda}_T, \lambda_T')$ and noting that $\gamma \in (1/2, 1]$, we derive from Hölder inequality that

$$\begin{aligned}
&\mathbb{E}\Big\|\int_0^t \mathrm{e}^{(t-s)A}(-A)\{u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta})\}\mathrm{d}s\Big\|_{\mathbb{H}}^2 \\
&\le \int_0^t \left\|\mathrm{e}^{(t-s)A}(-A)^{1-\gamma}\right\|^2 \mathrm{d}s\mathbb{E}\int_0^t \|(-A)^\gamma\{u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta})\}\|_{\mathbb{H}}^2\mathrm{d}s \\
&\le \int_0^t s^{-2(1-\gamma)}\mathrm{d}s\|(-A)^\gamma\nabla u^\lambda\|_{T,\infty}^2\int_0^t \mathbb{E}\|X_s - Y_s^{(n),\delta}\|_{\mathbb{H}}^2\mathrm{d}s \\
&\le C\int_0^t \mathbb{E}\|X_s - Y_s^{(n),\delta}\|_{\mathbb{H}}^2\mathrm{d}s.
\end{aligned} \tag{4.14}$$

So we have

$$\begin{aligned}
\Lambda_t^{(3)} &\le C_0\int_0^t \mathbb{E}\|X_s - Y_s^{(n),\delta}\|_{\mathbb{H}}^2\mathrm{d}s \\
&\quad + C_0\mathbb{E}\Big\|\int_0^t \mathrm{e}^{(t-s)A}(-A)\{u_s^\lambda(X_s) - u_s^\lambda(Y_s^{(n),\delta})\}\mathrm{d}s\Big\|_{\mathbb{H}}^2 \\
&\le C\int_0^t \mathbb{E}\|X_s - Y_s^{(n),\delta}\|_{\mathbb{H}}^2\mathrm{d}s.
\end{aligned} \tag{4.15}$$

On the other hand,

$$\begin{aligned}
\Theta_t &\le \mathbb{E}\int_0^t \|\mathrm{e}^{(t-r)A}\|_{\mathscr{L}_2}^2\|\nabla^2 u^\lambda\|_{T,\infty}^2\|X_r - Y_r^{(n),\delta}\|_{\mathbb{H}}^2\mathrm{d}r \\
&\le \sup_{r\in[0,t]}\mathbb{E}\|X_r - Y_r^{(n),\delta}\|_{\mathbb{H}}^2\left(\int_0^T \|\mathrm{e}^{rA}\|_{\mathscr{L}_2}^2\mathrm{d}r\|\nabla^2 u^\lambda\|_{T,\infty}^2\right)
\end{aligned}$$

Combining this with (4.11) and (4.1), we obtain

$$\begin{aligned}
\Lambda_t^{(9)} &\le 3\left(\int_0^T \|\mathrm{e}^{sA}\|_{\mathscr{L}_2}^2\mathrm{d}s\|\nabla^2 u^\lambda\|_{T,\infty}^2\right)\sup_{r\in[0,t]}\mathbb{E}\|X_r - Y_r^{(n),\delta}\|_{\mathbb{H}}^2 + C\{\delta^\theta + \lambda_n^{-\theta}\} \\
&\le \frac{1}{38}\sup_{r\in[0,t]}\mathbb{E}\|X_r - Y_r^{(n),\delta}\|_{\mathbb{H}}^2 + C\{\delta^\theta + \lambda_n^{-\theta}\}.
\end{aligned} \tag{4.16}$$

Then (4.4), (4.15) and (4.16) yield that

$$\sup_{r\in[0,t]}\Gamma_r^{(n),\delta}\le\frac{1}{2}\sup_{r\in[0,t]}\Gamma_r^{(n),\delta}+C\{\delta^\theta+\lambda_n^{-\theta}\}+C\int_0^t\sup_{r\in[0,s]}\Gamma_r^{(n),\delta}\mathrm{d}s+19\sum_{i=1,i\neq 3}^{8}\sup_{r\in[0,t]}\Lambda_r^{(i)}.$$

We therefore obtain that

$$\sup_{r\in[0,t]}\Gamma_r^{(n),\delta}\le 38\sum_{i=1,i\neq 3}^{8}\sup_{r\in[0,t]}\Lambda_r^{(i)}+C\{\delta^\theta+\lambda_n^{-\theta}\}+C\int_0^t\sup_{r\in[0,s]}\Gamma_r^{(n),\delta}\mathrm{d}s. \tag{4.17}$$

Again, taking $\theta=\alpha\varepsilon$, combining (4.5)-(4.10) with (4.17), noting $2\nu\in(0,\alpha\varepsilon)$, we arrive from Gronwall's inequality that at

$$\sup_{r\in[0,t]}\Gamma_r^{(n),\delta}\lesssim_T\delta^{2\nu}+\frac{1}{\lambda_n^{2\nu}}.$$

Thus, (3.7) holds. This therefore implies the desired assertion.

Finally, we give an example.

Example 4.1. (**Stochastic reaction-diffusion equations**) Let $(\Delta,\mathscr{D}(\Delta))$ be the Dirichlet Laplacian on a bounded domain $D\subset\mathbb{R}^d$, and let $\epsilon>\frac{d}{2}$ be a constant. Let $\mathbb{H}=L^2(\mathbf{m})$ for $\mathbf{m}$ be normalized Lebesgue measure on D. Let $-A=(-\Delta)^\epsilon$ has a discrete spectrum with eigenvalues $\{\lambda_n\}_{n\ge1}$ satisfying

$$C^{-1}n^{\frac{2\epsilon}{d}}\le\lambda_n\le Cn^{\frac{2\epsilon}{d}},\quad n\ge1,$$

for some constant $C>1$. Consider the following semi-linear SPDE:

$$\mathrm{d}X_t=\{AX_t+b_t(X_t)\}\mathrm{d}t+\mathrm{d}W_t,\quad t\in[0,T],\quad X_0=x\in\mathbb{H}. \tag{4.18}$$

Since $\epsilon>\frac{d}{2}$, then we can take $\alpha\in(0,1)$ with $1-\alpha>\frac{d}{2\epsilon}$ such that **(A1)** holds.

(1) If $\epsilon>d$, then we can take $\alpha\in(0,1)$ with $\frac{d}{2\epsilon}<1-\alpha<\frac{1}{2}$ such that **(A1)** holds. By Remark 3.2, we can take $\beta=0$. In this case, we can choose $\varepsilon\in(0,1)$ such that $\frac{\varepsilon}{2}+\alpha-1>0$. We assume that there exists $c>0$ such that

$$\|b_t(x)-b_t(y)\|_{\mathbb{H}}\le c\|x-y\|_{\mathbb{H}}^{\varepsilon},\quad t\in[0,T],\quad x,y\in\mathbb{H}. \tag{4.19}$$

Let $\delta=Tn^{-\frac{2\epsilon}{d}}$. Then the following assertions hold.

(i) If $\|b\|_{T,\infty}<\infty$, the average L^2-error on $[0,T]$ satisfies

$$\frac{1}{T}\int_0^T\mathbb{E}\|X_t-Y_t^{(n),\delta}\|_{\mathbb{H}}^2\mathrm{d}t\le Cn^{-\frac{2\epsilon}{d}(\varepsilon+2\alpha-2)}.$$

(ii) If there exists an $\gamma \in (\frac{1}{2}, 1]$ such that $\|(-A)^\gamma b\|_{T,\infty} < \infty$, then the strong convergence rate has the following estimate:

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 \le Cn^{-\frac{2\epsilon}{d}(\varepsilon+2\alpha-2)}.$$

(2) If $\epsilon \in (\frac{d}{2}, d]$, then $1-\alpha > \frac{d}{2\epsilon}$ means $1-\alpha > \frac{1}{2}$. By Remark 3.2, β has to be strictly positive such that $\frac{\varepsilon+2\beta\wedge\alpha\varepsilon^2}{2} + \alpha - 1 > 0$ (the inequality: $\frac{\varepsilon+2\beta\wedge\alpha\varepsilon^2}{2} + \alpha - 1 < \frac{1}{2}$ is always true). In this case, we assume that there exists $c > 0$ such that

$$\|b_t(x) - b_t(y)\|_{\mathbb{H}} \le c\|x-y\|_{-\beta}^{\varepsilon}, \quad t \in [0,T], \quad x, y \in \mathbb{H}. \tag{4.20}$$

Let $\delta = Tn^{-\frac{2\epsilon}{d}}$. Then the following assertions hold.

(i) If $\|b\|_{T,\infty} < \infty$, the average L^2-error on $[0,T]$ satisfies

$$\frac{1}{T}\int_0^T \mathbb{E}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 \mathrm{d}t \le Cn^{-\frac{2\epsilon}{d}(\varepsilon+2\alpha-2+2\beta\wedge\alpha\varepsilon^2)}.$$

(ii) If there exists an $\gamma \in (\frac{1}{2}, 1]$ such that $\|(-A)^\gamma b\|_{T,\infty} < \infty$, then the strong convergence rate has the following estimate:

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t - Y_t^{(n),\delta}\|_{\mathbb{H}}^2 \le Cn^{-\frac{2\epsilon}{d}(\varepsilon+2\alpha-2+2\beta\wedge\alpha\varepsilon^2)}.$$

Remark 4.2. In this paper, we have only studied the average L^2-error and strong convergence for the EI scheme. In the future, on the one hand, we may extend the EI scheme to some other equations with singular drift, for example: the stochastic Hamiltonian system in infinite dimension with singular drift; on the other hand, we may develop other schemes to SPDEs with singular drift, for instance: multilevel Monte Carlo method (see [3]).

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