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• **ARTICLES** •

Ramanujan-type congruences for broken 2-diamond partitions modulo 3

CHEN William Y.C.[1], FAN Anna R.B.[2] & YU Rebecca T.[2]

[1]*Center for Applied Mathematics, Tianjin University, Tianjin* 300072, *China;*
[2]*Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin* 300071, *China*
Email: chenyc@tju.edu.cn, fanruice@mail.nankai.edu.cn, yuting_shuxue@mail.nankai.edu.cn

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Abstract The notion of broken k-diamond partitions was introduced by Andrews and Paule. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. Andrews and Paule also posed three conjectures on the congruences of $\Delta_2(n)$ modulo 2, 5 and 25. Hirschhorn and Sellers proved the conjectures for modulo 2, and Chan proved the two cases of modulo 5. For the case of modulo 3, Radu and Sellers obtained an infinite family of congruences for $\Delta_2(n)$. In this paper, we obtain two infinite families of congruences for $\Delta_2(n)$ modulo 3 based on a formula of Radu and Sellers, a 3-dissection formula of the generating function of triangular number due to Berndt, and the properties of the U-operator, the V-operator, the Hecke operator and the Hecke eigenform. For example, we find that $\Delta_2(243n + 142) \equiv \Delta_2(243n + 223) \equiv 0 \pmod 3$. The infinite family of Radu and Sellers and the two infinite families derived in this paper have two congruences in common, namely, $\Delta_2(27n + 16) \equiv \Delta_2(27n + 25) \equiv 0 \pmod 3$.

Keywords broken k-diamond partition, modular form, Ramanujan-type congruence, Hecke eigenform

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1 Introduction

The objective of this paper is to derive two infinite families of congruences for the number of broken 2-diamond partitions modulo 3. The notion of broken k-diamond partitions was introduced by Andrews and Paule [1]. A broken k-diamond partition $\pi = (a_1, a_2, a_3, \ldots; b_2, b_3, b_4, \ldots)$ is a plane partition satisfying the relations illustrated in Figure 1, where a_i, b_i are nonnegative integers and $a_i \to a_j$ means $a_i \geqslant a_j$. More precisely, the building blocks in Figure 1, except for the broken block $(b_2, b_3, \ldots, b_{2k+2})$, have the same order structure as shown in Figure 2. We call each block a k-elongated partition diamond of length 1, or a k-elongated diamond, for short.

For example, Figure 3 gives a broken 2-diamond partition

$$\pi = (10, 8, 9, 7, 6, 3, 2, 1, 0, 1; 3, 5, 2, 1, 1, 1).$$

Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. For convenience, define $\Delta_k(0) = 1$ and $\Delta_k(n) = 0$ for $n < 0$. Let $B_k(q)$ denote the generating function of $\Delta_k(n)$, i.e.,

$$B_k(q) = \sum_{n \geqslant 0} \Delta_k(n) q^n.$$

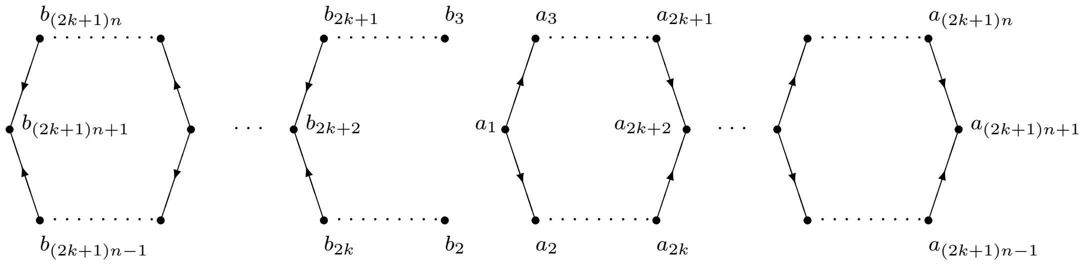

Figure 1 A broken k-diamond of length $2n$

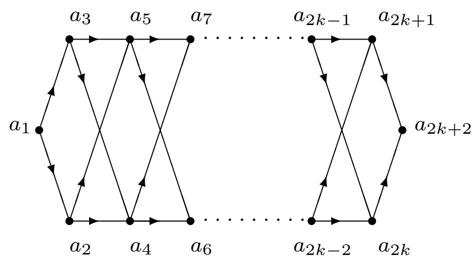

Figure 2 A k-elongated diamond

Andrews and Paule [1] obtained the following formula,

$$B_k(q) = \frac{(-q;q)_\infty}{(q;q)_\infty^2(-q^{2k+1};q^{2k+1})_\infty}. \tag{1.1}$$

Note that the above formula can also be written in terms of eta-quotients related to modular forms

$$B_k(q) = q^{(k+1)/12}\frac{\eta(2z)\eta((2k+1)z)}{\eta(z)^3\eta((4k+2)z)},$$

where $q = \mathrm{e}^{2\pi \mathrm{i} z}$.

From (1.1), Andrews and Paule proved that for $n \geqslant 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod 3. \tag{1.2}$$

They also posed three conjectures on congruences for the number of broken 2-diamond partitions of n. Fu [5] and Mortenson [8] found combinatorial proofs of congruence (1.2). Meanwhile, Hirschhorn and Sellers [6] gave a proof of (1.2) by deriving the following generating function for $\Delta_1(2n+1)$,

$$\sum_{n\geqslant 0}\Delta_1(2n+1)q^n = 3\frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2}{(q;q)_\infty^6}.$$

Hirschhorn and Sellers [6] also obtained the following congruences modulo 2,

$$\Delta_1(4n+2) \equiv \Delta_1(4n+3) \equiv 0 \pmod 2, \tag{1.3}$$

$$\Delta_2(10n+2) \equiv \Delta_2(10n+6) \equiv 0 \pmod 2, \tag{1.4}$$

where $n \geqslant 1$. The congruences in (1.4) were conjectured by Andrews and Paule [1].

Chan [4] found two infinite families of congruences modulo 5 for broken 2-diamond partitions

$$\Delta_2\left(5^{l+1}n + \frac{3}{4}(5^l-1) + 2\cdot 5^l + 1\right) \equiv 0 \pmod 5, \tag{1.5}$$

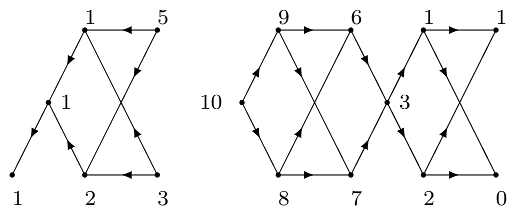

Figure 3 A broken 2-diamond partition of 60

$$\Delta_2\left(5^{l+1}n+\frac{3}{4}(5^l-1)+4\cdot 5^l+1\right)\equiv 0 \pmod 5, \tag{1.6}$$

where $l \geqslant 1$ and $n \geqslant 0$. The two cases for $l = 1$ in (1.5) and (1.6) were conjectured by Andrews and Paule [1], namely,

$$\Delta_2(25n+14)\equiv \Delta_2(25n+24)\equiv 0 \pmod 5.$$

Paule and Radu [10] obtained an infinite family of congruences modulo 5 for broken 2-diamond partitions. They showed that for any prime p with $p \equiv 13, 17 \pmod{20}$ and any nonnegative integer n,

$$\Delta_2\left(5p^2n+4p-\frac{1}{4}(p-1)\right)\equiv 0 \pmod 5. \tag{1.7}$$

Moreover, they posed four conjectures on congruences for broken 3-diamond partitions and broken 5-diamond partitions, which have been confirmed by Xiong [12] and Jameson [7].

For broken 2-diamond partitions, Radu and Sellers [11] showed that

$$\sum_{n\geqslant 0}\Delta_2(3n+1)q^n \equiv 2q\prod_{n\geqslant 1}\frac{(1-q^{10n})^4}{(1-q^{5n})^2} \pmod 3, \tag{1.8}$$

which implies the following congruences,

$$\Delta_2(15n+1)\equiv \Delta_2(15n+7)\equiv 0 \pmod 3, \tag{1.9}$$

$$\Delta_2(15n+10)\equiv \Delta_2(15n+13)\equiv 0 \pmod 3, \tag{1.10}$$

and

$$\Delta_2\left(3p^2n+\frac{3}{4}\left(p(4k+3)-1\right)+1\right)\equiv 0 \pmod 3, \tag{1.11}$$

where $p \equiv 3 \pmod 4$ is a prime,

$$0 \leqslant k \leqslant p-1 \quad \text{and} \quad k \neq \frac{p-3}{4}.$$

In this paper, we use (1.8) to establish two new infinite families of congruences of $\Delta_2(n)$ modulo 3 by using a 3-dissection formula of the generating function of triangular numbers and properties of the U-operator, the V-operator, the Hecke operator and the Hecke eigenform.

Theorem 1.1. *For $l \geqslant 1$, we have*

$$\Delta_2\left(3^{2l+1}n+\frac{3}{4}(3^{2l}-1)+3^{2l}+1\right)\equiv 0 \pmod 3, \tag{1.12}$$

$$\Delta_2\left(3^{2l+1}n+\frac{3}{4}(3^{2l}-1)+2\cdot 3^{2l}+1\right)\equiv 0 \pmod 3. \tag{1.13}$$

2 Preliminaries

In this section, we give an overview of some definitions and properties of modular forms which will be used in the proof of Theorem 1.1. Let N be a positive integer. We shall use modular forms in the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, where

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| c \equiv 0 \pmod{N} \right\}.$$

We shall also make use of the U-operator $U(d)$, the V-operator $V(d)$, the Hecke operator $T_{m,k}$ acting on modular forms and the Hecke eigenform with respect to the Hecke operator $T_{m,k}$, see Ono [9].

Definition 2.1. Let N be a positive integer, and k be an integer. Let

$$f(z) = \sum_{n\geqslant 0} a(n)q^n$$

be a modular form in $M_k(\Gamma_0(N))$. If d is a positive integer, then the U-*operator* $U(d)$ and the V-*operator* $V(d)$ acting on $f(z)$ are defined by

$$f(z)|U(d) := \sum_{n\geqslant 0} a(dn)q^n \tag{2.1}$$

and

$$f(z)|V(d) := \sum_{n\geqslant 0} a(n)q^{dn}. \tag{2.2}$$

For any positive integer m, the action of the Hecke operator $T_{m,k}$ on $f(z)$ is given by

$$f(z)|T_{m,k} := \sum_{n\geqslant 0} \left(\sum_{d|\,\gcd(m,n)} d^{k-1} a(mn/d^2) \right) q^n.$$

In particular, for any prime p,

$$f(z)|T_{p,k} := \sum_{n\geqslant 0} (a(pn) + p^{k-1}a(n/p))q^n. \tag{2.3}$$

Moreover, by (2.1) and (2.3), for any prime p and $k > 1$, we have

$$f(z)|U(p) \equiv f(z)|T_{p,k} \pmod{p}. \tag{2.4}$$

The operators $U(d)$, $V(d)$ and the Hecke operator $T_{m,k}$ have the following properties.

Proposition 2.2. *Let N be a positive integer, and k be an integer. Let $f(z)$ be a modular form in $M_k(\Gamma_0(N))$.*

(1) *If d is a positive integer, then*

$$f(z)|V(d) \in M_k(\Gamma_0(Nd)).$$

(2) *If d is a positive integer and $d|N$, then*

$$f(z)|U(d) \in M_k(\Gamma_0(N)).$$

(3) *If m is a positive integer, then*

$$f(z)|T_{m,k} \in M_k(\Gamma_0(N)).$$

A Hecke eigenform associated with the Hecke operator $T_{m,k}$ is defined as follows.

Definition 2.3. A modular form $f(z) \in M_k(\Gamma_0(N))$ is called a Hecke eigenform associated with the Hecke operator $T_{m,k}$ if for every $m \geqslant 2$ there is a complex number $\lambda(m)$ such that

$$f(z)|T_{m,k} = \lambda(m)f(z).$$

If a Hecke eigenform with respect to the Hecke operator $T_{m,k}$ is a cusp form, then the following proposition can be used to compute $\lambda(m)$.

Proposition 2.4. *Suppose that*

$$f(z) = \sum_{n\geqslant 0} a(n)q^n$$

is a cusp form in $S_k(\Gamma_0(N))$ *with* $a(1) = 1$. *If* $f(z)$ *is a Hecke eigenform associated with the Hecke operator* $T_{m,k}$, *then for* $m \geqslant 1$,

$$f(z)|T_{m,k} = a(m)f(z).$$

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 by using the approach of Chan [4]. We first establish the following congruence on the generating function of $\Delta_2(3n+1)$. Let $\psi(q)$ be the generating function for the triangular numbers, i.e.,

$$\psi(q) = \sum_{n\geqslant 0} q^{n(n+1)/2} = \prod_{n\geqslant 1} \frac{(1-q^{2n})^2}{(1-q^n)}. \tag{3.1}$$

Lemma 3.1. *We have*

$$\psi(q^{15})^2 \sum_{n\geqslant 0} \Delta_2(3n+1)q^{n+4} \equiv 2q^5\psi(q^5)^8 \pmod 3. \tag{3.2}$$

Proof. By (1.8) and (3.1), we find that

$$\sum_{n\geqslant 0} \Delta_2(3n+1)q^n \equiv 2q \prod_{n\geqslant 1} \frac{(1-q^{10n})^4}{(1-q^{5n})^2} = 2q\psi(q^5)^2 \pmod 3. \tag{3.3}$$

It follows that

$$\begin{aligned} 2q^5\psi(q^5)^8 &= q^4\psi(q^5)^6 \cdot (2q\psi(q^5)^2) \\ &\equiv \psi(q^5)^6 \sum_{n\geqslant 0} \Delta_2(3n+1)q^{n+4} \pmod 3. \end{aligned} \tag{3.4}$$

Since

$$\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} \equiv 1 \pmod 3$$

(see Ono [9]), we deduce that

$$\psi(q^5)^6 \sum_{n\geqslant 0} \Delta_2(3n+1)q^{n+4} \equiv \psi(q^{15})^2 \sum_{n\geqslant 0} \Delta_2(3n+1)q^{n+4} \pmod 3. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain that

$$\psi(q^{15})^2 \sum_{n\geqslant 0} \Delta_2(3n+1)q^{n+4} \equiv 2q^5\psi(q^5)^8 \pmod 3,$$

as claimed. □

It is not difficult to show that $q^5\psi(q^5)^8$ is a Hecke eigenform in $M_4(\Gamma_0(10))$. For a formal power series $f(q)$, we use the common notation $[q^m]f(q)$ to denote the coefficient of q^m in $f(q)$.

Lemma 3.2. *The function $q^5\psi(q^5)^8$ is a Hecke eigenform in $M_4(\Gamma_0(10))$ associated with the Hecke operator $T_{m,4}$. More precisely, for $m \geqslant 2$, we have*

$$q^5\psi(q^5)^8|T_{m,4} = \lambda(m)q^5\psi(q^5)^8,$$

where $\lambda(m) = [q^m]q\psi(q)^8$.

Proof. It has been shown by Chan [3] that $q\psi(q)^8$ is a Hecke eigenform in $M_4(\Gamma_0(2))$ with respect to the Hecke operator $T_{m,4}$. By Definition 2.3, for $m \geqslant 2$, there exists a complex number $\lambda(m)$ such that

$$q\psi(q)^8|T_{m,4} = \lambda(m)q\psi(q)^8. \tag{3.6}$$

Observing that $q\psi(q)^8$ is a cusp form for which the coefficient of q is 1, by Proposition 2.4 we find that

$$\lambda(m) = [q^m]q\psi(q)^8. \tag{3.7}$$

Substituting q by q^5 in (3.6), we obtain that

$$q^5\psi(q^5)^8|T_{m,4} = \lambda(m)q^5\psi(q^5)^8.$$

Meanwhile, by Proposition 2.2(1), we deduce that

$$q^5\psi(q^5)^8 = q\psi(q)^8|V(5) \in M_4(\Gamma_0(10)).$$

Thus $q^5\psi(q^5)^8$ is a Hecke eigenform in $M_4(\Gamma_0(10))$ associated with the Hecke operator $T_{m,4}$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $f(z) = 2q^5\psi(q^5)^8$ with $q = \mathrm{e}^{2\pi \mathrm{i}z}$. Applying the U-operator $U(3)$ to $f(z)$ gives

$$\begin{aligned} 2q^5\psi(q^5)^8|U(3) &\equiv \left(\psi(q^{15})^2\sum_{n\geqslant 0}\Delta_2(3n+1)q^{n+4}\right)\Bigg|U(3) \pmod 3 \\ &= \psi(q^5)^2\sum_{n\geqslant 0}\Delta_2(9n-11)q^n \pmod 3. \end{aligned}$$

On the other hand, by Lemma 3.2, we see that

$$\begin{aligned} q^5\psi(q^5)^8|T_{3,4} &= ([q^3]q\psi(q)^8)q^5\psi(q^5)^8 \\ &= 28q^5\psi(q^5)^8 \equiv q^5\psi(q^5)^8 \pmod 3. \end{aligned}$$

Employing relation (2.4), we deduce that

$$\begin{aligned} \psi(q^5)^2\sum_{n\geqslant 0}\Delta_2(9n-11)q^n &\equiv 2q^5\psi(q^5)^8|U(3) \pmod 3 \\ &\equiv 2q^5\psi(q^5)^8|T_{3,4} \equiv 2q^5\psi(q^5)^8 \pmod 3. \end{aligned}$$

Substituting n by $n+2$ in the above congruence, we get

$$\sum_{n\geqslant 0}\Delta_2(9n+7)q^n \equiv 2q^3\psi(q^5)^6 \equiv 2q^3\psi(q^{15})^2 \pmod 3. \tag{3.8}$$

Using a 3-dissection formula of $\psi(q)$ due to Berndt [2, p. 49], $\psi(q)$ can be written in the following form,

$$\psi(q) = A(q^3) + q\psi(q^9), \tag{3.9}$$

where $A(q)$ is a power series in q.

Plugging (3.9) into the right-hand side of (3.8), we find that

$$\sum_{n\geqslant 0}\Delta_2(9n+7)q^n \equiv 2q^3(A(q^{45}) + q^{15}\psi(q^{135}))^2 \pmod 3$$

$$\equiv 2q^3(A(q^{45})^2 + 2q^{15}A(q^{45})\psi(q^{135}) + q^{30}\psi(q^{135})^2) \pmod 3. \tag{3.10}$$

Extracting those terms whose powers of q are congruent to 6 modulo 9 in (3.10), we obtain that

$$\sum_{n\geqslant 0} \Delta_2(9(9n+6)+7)q^{9n+6} \equiv 2q^{33}\psi(q^{135})^2 \pmod 3. \tag{3.11}$$

Dividing both sides of (3.11) by q^6 and replacing q^9 by q, we get

$$\sum_{n\geqslant 0} \Delta_2(3^4n+61)q^n \equiv 2q^3\psi(q^{15})^2 \pmod 3. \tag{3.12}$$

Combining (3.8) and (3.12), we arrive at

$$\sum_{n\geqslant 0} \Delta_2(3^2n+7)q^n \equiv \sum_{n\geqslant 0} \Delta_2(3^4n+61)q^n \pmod 3. \tag{3.13}$$

Iterating (3.13) with n replaced by $9n+6$, we conclude that for $l \geqslant 1$,

$$\sum_{n\geqslant 0} \Delta_2(3^2n+7)q^n \equiv \sum_{n\geqslant 0} \Delta_2\left(3^{2l}n + \frac{3}{4}(3^{2l}-1)+1\right)q^n \pmod 3.$$

Using (3.8) again, we get

$$\sum_{n\geqslant 0} \Delta_2\left(3^{2l}n + \frac{3}{4}(3^{2l}-1)+1\right)q^n \equiv 2q^3\psi(q^{15})^2 \pmod 3. \tag{3.14}$$

Since there are no terms with powers of q congruent to 1, 2 modulo 3 in $2q^3\psi(q^{15})^2$, we obtain the following infinite families of congruences,

$$\Delta_2\left(3^{2l+1}n + \frac{3}{4}(3^{2l}-1)+3^{2l}+1\right) \equiv 0 \pmod 3,$$
$$\Delta_2\left(3^{2l+1}n + \frac{3}{4}(3^{2l}-1)+2\cdot 3^{2l}+1\right) \equiv 0 \pmod 3.$$

This completes the proof. □

Here are some examples of Theorem 1.1. For $n \geqslant 0$, we have

$$\Delta_2(27n+16) \equiv \Delta_2(27n+25) \equiv 0 \pmod 3, \tag{3.15}$$
$$\Delta_2(243n+142) \equiv \Delta_2(243n+223) \equiv 0 \pmod 3, \tag{3.16}$$
$$\Delta_2(2187n+1276) \equiv \Delta_2(2187n+2005) \equiv 0 \pmod 3. \tag{3.17}$$

Notice that the congruences in (3.15) are also contained in the infinite family of congruences (1.11) derived by Radu and Sellers.

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