Time optimal sampled-data controls for the heat equation

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Abstract

In this paper, we first design a time optimal control problem for the heat equation with sampled-data controls, and then use it to approximate a time optimal control problem for the heat equation with distributed controls.

The study of such a time optimal sampled-data control problem is not easy, because it may have infinitely many optimal controls. We find connections among this problem, a minimal norm sampled-data control problem and a minimization problem. And obtain some properties on these problems. Based on these, we not only build up error estimates for optimal time and optimal controls between the time optimal sampled-data control problem and the time optimal distributed control problem, in terms of the sampling period, but also prove that such estimates are *optimal* in some sense.

Résumé

Contrôles à données échantionnées en temps optimal pour l'équation de la chaleur. Dans cet article, nous concevons d'abord un problème pour l'équation de la chaleur avec les contrôles à données échantillonnés, puis l'utiliser pour approacher un problème de contrôle en temps minimal pour l'équation de la chaleur avec des contrôles distribués.

L'étude d'un tel problème n'est pas facile puisqu'il peut avoir un nombre infini de contrôles optimals. Nous trouvons des connexions entre ce problème, un problème de contrôle à donn es chantillonnes, et un problème de minimisation, et nous obtenons des propriétés sur ces problmes. Selon ces résultats, nous établissons non seulemnt des estimations d'erreur entre les deux problèmes en question, en termes de période d'échantillonnage, mais aussi nous prouvons que ces estimations sont optimales dans certain sens.

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1. Introduction

1.1. *Motivation and problems*

In most published literature on time optimal control problems, controls are distributed in time, i.e., they can vary at each instant of time. However, in practical application, it is more convenient to use controls which vary only finite times. Sampled-data controls are such kind of controls (see for instance [1,8,13,19,23]). In this paper, we will design and study a time optimal control problem for the heat equation with sampled-data controls. And then we use it to approximate a time optimal control problem for the heat equation with distributed controls, through building up several error estimates for optimal time and optimal controls between these two problems, in terms of the sampling period. Such errors estimates have laid foundation for us to replace distributed controls by sampled-data controls in time optimal control problems for heat equations.

Throughout this paper, $\mathbb{R}^+ \triangleq (0,\infty)$; $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}^+ \triangleq \{1,2,\ldots\}$) is a bounded domain with a C^2 boundary $\partial\Omega$; $\omega \subset \Omega$ is an open and nonempty subset with its characteristic function χ_ω; λ_1 is the first eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition over Ω; $B_r(0)$ denotes the closed ball in $L^2(\Omega)$, centered at 0 and of radius $r>0$; for each measurable set $\mathcal{A}$ in $\mathbb{R}$, $|\mathcal{A}|$ denotes its Lebesgue measure; $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the usual inner product and norm of $L^2(\Omega)$, respectively; $\langle\cdot,\cdot\rangle_\omega$ and $\|\cdot\|_\omega$ stand for the usual inner product and norm in $L^2(\omega)$, respectively.

First, we introduce a time optimal distributed control problem for the heat equation. Throughout this paper, we fix the initial state y_0 and the target ball $B_r(0)$ in the following way:

$$r>0 \ \text{ and } \ y_0 \in L^2(\Omega)\setminus B_r(0). \tag{1}$$

For each $M>0$, we consider the following time optimal distributed control problem:

$$(\mathcal{TP})^M: \ \ \mathcal{T}(M) = \inf\left\{\hat{t}>0 \ : \ \exists\, \hat{u}\in\mathcal{U}^M \ \text{ s.t. } \ y(\hat{t};y_0,\hat{u})\in B_r(0)\right\}, \tag{2}$$

where

$$\mathcal{U}^M \triangleq \left\{u\in L^2(\mathbb{R}^+\times\Omega) \ : \ \|u\|_{L^2(\mathbb{R}^+\times\Omega)}\leq M\right\},$$

and where $y(\cdot;y_0,u)$ is the solution to the following distributed controlled heat equation:

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega u & \text{in } \ \mathbb{R}^+\times\Omega,\\ y=0 & \text{on } \mathbb{R}^+\times\partial\Omega,\\ y(0)=y_0 & \text{in } \ \Omega. \end{cases} \tag{3}$$

Since $y(t;y_0,0)\to 0$ as $t\to\infty$, we find that $\mathcal{T}(M)<\infty$ for all $M>0$. About $(\mathcal{TP})^M$, we introduce some concepts in the following definition:

Definition 1.1 *(i) The number $\mathcal{T}(M)$ is called the optimal time; $\hat{u}\in\mathcal{U}^M$ is called an admissible control if $y(\hat{t};y_0,\hat{u})\in B_r(0)$ for some $\hat{t}>0$; $u^*\in\mathcal{U}^M$ is called an optimal control if $y(\mathcal{T}(M);y_0,u^*)\in B_r(0)$. (ii) Two optimal controls are said to be different (or the same), if they are different (or the same) on their effective domain $\big(0,\mathcal{T}(M)\big)\times\Omega$.*

Several notes on the problem $(\mathcal{TP})^M$ are given in order:

- It is shown in Theorem 3.1 that for each $M>0$, $(\mathcal{TP})^M$ has a unique optimal control.

- In many time optimal distributed control problems for heat equations, controls are taken from $L^\infty(\mathbb{R}^+; L^2(\Omega))$. However, the current setting is also significant (see, for instance, [16] and [38]).

Next, we are going to design a time optimal sampled-data control problem for the heat equation. For this purpose, we define the following space of sampled-data controls (where $\delta > 0$ is arbitrarily fixed):

$$L^2_\delta(\mathbb{R}^+ \times \Omega) \triangleq \Big\{u_\delta \in L^2(\mathbb{R}^+ \times \Omega) \ : \ u_\delta \triangleq \sum_{i=1}^{\infty} \chi_{((i-1)\delta, i\delta]} u^i, \ \{u^i\}_{i=1}^\infty \subset L^2(\Omega)\Big\}, \tag{4}$$

endowed with the $L^2(\mathbb{R}^+ \times \Omega)$-norm. Here and in what follows, $\chi_{((i-1)\delta, i\delta]}$ denotes the characteristic function of the interval $\big((i-1)\delta, i\delta\big]$ for each $i \in \mathbb{N}^+$. The numbers $\delta, 2\delta, \ldots, i\delta, \ldots$ are called the sampling instants, while δ is called the sampling period. Each u_δ in the space $L^2_\delta(\mathbb{R}^+ \times \Omega)$ is called a sampled-data control. For each $u_\delta \in L^2_\delta(\mathbb{R}^+ \times \Omega)$ and each $z_0 \in L^2(\Omega)$, write $y(\cdot; z_0, u_\delta)$ for the solution to the following sampled-data controlled heat equation:

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega u_\delta & \text{in } \ \mathbb{R}^+ \times \Omega, \\ y = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ y(0) = z_0 & \text{in } \ \Omega. \end{cases} \tag{5}$$

For each $M > 0$ and $\delta > 0$, we consider the following time optimal sampled-data control problem:

$$(\mathcal{TP})^M_\delta : \quad \mathcal{T}_\delta(M) = \inf\big\{k\delta \ : \ \exists\, k \in \mathbb{N}^+, \exists\, u_\delta \in \mathcal{U}^M_\delta \text{ s.t. } y(k\delta; y_0, u_\delta) \in B_r(0)\big\}, \tag{6}$$

where

$$\mathcal{U}^M_\delta \triangleq \big\{u_\delta \in L^2_\delta(\mathbb{R}^+ \times \Omega) \ \ : \ \ \|u_\delta\|_{L^2(\mathbb{R}^+\times\Omega)} \le M\big\}. \tag{7}$$

Since $y(t; y_0, 0) \to 0$ as $t \to \infty$, we see that $\mathcal{T}_\delta(M) < \infty$ for all $M \ge 0$ and $\delta > 0$. With respect to $(\mathcal{TP})^M_\delta$, we introduce some concepts in the following definition:

Definition 1.2 *(i) The number $\mathcal{T}_\delta(M)$ is called the optimal time; $u_\delta \in \mathcal{U}^M_\delta$ is called an admissible control if $y(\hat{k}\delta; y_0, u_\delta) \in B_r(0)$ for some $\hat{k} \in \mathbb{N}^+$; $u^*_\delta \in \mathcal{U}^M_\delta$ is called an optimal control if $y(\mathcal{T}_\delta(M); y_0, u^*_\delta) \in B_r(0)$. (ii) A control u^*_δ is called the optimal control with the minimal norm, if u^*_δ is an optimal control and satisfies that $\|u^*_\delta\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \le \|v^*_\delta\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}$ for any optimal control v^*_δ. (iii) Two optimal controls are said to be different (or the same), if they are different (or the same) over $(0, \mathcal{T}_\delta(M)) \times \Omega$.*

Several notes on this problem are given in order:

- The optimal time $\mathcal{T}_\delta(M)$ is a multiple of δ (see (6)). For each $M > 0$ and each $\delta > 0$, $(\mathcal{TP})^M_\delta$ has a unique optimal control with the minimal norm (see (ii) in Theorem 3.1); There are infinitely many pairs (M, δ) so that $(\mathcal{TP})^M_\delta$ has infinitely many different optimal controls (see Theorem 3.2).
- We may design a time optimal sampled-data control problem in another way: To find a control u^*_δ in $\mathcal{U}^M_\delta$ so that $y(\cdot; y_0, u^*_\delta)$ enters $B_r(0)$ in the shortest time $\hat{\mathcal{T}}_\delta(M)$ (which may not be a multiple of δ). We denote this problem by $(\widehat{\mathcal{TP}})^M_\delta$. Several reasons for us to design time optimal sampled-data control problem to be $(\mathcal{TP})^M_\delta$ are as follows: (i) Each sampled-data control u_δ has the form: $\sum_{i=1}^\infty \chi_{((i-1)\delta, i\delta]} u^i$ with some $\{u^i\}_{i=1}^\infty \subset L^2(\Omega)$. From the perspective of sampled-data controls, each u^i should be active in the whole subinterval $((i-1)\delta, i\delta]$. Thus, our definition for $\mathcal{T}(M)$ is reasonable. (ii) In the definition $(\widehat{\mathcal{TP}})^M_\delta$, in order to make sure if the control process should be finished, we need to observe the solution (of the controlled equation) at each time. However, in our definition $(\mathcal{TP})^M_\delta$,

we only need to observe the solution at time points $i\delta$, with $i = 1, 2, \ldots$. (iii) Our design on $(\mathcal{TP})_\delta^M$ might provide a right way to approach numerically $(\mathcal{TP})^M$ via a discretized time optimal control problem. For instance, if we semi-discretize $(\mathcal{TP})^M$ in time variable, then our design on $(\mathcal{TP})_\delta^M$ can be borrowed to define a semi-discretized (in the time variable) time optimal control problem. The reason is as follows: For the problem $(\mathcal{TP})^M$, we do not know the optimal time $\mathcal{T}(M)$ before the computation. Thus, if we want to semi-discretize the problem in time, we do not know how to choose the mesh size δ so that $\mathcal{T}(M) = k\delta$ for integer $k \geq 1$. On the other hand, if borrow our definition $(\mathcal{TP})_\delta^M$, we can pass the above-mentioned barrier.

1.2. *Main results*

Recall that y_0 and r are given by (1). The main results of this paper are presented in the following three theorems.

Theorem 1.3 *Let $M > 0$. Then the following conclusions are true:*
(i) There is $\delta_0 \triangleq \delta_0(M, y_0, r) > 0$ so that

$$0 \leq \mathcal{T}_\delta(M) - \mathcal{T}(M) \leq 2\delta \ \text{ for all } \ \delta \in (0, \delta_0). \tag{8}$$

(ii) For each $\eta \in (0,1)$, there exists a measurable set $\mathcal{A}_{M,\eta} \subset (0,1)$ (depending also on y_0 and r) with $\lim_{h\to 0^+} \frac{1}{h}|\mathcal{A}_{M,\eta} \cap (0,h)| = \eta$ so that

$$\delta > \mathcal{T}_\delta(M) - \mathcal{T}(M) > (1-\eta)\delta \ \text{ for each } \ \delta \in \mathcal{A}_{M,\eta}. \tag{9}$$

Theorem 1.4 *Let $M > 0$ and u^* be the optimal control to $(\mathcal{TP})^M$. For each $\delta > 0$, let u_δ^* be the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. Then the following conclusions are true:*
(i) There is $C \triangleq C(M, y_0, r) > 0$ so that

$$\|u_\delta^* - u^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \leq C\delta \ \text{ for each } \ \delta > 0. \tag{10}$$

(ii) For each $\eta \in (0,1)$, there is a measurable set $\mathcal{A}_{M,\eta} \subset (0,1)$ (depending also on y_0 and r) with $\lim_{h\to 0^+} \frac{1}{h}|\mathcal{A}_{M,\eta} \cap (0,h)| = \eta$ so that

$$\|u_\delta^* - u^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \geq \frac{1}{2}\lambda_1^{3/2} r(1-\eta)\delta \ \text{ for each } \ \delta \in \mathcal{A}_{M,\eta}. \tag{11}$$

Theorem 1.5 *Let $M > 0$ and u^* be the optimal control to $(\mathcal{TP})^M$. Then the following conclusions are true:*
(i) There is $C \triangleq C(M, y_0, r) > 0$ so that

$$\|u_\delta - u^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \leq C\sqrt{\delta} \ \text{ for each } \ \delta > 0, \tag{12}$$

where u_δ is any optimal control to $(\mathcal{TP})_\delta^M$.
(ii) For each $\eta \in (0,1)$, there is a measurable set $\mathcal{A}_{M,\eta} \subset (0,1)$ (depending also on y_0 and r) with $\lim_{h\to 0^+} \frac{1}{h}|\mathcal{A}_{M,\eta} \cap (0,h)| = \eta$ so that for each $\delta \in \mathcal{A}_{M,\eta}$, there is an optimal control $\hat{u}_\delta$ to $(\mathcal{TP})_\delta^M$ so that

$$\|\hat{u}_\delta - u^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \geq C_M\sqrt{(1-\eta)\delta}, \tag{13}$$

for some positive constant $C_M \triangleq C_M(y_0, r)$.

Several remarks on the main results are given in order.

- Theorem 1.3 and Theorem 1.4 present two facts. First, the error between $\mathcal{T}_\delta(M)$ and $\mathcal{T}(M)$ and the error between u^*_M and $u^*_{M,\delta}$ have the order 1 with respect to the sampling period δ. Second, this order is *optimal*, because of the lower bound estimates (9) and (11), and because of the property that $\lim_{h\to 0^+}\frac{1}{h}|\mathcal{A}_{M,\eta}\cap(0,h)|=\eta$ with any $\eta\in(0,1)$. Notice that when $\delta\in(0,1)\setminus\mathcal{A}_{M,\eta}$, (9) may not be true (see Theorem 6.2, as well as Remark 6.3).
- Theorem 1.5, as well as Theorem 1.4, presents two facts. First, in $(\mathcal{TP})^M_\delta$, the optimal control with the minimal norm differs from some of other optimal controls, from perspective of the order of the errors. Second, the order of the error between any optimal control of $(\mathcal{TP})^M_\delta$ and the optimal control to $(\mathcal{TP})^M$ is 1/2, with respect to δ. Moreover, this order is *optimal* in the sense (ii) of Theorem 1.5.
- Since we aim to approximate u^* by u^*_δ and because the efficient domain of u^* is $(0,\mathcal{T}(M))\times\Omega$, we take the $L^2((0,\mathcal{T}(M))\times\Omega)$-norm in the estimates in Theorem 1.4 and Theorem 1.5.
- There have been many publications on optimal sampled-data control problems (with fixed ending time point). In [6] (see also [5]), the authors built up the Pontryagin maximum principle for some optimal sampled-data control problems. In [7], the authors showed that for some LQ problem, the optimal sampled-data control converges to the optimal distributed control as the sampling period tends to zero. In [33], the authors built up some error estimates between the optimal distributed control and the optimal sampled-data control for some periodic heat equations. About more works on sampled-data controls, we would like to mention [1,3,4,8,13,14,19,20,23,32] and the references therein.
- There have been some literatures on the approximations of time optimal control problems for the parabolic equations. We refer to [15,41] for semi-discrete finite element approximations, and [34,43] for perturbations of equations. About more works on time optimal control problems, we would like to mention [2,10,11,16,17,18,21,22,25,27,30,31,35,37,38,39,40,42,44] and the references therein.
- About approximations of time optimal sampled-data controls, we have not found any literature in the past publications.

1.3. *The strategy to get the main results*

The strategy to prove the main theorems is as follows: We first introduce two norm optimal control problems which correspond to time optimal control problems $(\mathcal{TP})^M$ and $(\mathcal{TP})^M_\delta$ respectively; then get error estimates between the above two norm optimal control problems (in terms of δ); finally, obtain the desired error estimates between $(\mathcal{TP})^M$ and $(\mathcal{TP})^M_\delta$ (in terms of δ), through using connections between the time optimal control problems and the corresponding norm optimal control problems (see (iii) of Theorem 3.1 and Theorem 4.1, respectively).

To explain our strategy more clearly, we will introduce two norm optimal control problems. The first one corresponds to $(\mathcal{TP})^M$ and is as:

$$(\mathcal{NP})^T:\ \mathcal{N}(T)\triangleq\inf\{\|v\|_{L^2((0,T)\times\Omega)}\ :\ y(T;y_0,v)\in B_r(0)\},\tag{14}$$

where $T>0$ and $y(\cdot;y_0,v)$ is the solution of (3) with u being replaced by the zero extension of v over $\mathbb{R}^+$. The second one corresponds to $(\mathcal{TP})^M_\delta$ and is defined by

$$(\mathcal{NP})^{k\delta}_\delta:\ \mathcal{N}_\delta(k\delta)\triangleq\inf\{\|v_\delta\|_{L^2_\delta((0,k\delta)\times\Omega)}\ :\ y(k\delta;y_0,v_\delta)\in B_r(0)\},\tag{15}$$

where $(\delta,k)\in\mathbb{R}^+\times\mathbb{N}^+$,

$$L^2_\delta((0,k\delta)\times\Omega)\triangleq\{f|_{(0,k\delta)\times\Omega}\ :\ f\in L^2_\delta(\mathbb{R}^+\times\Omega)\},\tag{16}$$

and $y(\cdot; y_0, v_\delta)$ is the solution of (3) with u being replaced by the zero extension of v_δ over $\mathbb{R}^+$. (In the definition of $(\mathcal{NP})_\delta^{k\delta}$, $k\delta$ denotes the length of the time interval and δ is the mesh size.)

Some concepts about the above two norm optimal control problems are given in the following definition:

Definition 1.6 *(i) In the problem $(\mathcal{NP})^T$, $\mathcal{N}(T)$ is called the optimal norm; $v \in L^2((0,T)\times\Omega)$ is called an admissible control if $y(T; y_0, v) \in B_r(0)$; v^* is called an optimal control if $y(T; y_0, v^*) \in B_r(0)$ and $\|v^*\|_{L^2((0,T)\times\Omega)} = \mathcal{N}(T)$.*

(ii) In the problem $(\mathcal{NP})_\delta^{k\delta}$, $\mathcal{N}_\delta(k\delta)$ is called the optimal norm; $v_\delta \in L^2_\delta((0,k\delta)\times\Omega)$ is called an admissible control if $y(k\delta; y_0, v_\delta) \in B_r(0)$; and v_δ^* is called an optimal control if $y(k\delta; y_0, v_\delta^*) \in B_r(0)$ and $\|v_\delta^*\|_{L^2_\delta((0,k\delta)\times\Omega)} = \mathcal{N}_\delta(k\delta)$.

We mention that both $(\mathcal{NP})^T$ and $(\mathcal{NP})_\delta^{k\delta}$ have unique nonzero solutions (see Theorems 4.2-4.3). Inspired by [9], we study the above two minimal norm control problems by two minimization problems. The first one corresponds to $(\mathcal{NP})^T$ and reads

$$(JP)^T:\ V(T) \triangleq \inf_{z\in L^2(\Omega)} J^T(z) \triangleq \inf_{z\in L^2(\Omega)} \Big[\frac{1}{2}\|\chi_\omega \varphi(\cdot; T, z)\|^2_{L^2((0,T)\times\Omega)} + \langle y_0, \varphi(0; T, z)\rangle + r\|z\|\Big], \tag{17}$$

where $\varphi(\cdot; T, z)$ is the solution to the adjoint heat equation:

$$\begin{cases} \partial_t\varphi + \Delta\varphi = 0 & \text{in } [0,T)\times\Omega, \\ \varphi = 0 & \text{on } [0,T)\times\partial\Omega, \\ \varphi(T) = z \in L^2(\Omega). \end{cases} \tag{18}$$

(Throughout this paper, we treat $\varphi(\cdot; T, z)$ as a function from $[0,T]$ to $L^2(\Omega)$.) The second minimization problem corresponds to $(\mathcal{NP})_\delta^{k\delta}$ and is as:

$$(JP)_\delta^{k\delta}:\ V_\delta(k\delta) \triangleq \inf_{z\in L^2(\Omega)} J_\delta^{k\delta}(z) \triangleq \inf_{z\in L^2(\Omega)} \Big[\frac{1}{2}\|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2((0,k\delta)\times\Omega)} + \langle y_0, \varphi(0; k\delta, z)\rangle + r\|z\|\Big], \tag{19}$$

where $\overline{\varphi}_\delta(\cdot; k\delta, z)$ is defined by

$$\overline{\varphi}_\delta(t; k\delta, z) \triangleq \sum_{i=1}^{k} \chi_{((i-1)\delta, i\delta]}(t)\frac{1}{\delta}\int_{(i-1)\delta}^{i\delta} \varphi(s; k\delta, z)\,\mathrm{d}s \ \text{ for each } t \in (0, k\delta]. \tag{20}$$

We mention that both $(JP)^T$ and $(JP)_\delta^{k\delta}$ have unique nonzero minimizers in $L^2(\Omega)$ (see Theorems 4.2-4.3).

We prove Theorem 1.3 by the following steps:

(a) Build up connections between $(\mathcal{TP})_\delta^M$ and $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ (see (iii) of Theorem 3.1); and connections between $(\mathcal{TP})^M$ and $(\mathcal{NP})^{\mathcal{T}(M)}$ (see Theorem 4.1).

(b) Obtain the lower and upper bounds of the error $\mathcal{N}(T_1) - \mathcal{N}(T_2)$ for two different time points T_1, T_2 (see Theorem 5.2).

(c) Compute the error estimate $|\mathcal{N}(k\delta) - \mathcal{N}_\delta(k\delta)|$ (see Theorem 5.3).

(d) Get (i) of Theorem 1.3, with the aid of the above (a)-(c).

(e) By using the above (a)-(c) again, we build up sets $\mathcal{A}_{M,\eta}$ and obtain related properties in Theorem 5.4, which leads to (ii) of Theorem 1.3.

The steps to prove Theorem 1.4 and Theorem 1.5 are as follows:

(1) Build up connections between $(\mathcal{TP})_\delta^M$ and $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ (see (iii) of Theorem 3.1), and connections between $(\mathcal{TP})^M$ and $(\mathcal{NP})^{\mathcal{T}(M)}$ (see Theorem 4.1).

(2) With the aid of the connections obtained in (1), we can transfer the estimate in (i) of Theorem 1.4 into an estimate between optimal controls of $(\mathcal{NP})^{\mathcal{T}(M)}$ and $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$.

(3) Find connections between $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ (or $(\mathcal{NP})^{\mathcal{T}(M)}$) and $(JP)_\delta^{\mathcal{T}_\delta(M)}$ (or $(JP)^{\mathcal{T}(M)}$) (see Theorem 4.3 and Theorem 4.2, respectively).

(4) Obtain the error estimate between the minimizers of $(JP)^{\mathcal{T}(M)}$ and $(JP)_\delta^{\mathcal{T}_\delta(M)}$.

(5) Using the connections obtained in (3) and the estimate obtained in (4), we get an error estimate between optimal controls of $(\mathcal{NP})^{\mathcal{T}(M)}$ and $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$. This, along with results in (2), leads to the estimate in (i) of Theorem 1.4.

(6) Using connections obtained in (1) and (3), and using Theorem 5.4, we prove the estimate in (ii) of Theorem 1.4.

(7) Obtain the least order of the diameter of the set $\mathcal{O}_{M,\delta}$ (in the space $L^2((0,\mathcal{T}(M))\times\Omega)$), in terms of δ, (see Lemma 6.1). Here,

$$\mathcal{O}_{M,\delta} \triangleq \{u_\delta|_{(0,\mathcal{T}(M))\times\Omega} \ : \ u_\delta \ \text{ is an optimal control to } \ (\mathcal{TP})_\delta^M\}. \tag{21}$$

(8) Derive the estimates in Theorem 1.5, with the aid of Lemma 6.1 and the estimates in Theorem 1.4.

We would like to give the following note:

– The above introduced strategy was used to study other properties of time optimal distributed control problems (see, for instance, [34] and [42]). It could be used to study numerical approximations of time optimal distributed control problems, via discrete time optimal control problems.

The rest of the paper is organized as follows: Section 2 shows a kind of approximate null controllability for the equation (5). Section 3 concerns with the existence and uniqueness of time optimal control problems. Section 4 provides some connections among time optimal control problems, norm optimal control problems and some minimization problems. Section 5 presents several auxiliary estimates. Section 6 proves the main results. Section 7 (Appendix) gives one lemma, which was taken from [36] and presents an equivalence between controllability and observability in an abstract setting. Since [36] has not appeared, we put it and the proof in Appendix.

2. L^2-approximate null controllability with a cost

In this section, we present a kind of approximate null controllability for the sampled-data controlled equation (5). Such controllability will be defined in the next Definition 2.1 and will play a key role in getting some estimates in Section 5.

Definition 2.1 *(i) Let $(\delta,k)\in\mathbb{R}^+\times\mathbb{N}^+$. Equation (5) is said to have the L^2-approximate null controllability with a cost over $[0,k\delta]$, if for any $\varepsilon>0$, there is $C(\varepsilon,\delta,k)>0$ so that for each $z_0\in L^2(\Omega)$, there is $u_\delta^{z_0}\in L^2_\delta((0,k\delta)\times\Omega)$ (see (16)) satisfying that*

$$\frac{1}{C(\varepsilon,\delta,k)}\|u_\delta^{z_0}\|^2_{L^2((0,k\delta)\times\Omega)}+\frac{1}{\varepsilon}\|y(k\delta;z_0,u_\delta^{z_0})\|^2\le\|z_0\|^2. \tag{22}$$

(ii) Equation (5) is said to have the L^2-approximate null controllability with a cost, if it has the L^2-approximate null controllability with a cost over $[0,k\delta]$, for each $(\delta,k)\in\mathbb{R}^+\times\mathbb{N}^+$.

To prove the L^2-approximate null controllability with a cost for Equation (5), we need some preliminaries. For each $f\in L^2(\mathbb{R}^+\times\Omega)$ and $\delta>0$, we let

$$\bar{f}_\delta(t)\triangleq\sum_{i=1}^{\infty}\chi_{((i-1)\delta,i\delta]}(t)\frac{1}{\delta}\int_{(i-1)\delta}^{i\delta}f(s)\,\mathrm{d}s\ \text{ for each }\ t\in\mathbb{R}^+. \tag{23}$$

Lemma 2.2 *For each f, $g\in L^2(\mathbb{R}^+\times\Omega)$ and each $\delta>0$,*

$$\langle\bar{f}_\delta,g\rangle_{L^2(\mathbb{R}^+\times\Omega)}=\langle f,\bar{g}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)}=\langle\bar{f}_\delta,\bar{g}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)}. \tag{24}$$

Proof. Arbitrarily fix $\delta>0$ and f, $g\in L^2(\mathbb{R}^+\times\Omega)$. To prove (24), it suffices to show

$$\langle\bar{f}_\delta,g\rangle_{L^2(\mathbb{R}^+\times\Omega)}=\langle\bar{f}_\delta,\bar{g}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)}. \tag{25}$$

By (23), one can directly check that

$$\langle\bar{f}_\delta,g\rangle_{L^2(\mathbb{R}^+\times\Omega)}=\sum_{i=1}^{\infty}\langle\bar{f}_\delta,g\rangle_{L^2(((i-1)\delta,i\delta)\times\Omega)}=\sum_{i=1}^{\infty}\Big\langle\bar{f}_\delta(i\delta),\int_{(i-1)\delta}^{i\delta}g(t)\,\mathrm{d}t\Big\rangle$$
$$=\sum_{i=1}^{\infty}\big\langle\bar{f}_\delta(i\delta),\bar{g}_\delta(i\delta)\big\rangle\delta=\sum_{i=1}^{\infty}\langle\bar{f}_\delta,\bar{g}_\delta\rangle_{L^2(((i-1)\delta,i\delta)\times\Omega)}=\langle\bar{f}_\delta,\bar{g}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)},$$

which leads to (25). This ends the proof of this lemma. □

The following interpolation inequality plays an important role in the proof of the L^2-approximate null controllability with a cost.

Lemma 2.3 *There exists $C\triangleq C(\Omega,\omega)>0$ so that for each S,T with $0<S<T$ and each $\theta\in(0,1)$,*

$$\|\varphi(0;T,z)\|\le e^{C(1+\frac{1}{\theta(T-S)})}\|z\|^\theta\Big\|\frac{1}{S}\int_0^S\chi_\omega\varphi(t;T,z)\,\mathrm{d}t\Big\|^{1-\theta}\ \text{ for all }\ z\in L^2(\Omega). \tag{26}$$

Proof. Let $0<S<T$. Arbitrarily fix $z\in L^2(\Omega)$. We define a function f^z over Ω by

$$f^z\triangleq\frac{1}{S}\int_0^S e^{\Delta(S-t)}z\,\mathrm{d}t. \tag{27}$$

By [29, (iii) of Theorem 2.1], there is $C\triangleq C(\Omega,\omega)>0$ so that for each $\theta\in(0,1)$,

$$\|e^{\Delta(T-S)}f^z\|\le e^{C(1+\frac{1}{\theta(T-S)})}\|f^z\|^\theta\|\chi_\omega e^{\Delta(T-S)}f^z\|^{1-\theta}. \tag{28}$$

Two facts are given in order: First, it follows from (27) that

$$\|f^z\| \leq \|z\| \text{ and } e^{\Delta(T-S)}f^z = \frac{1}{S}\int_0^S \varphi(t;T,z)\,\mathrm{d}t. \tag{29}$$

Second, write $\{\lambda_j\}_{j=1}^{\infty}$ for the family of all eigenvalues of $-\Delta$ with the zero Dirichlet boundary condition so that $\lambda_1 < \lambda_2 \leq \cdots$. Let $\{e_j\}_{j=1}^{\infty}$ be the family of the corresponding normalized eigenvectors. Let $z = \sum_{j=1}^{\infty} z_j e_j$ for some $\{z_j\}_{j=1}^{\infty} \in l^2$. Then it follows that

$$\frac{1}{S}\int_0^S e^{\Delta(T-t)}z\,\mathrm{d}t = \sum_{j=1}^{\infty}\Big(\frac{1}{S}\int_0^S e^{\lambda_j t}\,\mathrm{d}t\Big)e^{-\lambda_j T}z_j e_j.$$

Since $\frac{1}{S}\int_0^S e^{\lambda_j t}\,\mathrm{d}t \geq 1$ for each $j \in \mathbb{N}^+$, it follows from (27) and the above equality that

$$\|e^{\Delta(T-S)}f^z\| = \Big\|\frac{1}{S}\int_0^S e^{\Delta(T-t)}z\,\mathrm{d}t\Big\| \geq \|e^{\Delta T}z\| = \|\varphi(0;T,z)\|. \tag{30}$$

Finally, the facts (29) and (30), along with (28), lead to (26). This ends the proof.

□

The next Theorem 2.4 contains the main results of this section. The conclusion (iii) in Theorem 2.4 will play an important role in our further studies.

Theorem 2.4 *The following conclusions are true:*

(i) Equation (5) has the L^2-approximate null controllability with a cost if and only if given $\varepsilon > 0$, $\delta > 0$ and $k \in \mathbb{N}^+$, there is $C(\varepsilon,\delta,k) > 0$ (which also depends on Ω and ω) so that

$$\|\varphi(0;k\delta,z)\|^2 \leq C(\varepsilon,\delta,k)\|\chi_\omega\overline{\varphi}_\delta(\cdot;k\delta,z)\|^2_{L^2((0,k\delta)\times\Omega)} + \varepsilon\|z\|^2 \text{ for all } z \in L^2(\Omega), \tag{31}$$

where $\overline{\varphi}_\delta(\cdot;k\delta,z)$ is given by (20).

(ii) Given $\delta > 0$ and $k \geq 2$, Equation (5) has the L^2-approximate null controllability with a cost over $[0,k\delta]$.

(iii) Given $\varepsilon > 0$, $\delta > 0$ and $k \geq 2$, the constants $C(\varepsilon,\delta,k)$ in (31) and (22) can be taken as

$$C(\varepsilon,\delta,k) = e^{C[1+1/(k\delta)]}/\varepsilon \text{ with } C \triangleq C(\Omega,\omega). \tag{32}$$

Proof. We first prove the conclusion (i). Arbitrarily fix $\delta > 0$, $k \in \mathbb{N}^+$ and $\varepsilon > 0$. We will put our problems under the framework of [36, Lemma 5.1] (which is cited as Lemma 7.1 in our appendix) in the following manner: Let $X \triangleq L^2(\Omega)$, $Y \triangleq L^2_\delta((0,k\delta)\times\Omega)$ and $Z \triangleq L^2(\Omega)$. Define operators $\mathcal{R}: Z \to X$ and $\mathcal{O}: Z \to Y$ by

$$\mathcal{R}z \triangleq \varphi(0;k\delta,z) \text{ and } \mathcal{O}z \triangleq \chi_\omega\overline{\varphi}_\delta(\cdot;k\delta,z) \text{ for all } z \in Z.$$

One can directly check that $\mathcal{R}^*: X^* \to Z^*$ and $\mathcal{O}^*: Y^* \to Z^*$ are given respectively by

$$\mathcal{R}^*z_0 = y(k\delta;z_0,0),\ z_0 \in L^2(\Omega);\ \ \mathcal{O}^*u_\delta = y(k\delta;0,u_\delta),\ u_\delta \in L^2_\delta((0,k\delta)\times\Omega).$$

From these, Definition 2.1 and (31), we can apply [36, Lemma 5.1] (see also Lemma 7.1 in Appendix) to get the conclusion (i) of Theorem 2.4.

We next prove the conclusions (ii) and (iii). Arbitrarily fix $\varepsilon > 0$, $\delta > 0$ and $k \geq 2$. By the conclusion (i), we find that it suffices to show (31) with the triplet (ε,δ,k). To this end, we use (26) (where $T = k\delta$,

$S = [k/2]\delta$ and $\theta = 1/2$, with $[k/2]$ the integer so that $k/2 - 1 < [k/2] \le k/2$) to get that for each $z \in L^2(\Omega)$,

$$\|\varphi(0; k\delta, z)\|^2 \le e^{2C(1+\frac{2}{k\delta - [k/2]\delta})} \Big\| \frac{1}{[k/2]\delta} \int_0^{[k/2]\delta} \varphi(t; k\delta, z)\,\mathrm{d}t \Big\|_\omega \|z\|,$$

where C is given by (26). Then by Young's inequality, we find that for each $z \in L^2(\Omega)$,

$$\|\varphi(0; k\delta, z)\|^2 \le \frac{1}{\varepsilon} e^{4C(1+\frac{2}{k\delta - [k/2]\delta})} \frac{1}{([k/2]\delta)^2} \Big\| \int_0^{[k/2]\delta} \varphi(t; k\delta, z)\,\mathrm{d}t \Big\|_\omega^2 + \varepsilon\|z\|^2. \tag{33}$$

Two observations are given in order: First, it follows from (20) that for each $z \in L^2(\Omega)$,

$$\begin{aligned} &\Big\| \int_0^{[k/2]\delta} \varphi(t; k\delta, z)\,\mathrm{d}t \Big\|_\omega \le \sum_{i=1}^{[k/2]} \Big\| \frac{1}{\delta} \int_{(i-1)\delta}^{i\delta} \varphi(t; k\delta, z)\,\mathrm{d}t \Big\|_\omega \delta \\ &= \|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\|_{L^1(0,[k/2]\delta; L^2(\Omega))} \le \sqrt{[k/2]\delta}\,\|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\|_{L^2((0,k\delta)\times\Omega)}; \end{aligned} \tag{34}$$

Second, since

$$k/4 \le [k/2] \le k/2 \ \text{ and } \ 1/([k/2]\delta) \le e^{4+1/k\delta},$$

one can directly check that

$$e^{4C\left(1+\frac{2}{k\delta - [k/2]\delta}\right)} \frac{1}{[k/2]\delta} \le e^{16(C+1)(1+\frac{1}{k\delta})}. \tag{35}$$

Finally, from (33), (34) and (35), we get (31), with $C(\varepsilon, \delta, k)$ given by (32), where $C(\Omega, \omega)$ may differ from that in (35). This proves (ii), as well as (iii).

In summary, we end the proof of Theorem 2.4.

□

3. Existence and uniqueness of optimal controls

In this section, we will prove that for each $M > 0$, $(\mathcal{TP})^M$ has the unique optimal control, while for some (M, δ), $(\mathcal{TP})^M_\delta$ has infinitely many optimal controls. The later may cause difficulties in our studies. Fortunately, we observe that the optimal control with the minimal norm to $(\mathcal{TP})^M_\delta$ (see Definition 1.2) is unique. The first main theorem in this section is stated in the next Theorem 3.1. It deserves mentioning what follows: The conclusion (iii) of Theorem 3.1 should belong to the materials in the next section. The reason that we put it here is that we will use it in the proof of the non-uniqueness of optimal controls to $(\mathcal{TP})^M_\delta$. More precisely, in the proof of Lemma 3.4, we will use it.

Theorem 3.1 *Let $M > 0$. The following conclusions are true:*
(i) The problem $(\mathcal{TP})^M$ has a unique optimal control.
(ii) For each $\delta > 0$, $(\mathcal{TP})^M_\delta$ has a unique optimal control with the minimal norm.

(iii) Let u_δ^* (with $\delta > 0$) be the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. Then $u_\delta^*|_{(0,\mathcal{T}_\delta(M))\times\Omega}$ (the restriction of u_δ^* over $(0,\mathcal{T}_\delta(M))\times\Omega$) is an optimal control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ and the $L^2((0,\mathcal{T}_\delta(M))\times\Omega)$-norm of u_δ^* is $\mathcal{N}_\delta(\mathcal{T}_\delta(M))$.

Proof. Arbitrarily fix $M > 0$. We will prove conclusions (i), (ii) and (iii) one by one.

(i) Because $\lim_{t\to+\infty} y(t; y_0, 0) = 0$ in $L^2(\Omega)$, the null control is an admissible control to $(\mathcal{TP})^M$, which implies that $(\mathcal{TP})^M$ has an admissible control. Then by the standard way as that used in the proof of [10, Lemma 1.1], one can show that $(\mathcal{TP})^M$ has an optimal control.

To show the uniqueness of the optimal control to $(\mathcal{TP})^M$, we first notice that each optimal control u^* to $(\mathcal{TP})^M$ has the property that

$$\|u^*\|_{L^2(0,\mathcal{T}(M))\times\Omega)} = M. \tag{36}$$

(The property (36) can be proved by the same way as that used to show [16, Lemma 4.3].) Next, we notice that if u_1^* and u_2^* are optimal controls to $(\mathcal{TP})^M$, then $(u_1^* + u_2^*)/2$ is also an optimal control to $(\mathcal{TP})^M$. From this, (36) and the parallelogram law in $L^2((0,\mathcal{T}(M))\times\Omega)$, we can easily use the contradiction argument to get the uniqueness. This ends the proof of the conclusion (i).

(ii) Arbitrarily fix $\delta > 0$. We first show that $(\mathcal{TP})_\delta^M$ has an optimal control. Indeed, since the null control is clearly an admissible control to $(\mathcal{TP})_\delta^M$, it follows by the definition of $\mathcal{T}_\delta(M)$ (see (6)) that there exists $\hat{k} \in \mathbb{N}^+$ so that

$$\mathcal{T}_\delta(M) = \hat{k}\delta. \tag{37}$$

Meanwhile, since $y_0 \in L^2(\Omega) \setminus B_r(0)$ (see (1)), by the definition of the infimum in (6), we see that there is $k_0 \in \mathbb{N}^+$ and $u_\delta^0 \in L_\delta^2(\mathbb{R}^+\times\Omega)$ so that

$$\mathcal{T}_\delta(M) \le k_0\delta \le \mathcal{T}_\delta(M) + \delta/2; \tag{38}$$

$$y(k_0\delta; y_0, u_\delta^0) \in B_r(0) \;\text{ and }\; \|u_\delta^0\|_{L^2(\mathbb{R}^+\times\Omega)} \le M. \tag{39}$$

From (37) and (38), we find that $\hat{k}\delta \le k_0\delta \le \hat{k}\delta + \delta/2$, which leads to that $k_0 = \hat{k}$. This, along with (37) and (39), yields that $\mathcal{T}_\delta(M) = k_0\delta \in (0,\infty)$, which, together with (39), implies that u_δ^0 is an optimal control to $(\mathcal{TP})_\delta^M$.

Next, we will prove that $(\mathcal{TP})_\delta^M$ has a unique optimal control with the minimal norm. Indeed, since $L_\delta^2(\mathbb{R}^+\times\Omega)$ is a closed subspace of $L^2(\mathbb{R}^+\times\Omega)$, by Definition 1.2, one can use a standard way (i.e., taking a minimization sequence) to show the existence of the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. To show the uniqueness, we let u_1 and u_2 be two optimal controls with the minimal norm. By Definition 1.2, one can easily check that $(u_1+u_2)/2$ is also an optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. By making use of Definition 1.2 again, we find that

$$\|u_1\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} = \|u_2\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} = \|(u_1+u_2)/2\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}.$$

These, along with the parallelogram law for $L^2((0,\mathcal{T}_\delta(M))\times\Omega)$, yield that

$$(u_1 - u_2)/2 = 0 \;\text{ in }\; L^2((0,\mathcal{T}_\delta(M))\times\Omega), \;\text{ i.e., }\; u_1 = u_2.$$

So $(\mathcal{TP})_\delta^M$ has a unique optimal control with the minimal norm.

(iii) Let u_δ^* be the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$. We will show that $u_\delta^*|_{(0,\mathcal{T}_\delta(M))\times\Omega}$ is an optimal control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$. Indeed, we have that

$$y(\mathcal{T}_\delta(M); y_0, u_\delta^*) \in B_r(0) \ \text{ and } \ \|u_\delta^*\|_{L^2_\delta(\mathbb{R}^+\times\Omega)} \leq M, \tag{40}$$

from which, one can easily check that $u_\delta^*|_{(0,\mathcal{T}_\delta(M))\times\Omega}$ is an admissible control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$. Then by the optimality of $\mathcal{N}_\delta(\mathcal{T}_\delta(M))$ and the second inequality in (40), we see that

$$\mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq \|u_\delta^*|_{(0,\mathcal{T}_\delta(M))\times\Omega}\|_{L^2_\delta((0,\mathcal{T}_\delta(M))\times\Omega)} \leq M < \infty. \tag{41}$$

Meanwhile, since $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ has an admissible control, we can use a standard argument (see for instance the proof of [10, Lemma 1.1]) to show that $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ has an optimal control v_δ^*. Write $\widetilde{v}_\delta^*$ for the zero extension of v_δ^* over $\mathbb{R}^+\times\Omega$. Then we have that

$$y(\mathcal{T}_\delta(M); y_0, \widetilde{v}_\delta^*) \in B_r(0) \ \text{ and } \ \|\widetilde{v}_\delta^*\|_{L^2_\delta(\mathbb{R}^+\times\Omega)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)). \tag{42}$$

From (42) and (41), it follows that $\widetilde{v}_\delta^*$ is an optimal control to $(\mathcal{TP})_\delta^M$. Since u_δ^* is the optimal control with the minimal norm to $(\mathcal{TP})_\delta^M$, we see from (41), (ii) of Definition 1.2 and the second equality in (42) that

$$\begin{aligned}\mathcal{N}_\delta(\mathcal{T}_\delta(M)) &\leq \|u_\delta^*\|_{L^2_\delta((0,\mathcal{T}_\delta(M))\times\Omega)} \\ &\leq \|\widetilde{v}_\delta^*\|_{L^2_\delta((0,\mathcal{T}_\delta(M))\times\Omega)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)).\end{aligned}$$

The above, together with the first conclusion in (40), implies that $u_\delta^*|_{(0,\mathcal{T}_\delta(M))\times\Omega}$ is an optimal control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$ and that

$$\|u_\delta^*\|_{L^2_\delta((0,\mathcal{T}_\delta(M))\times\Omega)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)).$$

In summary, we end the proof of Theorem 3.1.

□

The next theorem concerns with the non-uniqueness of optimal controls to $(\mathcal{TP})_\delta^M$.

Theorem 3.2 *There are sequences $\{M_n\}$ dense in $\mathbb{R}^+$ and $\{\delta_n\} \subset \mathbb{R}^+$, with $\lim_{n\to\infty}\delta_n = 0$, so that for each n, the problem $(\mathcal{TP})_{\delta_n}^{M_n}$ has infinitely many different optimal controls.*

To prove Theorem 3.2, we need two lemmas.

Lemma 3.3 *For each $(M,\delta) \in \mathbb{R}^+\times\mathbb{R}^+$ with $2\delta \leq \mathcal{T}_\delta(M) < \infty$, it stands that*

$$\mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq M < \mathcal{N}_\delta(\mathcal{T}_\delta(M) - \delta). \tag{43}$$

Proof. Let $(M,\delta) \in \mathbb{R}^+\times\mathbb{R}^+$ so that $2\delta \leq \mathcal{T}_\delta(M) < \infty$. Then by (6), we see that

$$\mathcal{T}_\delta(M) = \hat{k}\delta \ \text{ for some integer } \ \hat{k} \geq 2. \tag{44}$$

Thus, (43) is equivalent to the following inequality:

$$\mathcal{N}_\delta(\hat{k}\delta) \leq M < \mathcal{N}_\delta((\hat{k}-1)\delta). \tag{45}$$

To prove (45), we let u_δ^1 be an optimal control to $(\mathcal{TP})_\delta^M$. Then we have that

$$\|u_\delta^1\|_{L^2_\delta(\mathbb{R}^+\times\Omega)} \leq M \ \text{ and } \ y(\mathcal{T}_\delta(M); y_0, u_\delta^1) \in B_r(0). \tag{46}$$

According to (46) and (44), $u_\delta^1|_{(0,\hat{k}\delta)\times\Omega}$ is an admissible control to $(\mathcal{NP})_\delta^{\hat{k}\delta}$. Then by the optimality of $\mathcal{N}_\delta(\hat{k}\delta)$ and the first inequality in (46), we get that

$$\mathcal{N}_\delta(\hat{k}\delta) \leq \|u_\delta^1\|_{L^2_\delta((0,\hat{k}\delta)\times\Omega)} \leq M,$$

which leads to the first inequality in (45).

We now show the second inequality in (45). By contradiction, we suppose that

$$\mathcal{N}_\delta((\hat{k}-1)\delta) \leq M. \tag{47}$$

Then we would obtain from (47) that $(\mathcal{NP})_\delta^{(\hat{k}-1)\delta}$ has an admissible control, since $M < \infty$. Thus, by a standard way (see for instance the proof of [10, Lemma 1.1]), one can prove that $(\mathcal{NP})_\delta^{(\hat{k}-1)\delta}$ has an optimal control v_δ^1. Hence,

$$\|v_\delta^1\|_{L^2_\delta((0,(\hat{k}-1)\delta)\times\Omega)} = \mathcal{N}_\delta((\hat{k}-1)\delta) \ \text{ and } \ y((\hat{k}-1)\delta; y_0, v_\delta^1) \in B_r(0). \tag{48}$$

Write $\widetilde{v}_\delta^1$ for the zero extension of v_δ^1 over $\mathbb{R}^+\times\Omega$. From (48) and (47), we find that $\widetilde{v}_\delta^1$ is an admissible control to $(\mathcal{TP})_\delta^M$. Then by the optimality of $\mathcal{T}_\delta(M)$, we get that $\mathcal{T}_\delta(M) \leq (\hat{k}-1)\delta$, which contradicts (44). Thus, the second inequality in (45) is true. We end the proof of this lemma.

□

Lemma 3.4 *For each $M > 0$ and $N > 0$, there exists an integer $n \geq N$ so that $2/2^n \leq \mathcal{T}_{1/2^n}(M) < \infty$.*

Proof. It is clear that $\mathcal{T}_{1/2^n}(M) < \infty$ for all $M > 0$ and $n \in \mathbb{N}^+$. Thus, we only need to show that for any $M > 0$ and $N > 0$, $2/2^n \leq \mathcal{T}_{1/2^n}(M)$ for some $n \geq N$. By contradiction, suppose that it were not true. Then there would be $M > 0$ and $N > 0$ so that

$$\mathcal{T}_{1/2^n}(M) < 2/2^n \ \text{ for all } \ n \geq N. \tag{49}$$

Let u_n^*, with $n \geq N$, be an optimal control to $(\mathcal{TP})_{1/2^n}^M$ (see (ii) of Theorem 3.1). Then we have that

$$y(\mathcal{T}_{1/2^n}(M); y_0, u_n^*) \in B_r(0) \ \text{ and } \ \|u_n^*\|_{L^2(\mathbb{R}^+\times\Omega)} \leq M \ \text{ for all } \ n \geq N. \tag{50}$$

By the last inequality in (50), Hölder's inequality and (49), we can easily check that

$$\int_0^{\mathcal{T}_{1/2^n}(M)} e^{\Delta(\mathcal{T}_{1/2^n}(M)-t)}\chi_\omega u_n^*(t,\cdot)\mathrm{d}t \to 0, \ \text{ as } \ n\to\infty.$$

This, along with (49) and the first conclusion in (50), yields that

$$y_0 = \lim_{n\to\infty} y(\mathcal{T}_{1/2^n}(M); y_0, u_n^*) \in B_r(0),$$

which contradicts the assumption that $y_0 \in L^2(\Omega)\setminus B_r(0)$ (see (1)). This ends the proof.

□

We are now on the position to prove Theorem 3.2.

Proof of Theorem 3.2. Choose a sequence $\{M_n\}_{n=1}^\infty$ dense in $\mathbb{R}^+$ so that

$$\{M_n\}_{n=1}^\infty \subset \mathbb{R}^+ \setminus \{\mathcal{N}_{1/2^k}(j/2^k) \ : \ k, j \in \mathbb{N}^+\}. \tag{51}$$

By Lemma 3.4, there exists an increasing subsequence $\{k_n\}_{n=1}^\infty$ (in $\mathbb{N}^+$), with $\lim_{n\to\infty} k_n = \infty$, so that

$$2/2^{k_n} \leq \mathcal{T}_{1/2^{k_n}}(M_n) < \infty \ \text{ for each } \ n \in \mathbb{N}^+. \tag{52}$$

Write $\delta_n \triangleq 1/2^{k_n}$, $n \in \mathbb{N}^+$. Then, by (52), we can apply Lemma 3.3 to get that

$$\mathcal{N}_{\delta_n}(\mathcal{T}_{\delta_n}(M_n)) \leq M_n < \mathcal{N}_{\delta_n}(\mathcal{T}_{\delta_n}(M_n) - \delta_n).$$

This, along with (51), yields that

$$\mathcal{N}_{\delta_n}(\mathcal{T}_{\delta_n}(M_n)) < M_n < \mathcal{N}_{\delta_n}(\mathcal{T}_{\delta_n}(M_n) - \delta_n). \tag{53}$$

The key to show Theorem 3.2 is to claim that for each $n \in \mathbb{N}^+$, $(\mathcal{TP})^{M_n}_{\delta_n}$ has at least two different optimal controls. By contradiction, we suppose that for some $n_0 \in \mathbb{N}^+$, $(\mathcal{TP})^{M_{n_0}}_{\delta_{n_0}}$ had a unique optimal control. To get a contradiction, we define two convex subsets in $L^2(\Omega)$ as follows:

$$A_{n_0} \triangleq \left\{ y(\mathcal{T}_{\delta_{n_0}}(M_{n_0}); y_0, u_{\delta_{n_0}}) \ : \ \|u_{\delta_{n_0}}\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} \leq \mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}}(M_{n_0})) \right\},$$
$$B_{n_0} \triangleq \left\{ y(\mathcal{T}_{\delta_{n_0}}(M_{n_0}); 0, v_{\delta_{n_0}}) \ : \ \|v_{\delta_{n_0}}\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} \leq M_{n_0} - \mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}}(M_{n_0})) \right\}.$$

We first show that

$$A_{n_0} \cap B_r(0) = \{\hat{\eta}\} \ \text{ for some } \ \hat{\eta} \in L^2(\Omega), \tag{54}$$

i.e., $A_{n_0} \cap B_r(0)$ contains only one element. In fact, by (ii) and (iii) of Theorem 3.1, the optimal control with the minimal norm $u^*_{\delta_{n_0}}$ to $(\mathcal{TP})^{M_{n_0}}_{\delta_{n_0}}$ satisfies that

$$y(\mathcal{T}_{\delta_{n_0}}(M_{n_0}); y_0, u^*_{\delta_{n_0}}) \in B_r(0) \ \text{ and } \ \|u^*_{\delta_{n_0}}\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} = \mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}}(M_{n_0})).$$

These imply that $A_{n_0} \cap B_r(0) \neq \emptyset$. We next show that $A_{n_0} \cap B_r(0)$ contains only one element. Suppose, by contradiction, that it contained two different elements y_1 and y_2. Then by the definition of A_{n_0}, there would be two different controls u_1 and u_2 so that

$$y_1 = y(\mathcal{T}_{\delta_{n_0}}(M_{n_0}); y_0, u_1), \ \ \|u_1\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} \leq \mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}}(M_{n_0})); \tag{55}$$

$$y_2 = y(\mathcal{T}_{\delta_{n_0}}(M_{n_0}); y_0, u_2), \ \ \|u_2\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} \leq \mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}}(M_{n_0})). \tag{56}$$

Since $y_1, y_2 \in B_r(0)$, we have that $(y_1 + y_2)/2 \in B_r(0)$. From this (55) and (56), one can easily check that

$$(y_1 + y_2)/2 \in A_{n_0} \cap B_r(0). \tag{57}$$

Meanwhile, since $u_1 \neq u_2$, by the second inequality in (55) and the second inequality in (56), using the parallelogram law, we find that

$$\|(u_1 + u_2)/2\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} < \mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}}(M_{n_0})), \tag{58}$$

which, together with (53), indicates that

$$\|(u_1 + u_2)/2\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} < M_{n_0}.$$

From this and (57), we see that $(u_1 + u_2)/2$ is an optimal control to $(\mathcal{TP})^{M_{n_0}}_{\delta_{n_0}}$. This, along with Definition 1.2 and the conclusions (ii) and (iii) of Theorem 3.1, yields that

$$\|(u_1 + u_2)/2\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)} \geq \mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}}(M_{n_0})),$$

which contradicts (58). Hence, (54) is true.

Next, by the definitions of A_{n_0} and B_{n_0}, one can easily check that each element of $(A_{n_0}+B_{n_0})\cap B_r(0)$ can be expressed as:

$$y(\mathcal{T}_{\delta_{n_0}}(M_{n_0});y_0,u^*_{n_0}) \ \text{ with } \ u^*_{n_0} \ \text{ an optimal control to } \ (\mathcal{TP})^{M_{n_0}}_{\delta_{n_0}}. \tag{59}$$

Since it was assumed that $(\mathcal{TP})^{M_{n_0}}_{\delta_{n_0}}$ had a unique optimal control, it follows from (59) that $(A_{n_0}+B_{n_0})\cap B_r(0)$ contains only one element. This, along with (54), yields that

$$(A_{n_0}+B_{n_0})\cap B_r(0)=A_{n_0}\cap B_r(0)=\{\hat{\eta}\} \ \text{ for some } \ \hat{\eta}\in L^2(\Omega). \tag{60}$$

By (60), we can apply the Hahn-Banach separation theorem to find $\eta^*\in L^2(\Omega)$, with $\|\eta^*\|=r>0$, so that

$$\sup_{w\in A_{n_0}+B_{n_0}}\langle w,\eta^*\rangle\le\inf_{z\in B_r(0)}\langle z,\eta^*\rangle.$$

This, along with (60), yields that

$$\sup_{w\in\hat{\eta}+B_{n_0}}\langle w,\eta^*\rangle\le\langle\hat{\eta},\eta^*\rangle, \ \text{ i.e., } \ \sup_{w\in B_{n_0}}\langle w,\eta^*\rangle\le 0. \tag{61}$$

From now on and throughout the proof of Theorem 3.2, we simply write $\mathcal{T}_{\delta_{n_0}}$ for $\mathcal{T}_{\delta_{n_0}}(M_{n_0})$; simply write $\varphi(\cdot)$ and $\overline{\varphi}_{\delta_{n_0}}(\cdot)$ for $\varphi(\cdot;\mathcal{T}_{\delta_{n_0}},\eta^*)$ (see (18)) and $\overline{\varphi}_{\delta_{n_0}}(\cdot;\mathcal{T}_{\delta_{n_0}},\eta^*)$ (see (20)), respectively.

Arbitrarily fix $u_{\delta_{n_0}}\in L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)$. Three facts are given in order. Fact one: Since $M_{n_0}>\mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}})$ (see (53)), it follows from the definition of B_{n_0} that

$$y(\mathcal{T}_{\delta_{n_0}};0,u_{\delta_{n_0}})\in\lambda B_{n_0}, \ \text{ with } \ \lambda=\frac{\|u_{\delta_{n_0}}\|_{L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)}}{M_{n_0}-\mathcal{N}_{\delta_{n_0}}(\mathcal{T}_{\delta_{n_0}})}.$$

This, along with (61), yields that

$$\langle y(\mathcal{T}_{\delta_{n_0}};0,u_{\delta_{n_0}}),\eta^*\rangle\le 0. \tag{62}$$

Fact two: One can directly check that

$$\langle u_{\delta_{n_0}},\chi_\omega\varphi\rangle_{L^2((0,\mathcal{T}_{\delta_{n_0}})\times\Omega)}=\langle y(\mathcal{T}_{\delta_{n_0}};0,u_{\delta_{n_0}}),\eta^*\rangle. \tag{63}$$

Fact three: we have that

$$\langle u_{\delta_{n_0}},\chi_\omega\varphi\rangle_{L^2((0,\mathcal{T}_{\delta_{n_0}})\times\Omega)}=\langle u_{\delta_{n_0}},\chi_\omega\overline{\varphi}_{\delta_{n_0}}\rangle_{L^2((0,\mathcal{T}_{\delta_{n_0}})\times\Omega)}. \tag{64}$$

The proof of (64) is as follows: Let $f=u_{\delta_{n_0}}$ and let g be the zero extension of $\chi_\omega\varphi$ over $\mathbb{R}^+$. Since $u_{\delta_{n_0}}\in L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)$, it follows by (4), (23) and (20) that $\bar{f}_{\delta_{n_0}}=u_{\delta_{n_0}}$ and $\bar{g}_{\delta_{n_0}}=\overline{\varphi}_{\delta_{n_0}}$ (where $\overline{\varphi}_{\delta_{n_0}}$ is treated as its zero extension over $\mathbb{R}^+$). Then, by Lemma 2.2, we obtain (64).

Now, from facts (62), (63) and (64), we see that

$$\langle u_{\delta_{n_0}},\chi_\omega\overline{\varphi}_{\delta_{n_0}}\rangle_{L^2((0,\mathcal{T}_{\delta_{n_0}})\times\Omega)}\le 0.$$

Since $u_{\delta_{n_0}}$ was arbitrarily taken from $L^2_{\delta_{n_0}}(\mathbb{R}^+\times\Omega)$, the above inequality implies that

$$\chi_\omega \overline{\varphi}_{\delta_{n_0}}(t) = 0 \ \text{ in } \ L^2(\Omega), \ \text{ a.e. } \ t \in (0, \mathcal{T}_{\delta_{n_0}}). \tag{65}$$

Since $\mathcal{T}_{\delta_{n_0}} \geq 2\delta_{n_0}$ (see (52)), we apply (65) and Lemma 2.3 (where $T = \mathcal{T}_{\delta_{n_0}}$ and $S = \delta_{n_0}$) to get that $\varphi(0) = 0$ in $L^2(\Omega)$. Then from the backward uniqueness property for the heat equation (see, for instance, [26]), we deduce that $\eta^* = 0$. This leads to a contradiction. Hence, we ends the proof of the key claim: For each $n \in \mathbb{N}^+$, $(\mathcal{TP})_{\delta_n}^{M_n}$ has at least two different optimal controls.

Finally, we observe that any convex combination of optimal controls to $(\mathcal{TP})_{\delta_n}^{M_n}$ (with $n \in \mathbb{N}^+$) is still an optimal control to $(\mathcal{TP})_{\delta_n}^{M_n}$. Therefore, for each $n \in \mathbb{N}^+$, $(\mathcal{TP})_{\delta_n}^{M_n}$ has infinitely many different optimal controls. This ends the proof of Theorem 3.2.

□

4. Connections among different problems

This section presents connections among $(\mathcal{TP})_\delta^M$, $(\mathcal{NP})_\delta^{k\delta}$ and $(JP)_\delta^{k\delta}$ (and among $(\mathcal{TP})^M$, $(\mathcal{NP})^T$ and $(JP)^T$). We define that

$$T^* \triangleq \sup\{t > 0 \ : \ e^{\Delta t} y_0 \notin B_r(0)\}; \tag{66}$$

$$\mathcal{P}_{T^*} \triangleq \{(\delta, k) \ : \ \delta > 0, \ k \in \mathbb{N}^+ \ \text{ s.t. } \ 2\delta \leq k\delta < T^*\}. \tag{67}$$

We mention that $0 < T^* < \infty$ because of (1) (since the semigroup $\{e^{\Delta t}\}_{t \geq 0}$ has the exponential decay).

4.1. *Connections between time optimal control problems and norm optimal control problems*

We first present the following equivalence theorem. We will omit its proof, because it can be proved by the same way as one of proofs of [34, Proposition 4.1], [43, Proposition 3.1] and [42, Theorem 1.1 and Theorem 2.1].

Theorem 4.1 *Let T^* be given by (66). Then the following conclusions are true:*

(i) The function $T \to \mathcal{N}(T)$ is strictly decreasing and continuous from $(0, T^*)$ onto $(0, +\infty)$. Moreover, $\lim_{T \to T^*-} \mathcal{N}(T) = 0$.

(ii) When $M > 0$ and $T \in (0, T^*)$, $\mathcal{N}(\mathcal{T}(M)) = M$ and $\mathcal{T}(\mathcal{N}(T)) = T$.

(iii) The function $M \to \mathcal{T}(M)$ is strictly decreasing and continuous from $(0, +\infty)$ onto $(0, T^*)$.

(iv) For each $M > 0$, the optimal control to $(\mathcal{TP})^M$, when restricted on $(0, \mathcal{T}(M)) \times \Omega$, is the optimal control to $(\mathcal{NP})^{\mathcal{T}(M)}$. For each $T \in (0, T^*)$, the zero extension of the optimal control to $(\mathcal{NP})^{\mathcal{T}(M)}$ is the optimal control to $(\mathcal{TP})^M$.

We next recall (iii) of Theorem 3.1 for the connections between $(\mathcal{TP})_\delta^M$ and $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$.

4.2. *Connections between norm optimal control problems and the minimization problems*

The first theorem of this subsection concerns with connections between problems $(\mathcal{NP})^T$ and $(JP)^T$ (given by (17)). Its proof can be done by the same methods as those in the proofs of of Lemma 3.5 and Proposition 3.6 in [34]. We omit it here.

Theorem 4.2 *Let $T \in (0, T^*)$ with T^* given by (66). Then the following conclusions are true:*

(i) The problem $(JP)^T$ has a unique nonzero minimizer z^* in $L^2(\Omega)$.

(ii) Problem $(\mathcal{NP})^T$ has a unique optimal control v^* (treated as a function from $(0, T)$ to $L^2(\Omega)$), which satisfies that

$$v^*(t) = \chi_\omega \varphi(t; T, z^*) \ \text{ in } \ L^2(\Omega), \quad a.e. \ t \in (0, T), \tag{68}$$

and that

$$y(T; y_0, v^*) = -rz^*/\|z^*\|. \tag{69}$$

(iii) It holds that $V(T) = -\frac{1}{2}\mathcal{N}(T)^2 = -\frac{1}{2}\|\chi_\omega \varphi(\cdot; T, z^*)\|^2_{L^2((0,T)\times\Omega)}$.

The next theorem deals with connections between $(\mathcal{NP})^{k\delta}_\delta$ (given by (15)) and $(JP)^{k\delta}_\delta$ (given by (19)). Recall (20) for the definition of $\overline{\varphi}(\cdot; k\delta, z)$.

Theorem 4.3 *Let $(\delta, k) \in \mathcal{P}_{T^*}$ (given by (67)). Then the following conclusions are true:*

(i) The problem $(JP)^{k\delta}_\delta$ has a unique minimizer z^*_δ in $L^2(\Omega)$. Moreover, $z^*_\delta \neq 0$ and

$$\chi_\omega \overline{\varphi}_\delta(t; k\delta, z^*_\delta) \neq 0 \ \text{ for all } \ t \in \big(0, (k-1)\delta\big]. \tag{70}$$

(ii) Problem $(\mathcal{NP})^{k\delta}_\delta$ has a unique optimal control v^*_δ (treated as a piece-wise constant function from $(0, k\delta]$ to $L^2(\Omega)$), which verifies that

$$v^*_\delta(t) = \chi_\omega \overline{\varphi}_\delta(t; k\delta, z^*_\delta) \ \text{ in } \ L^2(\Omega), \quad a.e. \ t \in (0, k\delta], \tag{71}$$

(where z^*_δ is the minimizer of $(JP)^{k\delta}_\delta$) and that

$$y(k\delta; y_0, v^*_\delta) = -rz^*_\delta/\|z^*_\delta\|. \tag{72}$$

(iii) $V_\delta(k\delta) = -\frac{1}{2}\mathcal{N}_\delta(k\delta)^2 = -\frac{1}{2}\|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z^*_\delta)\|^2_{L^2((0,k\delta)\times\Omega)}$.

Proof. (i) First of all, we show the existence of minimizers of $(JP)^{k\delta}_\delta$. Indeed, by (19), one can easily see that $J^{k\delta}_\delta$ is continuous and convex over $L^2(\Omega)$. We now show its coercivity. Since $(\delta, k) \in \mathcal{P}_{T^*}$ (given by (67)), we have that $k \geq 2$. Thus, we can apply Theorem 2.4 to see that both (31) and (32) are true. By taking $\varepsilon = \big(\frac{r}{2\|y_0\|}\big)^2$ in (31), we find that for each $z \in L^2(\Omega)$,

$$\begin{aligned}
\|\varphi(0; k\delta, z)\|^2 &\leq e^{C(1+\frac{1}{k\delta})}\Big(\frac{2\|y_0\|}{r}\Big)^2 \|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2((0,k\delta)\times\Omega)} + \Big(\frac{r}{2\|y_0\|}\Big)^2 \|z\|^2 \\
&\leq \Big(e^{\frac{C}{2}(1+\frac{1}{k\delta})} \frac{2\|y_0\|}{r} \|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\|_{L^2((0,k\delta)\times\Omega)} + \frac{r}{2\|y_0\|}\|z\|\Big)^2,
\end{aligned}$$

where $C \triangleq C(\Omega, \omega)$ is given by (32). The above, along with the Cauchy-Schwarz inequality, yields that for each $z \in L^2(\Omega)$,

$$\langle y_0, \varphi(0; k\delta, z)\rangle \geq -\big(2e^{\frac{C}{2}(1+\frac{1}{k\delta})}\|y_0\|^2 r^{-1}\big)\|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\|_{L^2((0,k\delta)\times\Omega)} - \frac{r}{2}\|z\|.$$

From this and (19), one can easily check that

$$J^{k\delta}_\delta(z) \geq \frac{r}{2}\|z\| - 2e^{C(1+\frac{1}{k\delta})}\|y_0\|^4 r^{-2} \ \text{ for each } \ z \in L^2(\Omega), \tag{73}$$

which leads to the coercivity of $J^{k\delta}_\delta$ over $L^2(\Omega)$. Hence, $J^{k\delta}_\delta$ has at least one minimizer in $L^2(\Omega)$.

Next, we claim that 0 is not a minimizer of $J_\delta^{k\delta}$. By contradiction, suppose that it were not true. Then we would find from (19) that for all $z \in L^2(\Omega)$ and $\varepsilon > 0$,

$$0 \le \frac{J_\delta^{k\delta}(\varepsilon z) - J_\delta^{k\delta}(0)}{\varepsilon} = \frac{\varepsilon}{2}\|\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\|^2_{L^2((0,k\delta)\times\Omega)} + \langle y_0, \varphi(0; k\delta, z)\rangle + r\|z\|.$$

Sending ε to 0 in the above leads to that

$$\langle e^{\Delta k\delta} y_0, z\rangle + r\|z\| = \langle y_0, \varphi(0; k\delta, z)\rangle + r\|z\| \ge 0 \text{ for all } z \in L^2(\Omega).$$

This yields that

$$\|e^{\Delta k\delta} y_0\| = \sup_{z\in L^2(\Omega)\setminus\{0\}} \langle e^{\Delta k\delta} y_0, z\rangle / \|z\| \le r.$$

Since $y_0 \in L^2(\Omega) \setminus B_r(0)$ (see (1)), the above, along with (66), indicates that $k\delta \ge T^*$, which contradicts the assumption that $(\delta, k) \in \mathcal{P}_{T^*}$ (given by (67)). Thus, 0 is not a minimizer of $J_\delta^{k\delta}$.

We now show the uniqueness of the minimizer of $J_\delta^{k\delta}$. To this end, we claim that the first term on the right hand side of (19) is strictly convex. When this claim is proved, it follows from (19) that $J_\delta^{k\delta}$ is strictly convex over $L^2(\Omega)$. So its minimizer is unique.

To show the above claim, we first observe from (20) that

$$\overline{\varphi}_\delta(t; k\delta, \lambda z_1 + \mu z_2) = \lambda\overline{\varphi}_\delta(t; k\delta, z_1) + \mu\overline{\varphi}_\delta(t; k\delta, z_2) \text{ for all } \lambda, \mu \in \mathbb{R}. \tag{74}$$

By this, we see that the first term on the right hand side of (19) is convex. Next, we suppose, by contradiction, that this term were not strictly convex. Then, by the convexity of this term, there would be $\hat\lambda \in (0, 1)$ and z_1, $z_2 \in L^2(\Omega)$, with $z_1 \ne z_2$, so that

$$\int_0^{k\delta} \|\chi_\omega\overline{\varphi}_\delta(t; k\delta, \hat\lambda z_1 + (1-\hat\lambda)z_2)\|^2 \,\mathrm{d}t$$
$$= \hat\lambda \int_0^{k\delta} \|\chi_\omega\overline{\varphi}_\delta(t; k\delta, z_1)\|^2 \,\mathrm{d}t + (1-\hat\lambda)\int_0^{k\delta} \|\chi_\omega\overline{\varphi}_\delta(t; k\delta, z_2)\|^2 \,\mathrm{d}t,$$

which, along with (74), yields that for each $t \in (0, k\delta)$,

$$\|\hat\lambda\chi_\omega\overline{\varphi}_\delta(t; k\delta, z_1) + (1-\hat\lambda)\chi_\omega\overline{\varphi}_\delta(t; k\delta, z_1)\|^2 = \hat\lambda\|\chi_\omega\overline{\varphi}_\delta(t; k\delta, z_1)\|^2 + (1-\hat\lambda)\|\chi_\omega\overline{\varphi}_\delta(t; k\delta, z_2)\|^2.$$

From this and the strict convexity of $\|\cdot\|^2$, we see that for each $t \in (0, k\delta)$,

$$\chi_\omega\overline{\varphi}_\delta(t; k\delta, z_1) = \chi_\omega\overline{\varphi}_\delta(t; k\delta, z_2), \text{ i.e., } \chi_\omega\overline{\varphi}_\delta(t; k\delta, z_1 - z_2) = 0. \tag{75}$$

Notice that $k \ge 2$. Thus, we can apply Lemma 2.3 (where $S = (k-1)\delta$, $T = k\delta$ and $z = z_1 - z_2$), and use (75) to obtain that $\varphi(0; k\delta, z_1 - z_2) = 0$. This, together with the backward uniqueness of the heat equation, yields that $z_1 = z_2$ in $L^2(\Omega)$, which leads to a contradiction. Hence, the first term on the right hand side of (19) is strictly convex.

In summary, conclude that $J_\delta^{k\delta}$ has a unique minimizer $z_\delta^* \ne 0$.

Finally, we prove that the minimizer z_δ^* satisfies (70). By contradiction, suppose that it were not true. Then we would have that

$$\chi_\omega \overline{\varphi}_\delta(t_0; k\delta, z_\delta^*) = 0 \text{ for some } t_0 \in \big(0, (k-1)\delta\big]. \tag{76}$$

Since $\overline{\varphi}_\delta(\cdot; k\delta, z_\delta^*)$ is a piece-wise constant function from $(0, k\delta]$ to $L^2(\Omega)$ (see (20)), it follows from (76) that

$$\chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z_\delta^*) = 0 \text{ over } \big((i_0-1)\delta, i_0\delta\big] \text{ for some } i_0 \in \{1, \cdots, k-1\}. \tag{77}$$

By (77), we can apply Lemma 2.3 (where $T = (k+1-i_0)\delta$, $S = \delta$ and $z = e^{\Delta(i_0-1)\delta)} z_\delta^*$) to get that

$$0 = \varphi(0; (k+1-i_0)\delta, e^{\Delta(i_0-1)\delta)} z_\delta^*) = \varphi((i_0-1)\delta; k\delta, z_\delta^*).$$

This, along with the backward uniqueness for the heat equation, yields that $z_\delta^* = 0$ in $L^2(\Omega)$, which leads to a contradiction. Therefore, (70) holds. This ends the proof of the conclusion (i) of Theorem 4.3.

(ii) Let z_δ^* be the minimizer of $J_\delta^{k\delta}$. Let v_δ^* be given by (71). It suffices to show that v_δ^* is the unique optimal control to $(\mathcal{NP})_\delta^{k\delta}$ and satisfies (72). *From now on and throughout the proof of Theorem 4.3, we simply write $\varphi(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ for $\varphi(\cdot; k\delta, z_\delta^*)$ and $\overline{\varphi}_\delta(\cdot; k\delta, z_\delta^*)$.*

We first show that v_δ^* is an admissible control to $(\mathcal{NP})_\delta^{k\delta}$ and satisfies (72). By (19), one can easily check that the Euler-Lagrange equation associated with the minimizer z_δ^* is as follows:

$$\langle \chi_\omega \overline{\varphi}_\delta(\cdot), \overline{\varphi}_\delta(\cdot; k\delta, z)\rangle_{L^2((0,k\delta)\times\Omega)} + \langle y_0, e^{\Delta k\delta} z\rangle + \langle r\frac{z_\delta^*}{\|z_\delta^*\|}, z\rangle = 0, \ \forall\, z \in L^2(\Omega). \tag{78}$$

We claim that for each $z \in L^2(\Omega)$,

$$\langle \chi_\omega \overline{\varphi}_\delta(\cdot), \chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\rangle_{L^2((0,k\delta)\times\Omega)} = \langle \chi_\omega \overline{\varphi}_\delta(\cdot), \chi_\omega \varphi(\cdot; k\delta, z)\rangle_{L^2((0,k\delta)\times\Omega)}. \tag{79}$$

To this end, we arbitrarily fix $z \in L^2(\Omega)$. Let $f(\cdot)$ and $g(\cdot)$ be the zero extensions of $\chi_\omega\varphi(\cdot)$ and $\chi_\omega\varphi(\cdot; k\delta, z)$ over $\mathbb{R}^+$. Then by (23) and (20), we see that

$$\bar{f}_\delta(\cdot) = \chi_\omega \overline{\varphi}_\delta(\cdot) \text{ and } \bar{g}_\delta(\cdot) = \chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z) \text{ over } \mathbb{R}^+,$$

where $\overline{\varphi}_\delta(\cdot)$ and $\overline{\varphi}_\delta(\cdot; k\delta, z)$ are treated as their zero extensions over $\mathbb{R}^+$. Then by Lemma 2.2, we have that

$$\langle \bar{f}_\delta, \bar{g}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)} = \langle \bar{f}_\delta, g\rangle_{L^2(\mathbb{R}^+\times\Omega)},$$

which leads to (79). Now, from (78) and (79), it follows that for each $z \in L^2(\Omega)$,

$$\langle \chi_\omega \overline{\varphi}_\delta(\cdot), \chi_\omega \overline{\varphi}_\delta(\cdot; k\delta, z)\rangle_{L^2((0,k\delta)\times\Omega)} = \langle v_\delta^*(\cdot), \chi_\omega \varphi(\cdot; k\delta, z)\rangle_{L^2((0,k\delta)\times\Omega)} = \langle y(k\delta; 0, v_\delta^*), z\rangle.$$

This, along with (78), yields that

$$y(k\delta; y_0, v_\delta^*) + r z_\delta^*/\|z_\delta^*\| = 0. \tag{80}$$

From (80), v_δ^* is an admissible control to $(\mathcal{NP})_\delta^{k\delta}$, and satisfies (72).

We next prove that v_δ^* is an optimal control to $(\mathcal{NP})_\delta^{k\delta}$. To this end, we arbitrarily fix an admissible control v_δ to $(\mathcal{NP})_\delta^{k\delta}$. Then we have that $\|y(k\delta; y_0, v_\delta)\| \le r$. This, together with (80), implies that

$$\begin{aligned} &\langle y(k\delta; 0, v_\delta^*), z_\delta^*\rangle = \langle y(k\delta; y_0, v_\delta^*), z_\delta^*\rangle - \langle e^{\Delta k\delta} y_0, z_\delta^*\rangle = -r\|z_\delta^*\| - \langle e^{\Delta k\delta} y_0, z_\delta^*\rangle \\ \le\ & \langle y(k\delta; y_0, v_\delta), z_\delta^*\rangle - \langle e^{\Delta k\delta} y_0, z_\delta^*\rangle = \langle y(k\delta; 0, v_\delta), z_\delta^*\rangle. \end{aligned} \tag{81}$$

Meanwhile, by Lemma 2.2 (where (f, g) are taken as the zero extensions of $(v_\delta^*, \chi_\omega\varphi)$ and $(v_\delta, \chi_\omega\varphi)$, respectively), and by (23) and (20), one can easily verify that

$$\langle v_\delta^*, \chi_\omega \overline{\varphi}_\delta\rangle_{L^2((0,k\delta)\times\Omega))} = \langle v_\delta^*, \chi_\omega \varphi\rangle_{L^2((0,k\delta)\times\Omega)};\ \langle v_\delta, \chi_\omega \overline{\varphi}_\delta\rangle_{L^2((0,k\delta)\times\Omega))} = \langle v_\delta, \chi_\omega \varphi\rangle_{L^2((0,k\delta)\times\Omega)}. \quad (82)$$

Since v_δ and v_δ^* are piece-wise constant functions (see (16) and (4)), it follows from (71), (82) and (81) that

$$\begin{aligned} &\|v_\delta^*\|_{L^2((0,k\delta)\times\Omega)}\|\chi_\omega\overline{\varphi}_\delta\|_{L^2((0,k\delta)\times\Omega)} = \langle v_\delta^*, \chi_\omega\overline{\varphi}_\delta\rangle_{L^2((0,k\delta)\times\Omega)} \\ &= \langle v_\delta^*, \chi_\omega\varphi\rangle_{L^2((0,k\delta)\times\Omega)} = \langle y(k\delta; 0, v_\delta^*), z_\delta^*\rangle \\ &\le \langle y(k\delta; 0, v_\delta), z_\delta^*\rangle = \langle v_\delta, \chi_\omega\varphi\rangle_{L^2((0,k\delta)\times\Omega)} \\ &= \langle v_\delta, \chi_\omega\overline{\varphi}_\delta\rangle_{L^2((0,k\delta)\times\Omega)} \le \|v_\delta\|_{L^2((0,k\delta)\times\Omega)}\|\chi_\omega\overline{\varphi}_\delta\|_{L^2((0,k\delta)\times\Omega)}. \end{aligned}$$

This, along with (70), yields that $\|v_\delta^*\|_{L^2((0,k\delta)\times\Omega)} \le \|v_\delta\|_{L^2((0,k\delta)\times\Omega)}$. Because v_δ is an arbitrarily fixed admissible control to $(\mathcal{NP})_\delta^{k\delta}$, we see that v_δ^* is an optimal control to $(\mathcal{NP})_\delta^{k\delta}$.

Finally, we prove the uniqueness of the optimal control to $(\mathcal{NP})_\delta^{k\delta}$. By contradiction, we suppose that $(\mathcal{NP})_\delta^{k\delta}$ had two different optimal controls $v_{\delta,1}^*$ and $v_{\delta,2}^*$. Then one could easily check that $(v_{\delta,1}^* + v_{\delta,2}^*)/2$ is still an optimal control. Since $v_{\delta,1}^* \neq v_{\delta,2}^*$, we can use the parallelogram law to get that

$$\|(v_{\delta,1}^* + v_{\delta,2}^*)/2\|_{L^2((0,k\delta)\times\Omega)} < \mathcal{N}_\delta(k\delta),$$

which contradicts the optimality of $\mathcal{N}_\delta(k\delta)$ to $(\mathcal{NP})_\delta^{k\delta}$. This proves the conclusion (ii) of Theorem 4.3.

(iii) Taking $z = z_\delta^*$ in (78) leads to that

$$\langle y_0, \varphi(0)\rangle + r\|z_\delta^*\| = -\|\chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)}.$$

Since z_δ^* is the minimizer of $J_\delta^{k\delta}$, the above equality, along with (19), indicates that

$$V_\delta(k\delta) = J_\delta^{k\delta}(z_\delta^*) = -\frac{1}{2}\|\chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)}. \quad (83)$$

Meanwhile, from (ii) of Theorem 4.3, we see that

$$\mathcal{N}_\delta(k\delta) = \|v_\delta^*\|_{L^2((0,k\delta)\times\Omega)} = \|\chi_\omega\overline{\varphi}_\delta\|_{L^2((0,k\delta)\times\Omega)}.$$

This, along with (83) leads to the conclusion (iii) of Theorem 4.3.

In summary, we end the proof of Theorem 4.3.

□

5. Several auxiliary estimates

This section presents several estimates, as well as properties, on minimizers (of $J_\delta^{k\delta}$ and J^T), the minimal norm functions and the minimal time functions. These estimate will play important roles in the proofs of the main theorems.

5.1. *Some estimates on minimizers*

The following theorem concerns with the $H_0^1(\Omega)$-estimates on the minimizers of the functionals $J_\delta^{k\delta}$ and J^T.

Theorem 5.1 *Let $(\delta, k) \in \mathcal{P}_{T^*}$ (given by (67)) and $0 < T < T^*$ (given by (66)). Write z^*_δ and z^* for the minimizers of $J^{k\delta}_\delta$ and J^T, respectively. Then the following conclusions are true:*

(i) There is a positive constant $C_1 \triangleq C_1(\Omega, \omega)$ so that

$$\|z^*_\delta\| \leq e^{C_1(1+\frac{1}{k\delta})}\|y_0\|^4 r^{-3}; \tag{84}$$

$$\|\partial_t \varphi(\cdot; k\delta, z^*_\delta)\|_{L^2(0,k\delta;L^2(\Omega))} \leq \|z^*_\delta\|_{H^1_0(\Omega)} \leq e^{C_1(1+\frac{1}{k\delta})}\|y_0\|^6 r^{-5}. \tag{85}$$

(ii) There is a positive constant $C_2 \triangleq C_2(\Omega, \omega)$ so that

$$\|z^*\| \leq e^{C_2(1+\frac{1}{T})}\|y_0\|^4 r^{-3}; \tag{86}$$

$$\|\partial_t \varphi(\cdot; k\delta, z^*)\|_{L^2(0,T;L^2(\Omega))} \leq \|z^*\|_{H^1_0(\Omega)} \leq e^{C_2(1+\frac{1}{T})}\|y_0\|^6 r^{-5}. \tag{87}$$

Proof. Throughout the proof, $C(\Omega, \omega)$ stands for a positive constant depending only on Ω and ω. It may vary in different contexts.

(i) We begin with proving (84). From (73), we find that

$$\frac{r}{2}\|z^*_\delta\| - 2e^{C(1+\frac{1}{k\delta})}\|y_0\|^4 r^{-2} \leq J^{k\delta}_\delta(z^*_\delta), \quad \text{for some } C = C(\Omega, \omega).$$

Since z^*_δ is the minimizer of $J^{k\delta}_\delta$, the above inequality, along with (19), implies that

$$\frac{r}{2}\|z^*_\delta\| - 2e^{C(1+\frac{1}{k\delta})}\|y_0\|^4 r^{-2} \leq J^{k\delta}_\delta(0) = 0,$$

which leads to (84).

To show (85), we need two estimates related to the optimal control u^*_δ of $(\mathcal{NP})^{k\delta}_\delta$. We first claim that

$$\|u^*_\delta\|_{L^2((0,k\delta)\times\Omega)} \leq e^{C(1+\frac{1}{k\delta})}\|y_0\|^2 r^{-1} \quad \text{for some } C \triangleq C(\Omega, \omega). \tag{88}$$

Indeed, since $(\delta, k) \in \mathcal{P}_{T^*}$ (given by (67)), we have that $k \geq 2$. Thus, by (ii) of Theorem 2.4, Equation (5) has the L^2-approximate null controllability with a cost. From this, Definition 2.1 (see (22)), and (iii) of Theorem 2.4 (see (32)), we find that for $\varepsilon_0 = (r/\|y_0\|)^2$, there is $u_\delta \in L^2_\delta((0, k\delta) \times \Omega)$ so that

$$\frac{\varepsilon_0}{e^{C(1+\frac{1}{k\delta})}}\|u_\delta\|^2_{L^2((0,k\delta)\times\Omega)} + \frac{1}{\varepsilon_0}\|y(k\delta; y_0, u_\delta)\|^2 \leq \|y_0\|^2 \quad \text{for some } C \triangleq C(\Omega, \omega). \tag{89}$$

Since $\varepsilon_0 = (r/\|y_0\|)^2$, it follows from (89) that u_δ is an admissible control to $(\mathcal{NP})^{k\delta}_\delta$. Then by the optimality of u^*_δ and $\mathcal{N}_\delta(k\delta)$, and by (89), we find that

$$\|u^*_\delta\|_{L^2((0,k\delta)\times\Omega)} = \mathcal{N}_\delta(k\delta) \leq \|u_\delta\|_{L^2((0,k\delta)\times\Omega)} \leq e^{\frac{C}{2}(1+\frac{1}{k\delta})}\|y_0\|^2 r^{-1},$$

which leads to (88).

Next, we claim that

$$\|y(k\delta; y_0, u^*_\delta)\|_{H^1_0(\Omega)} \leq e^{C(1+\frac{1}{k\delta})}\|y_0\|^2 r^{-1} \quad \text{for some } C \triangleq C(\Omega, \omega). \tag{90}$$

For this purpose, we consider the following equation:

$$\begin{cases} \partial_t y - \Delta y = f & \text{in } \mathbb{R}^+ \times \Omega, \\ y = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ y(0) = z & \text{in } \Omega, \end{cases} \tag{91}$$

where $z \in C_0^\infty(\Omega)$ and $f \in C_0^\infty(\mathbb{R}^+ \times \Omega)$. Multiplying y on both sides of Equation (91), by the Poincaré and Cauchy-Schwarz inequalities, we obtain that there exists $\widehat{C} \triangleq \widehat{C}(\Omega) > 0$ so that for each $S > 0$,

$$\int_0^S \int_\Omega |\nabla y(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \le \widehat{C} \int_0^S \int_\Omega |f(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + \int_\Omega |z(x)|^2 \,\mathrm{d}x. \tag{92}$$

Meanwhile, multiplying $-t\Delta y$ on both sides of Equation (91) and then integrating it over Ω, after some computations, we obtain that for each $S > 0$,

$$\int_\Omega S|\nabla y(S,x)|^2 \,\mathrm{d}x \le \int_0^S \int_\Omega t|f(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^S \int_\Omega |\nabla y(t,x)|^2 \,\mathrm{d}x\mathrm{d}t. \tag{93}$$

From (93) and (92), we deduce that for each $S > 0$, $z \in C_0^\infty(\Omega)$ and $f \in C_0^\infty(\mathbb{R}^+ \times \Omega)$,

$$\int_\Omega |\nabla y(S,x)|^2 \,\mathrm{d}x \le e^{\widehat{C}(1+1/S)} \Big[\int_0^S \int_\Omega |f(t,x)|^2 \,\mathrm{d}x\mathrm{d}t + \int_\Omega |z(x)|^2 \,\mathrm{d}x \Big].$$

Then by a standard density argument, we can easily derive from the above inequality that

$$\|y(k\delta; y_0, u_\delta^*)\|_{H_0^1(\Omega)} \le e^{\widehat{C}(1+\frac{1}{k\delta})} \big(\|u_\delta^*\|_{L^2((0,k\delta)\times\Omega)} + \|y_0\| \big).$$

Since $\|y_0\| > r$, the above, along with (88), leads to (90).

We now show the second inequality in (85). From (72), we see that

$$\|z_\delta^*\|_{H_0^1(\Omega)} = \frac{\|z_\delta^*\|}{r} \|y(k\delta; y_0, u_\delta^*)\|_{H_0^1(\Omega)},$$

which, together with (84) and (90), leads to the second inequality in (85).

Then, we show the first inequality in (85). Simply write $\varphi(\cdot)$ for $\varphi(\cdot; k\delta, z_\delta^*)$. Multiplying by $\Delta\varphi$ on both sides of the equation satisfied by $\varphi(\cdot; k\delta, z_\delta^*)$, and then integrating it over Ω, after some computations, we obtain that

$$\int_\Omega |\nabla \varphi(0,x)|^2 \,\mathrm{d}x + \int_0^{k\delta} \int_\Omega |\Delta\varphi(t,x)|^2 \,\mathrm{d}x\mathrm{d}t = \int_\Omega |\nabla \varphi(k\delta,x)|^2 \,\mathrm{d}x.$$

From this, it follows that

$$\int_0^{k\delta} \int_\Omega |\partial_t \varphi(t,x)|^2 \,\mathrm{d}x\mathrm{d}t = \int_0^{k\delta} \int_\Omega |\Delta\varphi(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \le \int_\Omega |\nabla z_\delta^*(x)|^2 \,\mathrm{d}x,$$

which leads to the first inequality in (85). This ends the proof of the conclusion (i).

(ii) Arbitrarily fix $k_0 \in \mathbb{N}^+$ so that $k_0 \ge \max\{2, 2/T\}$. For each integer $k \ge k_0$, let n_k be the integer so that

$$kT - 1 < n_k \le kT. \tag{94}$$

We first claim

$$\liminf_{k\to\infty} V_{1/k}(n_k/k) \le V(T) \;\text{ for all }\; k \ge k_0. \tag{95}$$

In fact, for each $k \ge k_0$, $(\mathcal{NP})_{1/k}^{n_k/k}$ has a unique optimal control v_k^* (see (ii) of Theorem 4.3). Then, by (94), one can easily check that the zero extension of v_k^* over $(0,T)$ is an admissible control to $(\mathcal{NP})^T$. From this and the optimality of $\mathcal{N}(T)$, one can easily check that

$$\mathcal{N}(T) \le \mathcal{N}_{1/k}(n_k/k) \;\text{ for all }\; k \ge k_0. \tag{96}$$

Since $0 < T < T^*$ and because $(1/k, n_k) \in \mathcal{P}_{T^*}$ (given by (67)) for all $k \ge k_0$ (which follows from (94) and (67)), we can apply (iii) of Theorem 4.2 and (iii) of Theorem 4.3 (with (δ,k)=$(1/k,n_k)$), and use (96) to obtain (95).

For each integer $k \ge k_0$, write $z_{1/k}^*$ for the minimizer of $(JP)_{1/k}^{n_k/k}$. The key is to show that on a subsequence of $\{z_{1/k}^*\}_{k\ge k_0}$, still denoted in the same manner,

$$z_{1/k}^* \to z^* \;\text{ weakly in }\; H_0^1(\Omega);\;\text{ strongly in }\; L^2(\Omega), \;\text{ as }\; k \to \infty. \tag{97}$$

(Here, z^* is the minimizer of $(JP)^T$.) To this end, we notice that $(1/k, n_k) \in \mathcal{P}_{T^*}$ (given by (67)) for all $k \ge k_0$ (which follows from (94) and (67)). Thus, we can use the second inequality in (85) (where $\delta = 1/k$; $k = n_k$) to find that $\{z_{1/k}^*\}_{k\ge k_0}$ is bounded in $H_0^1(\Omega)$. So there exists a subsequence of $\{z_{1/k}^*\}_{k\ge k_0}$, still denoted in the same manner, and some $\hat{z} \in H_0^1(\Omega)$ so that

$$z_{1/k}^* \to \hat{z} \;\text{ weakly in }\; H_0^1(\Omega);\;\text{ strongly in }\; L^2(\Omega), \;\text{ as }\; k \to \infty. \tag{98}$$

From the above, we see that in order to show (97), it suffices to prove that $z^* = \hat{z}$. For this purpose, we first claim that for each $k \ge k_0$,

$$\begin{aligned} &\|\varphi(0;T,\hat{z}) - \varphi(0;n_k/k, z_{1/k}^*)\| \\ \le& \sup_{0\le s\le \hat{t}\le s+\frac{1}{k}\le T} \|\varphi(\hat{t};T,\hat{z}) - \varphi(s;T,\hat{z})\| + \|\hat{z} - z_{1/k}^*\|; \end{aligned} \tag{99}$$

$$\begin{aligned} &\|\varphi(t;T,\hat{z}) - \overline{\varphi}_{1/k}(t;n_k/k, z_{1/k}^*)\| \\ \le& 2 \sup_{0\le s\le \hat{t}\le s+\frac{1}{k}\le T} \|\varphi(\hat{t};T,\hat{z}) - \varphi(s;T,\hat{z})\| + \|\hat{z} - z_{1/k}^*\|, \;\forall\, t \in (0, n_k/k). \end{aligned} \tag{100}$$

To show (99), we arbitrarily fix $k \ge k_0$. By (94), we see that $0 \le T - n_k/k \le 1/k$. This, along with the time-invariance of Equation (18), yields

$$\begin{aligned} \|\varphi(0;T,\hat{z}) - \varphi(0;n_k/k,\hat{z})\| &= \|\varphi(0;T,\hat{z}) - \varphi(T-n_k/k;T,\hat{z})\| \\ &\le \sup_{0\le s\le t\le s+\frac{1}{k}\le T} \|\varphi(t;T,\hat{z}) - \varphi(s;T,\hat{z})\| \end{aligned} \tag{101}$$

Meanwhile, since $\{e^{t\Delta} : t \ge 0\}$ is contractive, we have that

$$\|\varphi(0;n_k/k,\hat{z}) - \varphi(0;n_k/k, z_{1/k}^*)\| \le \|\hat{z} - z_{1/k}^*\|. \tag{102}$$

Using the triangle inequality, by (101) and (102), we obtain (99).

To show (100), we arbitrarily fix $k \ge k_0$ and $t \in (0, n_k/k)$. Three facts are given in order. Fact one: Since $0 \le T - n_k/k \le 1/k$, we can use the time-invariance of Equation (18) to get that

$$\|\varphi(t;T,\hat{z})-\varphi(t;n_k/k,\hat{z})\| = \|\varphi(t;T,\hat{z})-\varphi(T-n_k/k+t;T,\hat{z})\|$$
$$\leq \sup_{0\leq s\leq \hat{t}\leq s+\frac{1}{k}\leq T} \|\varphi(\hat{t};T,\hat{z})-\varphi(s;T,\hat{z})\|. \tag{103}$$

Fact two: Since $0 \leq T - n_k/k \leq 1/k$, by (20) and the time-invariance of Equation (18), we can easily check that

$$\|\varphi(t;n_k/k,\hat{z})-\overline{\varphi}_{1/k}(t;n_k/k,\hat{z})\| \tag{104}$$
$$= \Big\| \sum_{i=1}^{n_k} \chi_{((i-1)/k,i/k]}(t)k \int_{(i-1)/k}^{i/k} \left[\varphi(t;n_k/k,\hat{z})-\varphi(s;n_k/k,\hat{z})\right] \mathrm{d}s \Big\|$$
$$\leq \sup_{0\leq s\leq \hat{t}\leq s+\frac{1}{k}\leq n_k/k} \|\varphi(\hat{t};n_k/k,\hat{z})-\varphi(s;n_k/k,\hat{z})\|$$
$$\leq \sup_{0\leq s\leq \hat{t}\leq s+\frac{1}{k}\leq T} \|\varphi(\hat{t};T,\hat{z})-\varphi(s;T,\hat{z})\|.$$

Fact three: Since $\{e^{t\Delta} : t \geq 0\}$ is contractive, by (20), we see that

$$\|\overline{\varphi}_{1/k}(t;n_k/k,\hat{z})-\overline{\varphi}_{1/k}(t;n_k/k,z^*_{1/k})\| = \|\overline{\varphi}_{1/k}(t;n_k/k,\hat{z}-z^*_{1/k})\| \leq \|\hat{z}-z^*_{1/k}\|. \tag{105}$$

The above three facts (103), (104) and (105), together with the triangle inequality, leads to (100).

Two observations are given in order: First, since $\varphi(\cdot;T,\hat{z})$ is uniformly continuous on $[0,T]$, we see that two supremums in (99) and (100) tend to zero as $k\to\infty$. Second, it follows by (94) that $\lim_{k\to\infty} n_k/k = T$. From these two observations, (98), (99) and (100), one can easily check that

$$\langle y_0, \varphi(0;T,\hat{z})\rangle = \lim_{k\to\infty} \langle y_0, \varphi(0;n_k/k,z^*_{1/k})\rangle;$$
$$\int_0^T \|\chi_\omega\varphi(t;T,\hat{z})\|^2\,\mathrm{d}t = \lim_{k\to\infty} \int_0^{n_k/k} \|\chi_\omega\overline{\varphi}_{1/k}(t;n_k/k,z^*_{1/k})\|^2\,\mathrm{d}t.$$

These, together with (17), (19) and (98), indicate that

$$J^T(\hat{z}) = \lim_{k\to\infty} J^{n_k/k}_{1/k}(z^*_{1/k}) = \lim_{k\to\infty} V_{1/k}(n_k/k).$$

This, along with (95) and (17), yields that

$$J^T(\hat{z}) = V(T) = \inf_{z\in L^2(\Omega)} J^T(z).$$

Hence, $\hat{z}$ is a minimizer of J^T. Then, by the uniqueness of the minimizer, we see that $\hat{z} = z^*$. Hence, (97) is true.

Finally, since $0 < T < T^*$ and because $(1/k, n_k) \in \mathcal{P}_{T^*}$ (given by (67)) for all $k \geq k_0$ (which follows from (94) and (67)), the conclusion (i) in Theorem 5.1 is available for $(\delta,k)=(1/k,n_k)$. Thus, by (84), the second inequality in (85) (with $(\delta,k)=(1/k,n_k)$) and (97), using the fact that $n_k/k \to T$ (see (94)), we can easily obtain (86) and the second inequality in (87). Besides, by the same way as that used to prove the first inequality in (85), we get the first inequality in (87).

In summary, we end the proof of Theorem 5.1.

□

5.2. *Some estimates related to minimal norm functions*

Several inequalities related to the minimal norm functions $T \to \mathcal{N}(T)$ and $k\delta \to \mathcal{N}_\delta(k\delta)$ will be presented in the following two theorems.

Theorem 5.2 *There is $C_3 \triangleq C_3(\Omega,\omega) > 0$ so that for each pair (T_1,T_2), with $0 < T_1 \leq T_2 < T^*$ (given by (66)),*

$$\lambda_1^{3/2} r(T_2 - T_1) \leq \mathcal{N}(T_1) - \mathcal{N}(T_2) \leq e^{C_3(1+\frac{1}{T_1})}\|y_0\|(T_2 - T_1). \tag{106}$$

Proof. Arbitrarily fix a pair (T_1,T_2), with $0 < T_1 < T_2 < T^*$ (where T^* is given by (66)). The proof is organized by the following two steps:

Step 1. To show the first inequality in (106)
By (i) of Theorem 4.1, we have that

$$M_1 \triangleq \mathcal{N}(T_1) > \mathcal{N}(T_2) \triangleq M_2. \tag{107}$$

Then by (iii) in Theorem 4.1, we see that

$$0 < \mathcal{T}(M_1) = T_1 < T_2 = \mathcal{T}(M_2) < T^*. \tag{108}$$

Let u_1^* be an optimal control to $(\mathcal{TP})^{M_1}$. Then we find that

$$\|y(\mathcal{T}(M_1); y_0, u_1^*)\| \leq r \ \text{ and } \ \|u_1^*\|_{L^2(\mathbb{R}^+\times\Omega)} \leq M_1. \tag{109}$$

It follows from the first inequality in (109) that

$$\begin{aligned}\Big\|y(\mathcal{T}(M_1); y_0, \frac{M_2}{M_1}u_1^*)\Big\| &\leq \Big\|y(\mathcal{T}(M_1); y_0, \frac{M_2}{M_1}u_1^*) - y(\mathcal{T}(M_1); y_0, u_1^*)\Big\| \\ &\quad + \|y(\mathcal{T}(M_1); y_0, u_1^*)\| \\ &\leq \frac{M_1 - M_2}{M_1} \int_0^{\mathcal{T}(M_1)} \|e^{\Delta(\mathcal{T}(M_1)-t)}\chi_\omega u_1^*(t,\cdot)\|\,\mathrm{d}t + r.\end{aligned}$$

Since

$$\|e^{\Delta t}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leq e^{-\lambda_1 t} \ \text{ for each } \ t \geq 0,$$

the above, along with Hölder's inequality and the second inequality in (109), yields that

$$\begin{aligned}\Big\|y(\mathcal{T}(M_1); y_0, \frac{M_2}{M_1}u_1^*)\Big\| &\leq r + \frac{M_1 - M_2}{M_1}\frac{1}{\sqrt{2\lambda_1}}M_1 \\ &\leq r + (M_1 - M_2)/\sqrt{\lambda_1}.\end{aligned} \tag{110}$$

Next, we define a control u_2 over $\mathbb{R}^+$ as follows:

$$u_2(t) = \begin{cases} \dfrac{M_2}{M_1}u_1^*(t), & t \in (0, \mathcal{T}(M_1)], \\ 0, & t \in (\mathcal{T}(M_1), \infty). \end{cases} \tag{111}$$

From (111) and the second inequality in (109), it follows that

$$\|u_2\|_{L^2(\mathbb{R}^+\times\Omega)} \leq M_2. \tag{112}$$

Meanwhile, we let

$$\widehat{T} \triangleq \frac{1}{\lambda_1}\ln\Big(1+\frac{1}{\lambda_1^{1/2}r}(M_1-M_2)\Big) \leq \frac{1}{\lambda_1^{3/2}r}(M_1-M_2). \tag{113}$$

Since $u_2=0$ over $\big(\mathcal{T}(M_1),\infty\big)$, by (111), (110) and (113), one can easily check that

$$\begin{aligned}\|y(\mathcal{T}(M_1)+\widehat{T};y_0,u_2)\| &\leq e^{-\lambda_1\widehat{T}}\|y(\mathcal{T}(M_1);y_0,u_2)\| \\ &\leq e^{-\lambda_1\widehat{T}}\big(r+(M_1-M_2)/\lambda_1^{1/2}\big)=r.\end{aligned} \tag{114}$$

Now, it follows from (112) and (114) that u_2 is an admissible control to $(\mathcal{TP})^{M_2}$, which drives the solution to $B_r(y_0)$ at time $\mathcal{T}(M_1)+\widehat{T}$. This, along with the optimality of $\mathcal{T}(M_2)$, yields that

$$\mathcal{T}(M_2) \leq \mathcal{T}(M_1)+\widehat{T}.$$

From this, (108) and (113), we find that

$$T_2-T_1=\mathcal{T}(M_2)-\mathcal{T}(M_1)\leq \widehat{T} \leq \frac{1}{\lambda_1^{3/2}r}(M_1-M_2).$$

Since $M_1 \triangleq \mathcal{N}(T_1)$ and $M_2 \triangleq \mathcal{N}(T_2)$ (see (107)), the above leads to the first inequality in (106). This ends the proof of Step 1.

Step 2. To show the second inequality in (106)
Let z_1^* be the minimizer of J^{T_1}. *Throughout this step, we simply write $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ for $\varphi(\cdot;T_1,z_1^*)$ and $\varphi(\cdot;T_2,z_1^*)$ respectively.* First, we claim that

$$\|\varphi_2(T_2-T_1)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq e^{C_{21}(1+\frac{1}{T_1})}\mathcal{N}(T_1) \ \text{ for some } \ C_{21}\triangleq C_{21}(\Omega,\omega). \tag{115}$$

(Here and throughout the proof, we take the norm of $H^2(\Omega)\cap H_0^1(\Omega)$ as: $\|f\|_{H^2(\Omega)\cap H_0^1(\Omega)} \triangleq \|\Delta f\|$.) Indeed, according to [28, Theorem 6.13 in Chapter 2], there is $C_{22}\triangleq C_{22}(\Omega)>0$ so that

$$\|\Delta e^{\Delta s}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leq C_{22}/s \ \text{ for each } \ s>0.$$

From this, we see that

$$\|\varphi_2(T_2-T_1)\|_{H^2(\Omega)\cap H_0^1(\Omega)} = \|\Delta\varphi_2(T_2-T_1)\| = \|\Delta e^{\Delta\frac{T_1}{2}}\varphi_2(T_2-T_1/2)\| \leq \frac{2C_{22}}{T_1}\|\varphi_2(T_2-T_1/2)\|.$$

This, along with [12, Proposition 3.1], yields that for some $C_{23}\triangleq C_{23}(\Omega,\omega)>0$,

$$\|\varphi_2(T_2-T_1)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq \frac{2C_{22}}{T_1}e^{C_{23}(1+\frac{2}{T_1})}\|\chi_\omega\varphi_2\|_{L^2((T_2-T_1/2,T_2)\times\Omega)}. \tag{116}$$

Meanwhile, by (ii) of Theorem 4.2 and the time-invariance of Equation (18), we see that

$$\mathcal{N}(T_1)=\|\chi_\omega\varphi_1\|_{L^2((0,T_1)\times\Omega)}=\|\chi_\omega\varphi_2\|_{L^2((T_2-T_1,T_2)\times\Omega)}.$$

This, along with (116), yields that

$$\|\varphi_2(T_2-T_1)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \le e^{2C_{22}}e^{\frac{1}{T_1}}e^{C_{23}(1+\frac{2}{T_1})}\mathcal{N}(T_1),$$

which leads to (115).

Next, since $0 < T_1 \le T_2 < T^*$ (given by (66)), it follows by (i) and (ii) of Theorem 4.1 that $\mathcal{N}(T_1) \ge \mathcal{N}(T_2)$. From this and (iii) of Theorem 4.2, it follows that

$$V(T_1) = -\frac{1}{2}\mathcal{N}(T_1)^2 \le -\frac{1}{2}\mathcal{N}(T_2)^2 = V(T_2). \tag{117}$$

This, along with (17), yields that

$$\begin{aligned}0 &\le V(T_2) - V(T_1) \le J^{T_2}(z_1^*) - J^{T_1}(z_1^*)\\ &\le \frac{1}{2}\Big[\int_0^{T_2}\|\chi_\omega\varphi_2(t)\|^2\,\mathrm{d}t - \int_0^{T_1}\|\chi_\omega\varphi_1(t)\|^2\,\mathrm{d}t\Big] + \langle y_0, \varphi_2(0) - \varphi_1(0)\rangle.\end{aligned} \tag{118}$$

At the same time, by the time-invariance of Equation (18), we have that

$$\varphi_1(t) = \varphi_2(t + T_2 - T_1) \ \text{ for each } \ t \in (0, T_1). \tag{119}$$

Since the semigroup $\{e^{\Delta t}\}_{t\ge 0}$ is contractive, from (119), we see that

$$\int_0^{T_2}\|\chi_\omega\varphi_2(t)\|^2\,\mathrm{d}t - \int_0^{T_1}\|\chi_\omega\varphi_1(t)\|^2\,\mathrm{d}t \le (T_2 - T_1)\|\varphi_2(T_2 - T_1)\|^2. \tag{120}$$

From (119), we also have that

$$\begin{aligned}&\langle y_0, \varphi_2(0) - \varphi_1(0)\rangle = \langle y_0, \varphi_2(0) - \varphi_2(T_2 - T_1)\rangle\\ &\le \|y_0\|\Big\|\int_0^{T_2-T_1}\partial_t\varphi_2(t)\,\mathrm{d}t\Big\| = \|y_0\|\Big\|\int_0^{T_2-T_1}e^{\Delta(T_2-T_1-t)}\Delta\varphi_2(T_2-T_1)\,\mathrm{d}t\Big\|\\ &\le (T_2 - T_1)\|y_0\|\|\varphi_2(T_2 - T_1)\|_{H^2(\Omega)\cap H_0^1(\Omega)}.\end{aligned} \tag{121}$$

Now, by (118), (120) and (121), we obtain that there exists $\widehat{C} \triangleq \widehat{C}(\Omega) > 0$ so that

$$\begin{aligned}0 &\le V(T_2) - V(T_1)\\ &\le \widehat{C}(T_2 - T_1)\Big[\|\varphi_2(T_2 - T_1)\|^2_{H^2(\Omega)\cap H_0^1(\Omega)} + \|y_0\|\|\varphi_2(T_2 - T_1)\|_{H^2(\Omega)\cap H_0^1(\Omega)}\Big].\end{aligned}$$

By this, (117) and (115), we get that

$$\begin{aligned}\mathcal{N}(T_1) - \mathcal{N}(T_2) &\le \frac{2}{\mathcal{N}(T_2) + \mathcal{N}(T_1)}\big(V(T_2) - V(T_1)\big)\\ &\le 2\widehat{C}e^{C_{21}(1+\frac{1}{T_1})}(\mathcal{N}(T_1) + \|y_0\|)(T_2 - T_1).\end{aligned} \tag{122}$$

Finally, by [12, Proposition 3.1], we can find $u_{T_1} \in L^2((0, T_1) \times \Omega)$ so that

$$y(T_1; y_0, u_{T_1}) = 0 \ \text{ and } \ \|u_{T_1}\|_{L^2((0,T_1)\times\Omega)} \le e^{C_{24}(1+\frac{1}{T_1})}\|y_0\| \ \text{ for some } \ C_{24} \triangleq C_{24}(\Omega, \omega).$$

From the first equality in the above, we see that u_{T_1} is an admissible to $(\mathcal{NP})^{T_1}$. This, along with the second inequality in the above and the optimality of $\mathcal{N}(T_1)$, indicates

$$\mathcal{N}(T_1) \le \|u_{T_1}\|_{L^2((0,T_1)\times\Omega)} \le e^{C_{24}(1+\frac{1}{T_1})}\|y_0\|,$$

which, along with (122), leads to the second inequality in (106) for some $C_3 \triangleq C_3(\Omega,\omega)$.

In summary, we finish the proof of Theorem 5.2.

□

Theorem 5.3 *Let $\mathcal{P}_{T^*}$ and T^* be given by (67) and (66), respectively. Then there is $C_4 \triangleq C_4(\Omega,\omega) > 0$ so that for each $(\delta,k) \in \mathcal{P}_{T^*}$,*

$$0 \le \mathcal{N}_\delta(k\delta) - \mathcal{N}(k\delta) \le e^{C_4(1+T^*+\frac{1}{k\delta}+\frac{1}{T^*-k\delta})}\|y_0\|^{12} r^{-11}\delta^2. \tag{123}$$

Proof. Arbitrarily fix $(\delta,k) \in \mathcal{P}_{T^*}$ (given by (67)). Let z_δ^* be the minimizer of $J_\delta^{k\delta}$. *Throughout the proof of Theorem 5.3, we simply write respectively $\varphi(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ for $\varphi(\cdot;k\delta,z_\delta^*)$ (see (18)) and $\overline{\varphi}(\cdot;k\delta,z_\delta^*)$ (see (20)).* We organize the proof by several steps as follows:

Step 1. To prove that

$$0 \le V(k\delta) - V_\delta(k\delta) \le \|\chi_\omega\varphi\|^2_{L^2((0,k\delta)\times\Omega)} - \|\chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)} \tag{124}$$

Since $L^2_\delta((0,k\delta)\times\Omega) \subset L^2((0,k\delta)\times\Omega)$ (see (4)), we find that each admissible control to $(\mathcal{NP})^{k\delta}_\delta$ is also an admissible control to $(\mathcal{NP})^{k\delta}$. This, along with (14) and (15), yields that $\mathcal{N}(k\delta) \le \mathcal{N}_\delta(k\delta)$, from which, as well as (iii) of Theorems 4.2 and (iii) of Theorem 4.3, it follows that

$$V_\delta(k\delta) = -\frac{1}{2}\mathcal{N}_\delta(k\delta)^2 \le -\frac{1}{2}\mathcal{N}(k\delta)^2 = V(k\delta). \tag{125}$$

This, along with (17) and (19), yields that

$$\begin{aligned} 0 \le V(k\delta) - V_\delta(k\delta) &\le J^{k\delta}(z_\delta^*) - J_\delta^{k\delta}(z_\delta^*) \\ &\le \frac{1}{2}\left[\|\chi_\omega\varphi\|^2_{L^2((0,k\delta)\times\Omega)} - \|\chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)}\right], \end{aligned}$$

which leads to (124).

Step 2. To show that

$$\|\chi_\omega\varphi\|^2_{L^2((0,k\delta)\times\Omega)} - \|\chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)} = \|\chi_\omega\varphi - \chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)} \tag{126}$$

First, we claim that for each $f \in L^2(\mathbb{R}^+\times\Omega)$,

$$\|f\|^2_{L^2(\mathbb{R}^+\times\Omega)} = \|\bar{f}_\delta\|^2_{L^2(\mathbb{R}^+\times\Omega)} + \|f-\bar{f}_\delta\|^2_{L^2(\mathbb{R}^+\times\Omega)}, \tag{127}$$

where $\bar{f}_\delta$ is given by (23). Indeed, for an arbitrarily fixed $f \in L^2(\mathbb{R}^+\times\Omega)$, one can directly check that

$$\|f\|^2_{L^2(\mathbb{R}^+\times\Omega)} = \|\bar{f}_\delta\|^2_{L^2(\mathbb{R}^+\times\Omega)} + \|f-\bar{f}_\delta\|^2_{L^2(\mathbb{R}^+\times\Omega)} + 2\langle\bar{f}_\delta, f-\bar{f}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)}. \tag{128}$$

Meanwhile, it follows by (23) that $\bar{g}_\delta = 0$, where $g \triangleq f - \bar{f}_\delta$. Then by Lemma 2.2, we obtain that

$$\langle\bar{f}_\delta, f-\bar{f}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)} = \langle\bar{f}_\delta, g\rangle_{L^2(\mathbb{R}^+\times\Omega)} = \langle\bar{f}_\delta, \bar{g}_\delta\rangle_{L^2(\mathbb{R}^+\times\Omega)} = 0.$$

This, along with (128), leads to (127).

Next, by taking f to be the zero extension of φ over $\mathbb{R}^+\times\Omega$ in (127), we obtain (126). Here, we used the fact that in this case, $\bar{f}_\delta$ is the zero extension of $\chi_\omega\overline{\varphi}_\delta$ over $\mathbb{R}^+\times\Omega$, which follows from (23) and (20).

Step 3. To verify that there exists $C_{41} \triangleq C_{41}(\Omega,\omega) > 0$ so that

$$\|\chi_\omega\varphi-\chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)}\leq e^{C_{41}(1+k\delta+\frac{1}{k\delta})}\|y_0\|^{12}r^{-10}\delta^2 \tag{129}$$

From (20), it follows that

$$\int_0^{k\delta}\|\chi_\omega\varphi(t)-\chi_\omega\overline{\varphi}_\delta(t)\|^2\,\mathrm{d}t=\sum_{j=1}^k\int_{(j-1)\delta}^{j\delta}\Big\|\chi_\omega\varphi(t)-\frac{1}{\delta}\int_{(j-1)\delta}^{j\delta}\chi_\omega\varphi(s)\,\mathrm{d}s\Big\|^2\,\mathrm{d}t$$

$$=\sum_{j=1}^k\int_{(j-1)\delta}^{j\delta}\Big\|\frac{1}{\delta}\int_{(j-1)\delta}^{j\delta}\int_s^t\chi_\omega\partial_\tau\varphi(\tau)\,\mathrm{d}\tau\mathrm{d}s\Big\|^2\,\mathrm{d}t\leq\sum_{j=1}^k\int_{(j-1)\delta}^{j\delta}\Big(\int_{(j-1)\delta}^{j\delta}\|\partial_\tau\varphi(\tau)\|\,\mathrm{d}\tau\Big)^2\,\mathrm{d}t.$$

Applying the Hölder inequality to the above leads to that

$$\|\chi_\omega\varphi-\chi_\omega\overline{\varphi}_\delta\|^2_{L^2((0,k\delta)\times\Omega)}\leq\delta^2\|\partial_t\varphi\|^2_{L^2((0,k\delta)\times\Omega)}.$$

This, along with (85), implies (129) for some $C_{41}\triangleq C_{41}(\Omega,\omega)$.

Step 4. To show (123)

We first claim that

$$\mathcal{N}(k\delta)\geq e^{-\frac{2}{\lambda_1}-\frac{1}{T^*-k\delta}}r. \tag{130}$$

In fact, by (i) of Theorem 4.1, we have that

$$\lim_{T_2\to T^{*-}}\mathcal{N}(T_2)=0.$$

This, along with the first inequality in (106) (where $T_1=k\delta$), yields that

$$\begin{aligned}\mathcal{N}(k\delta)&=\lim_{T_2\to T^{*-}}(\mathcal{N}(k\delta)-\mathcal{N}(T_2))\\&\geq\lim_{T_2\to T^{*-}}\lambda_1^{3/2}r(T_2-k\delta)=\lambda_1^{3/2}r(T^*-k\delta).\end{aligned} \tag{131}$$

Since we clearly have that

$$\lambda_1\geq e^{-\frac{1}{\lambda_1}}\ \text{ and }\ T^*-k\delta\geq e^{-\frac{1}{T^*-k\delta}},$$

(130) follows from (131) at once.

Meanwhile, from (124), (126) and (129), we obtain that

$$0\leq V(k\delta)-V_\delta(k\delta)\leq e^{C_{41}(1+k\delta+\frac{1}{k\delta})}\|y_0\|^{12}r^{-10}\delta^2.$$

From this, (125) and (130), we find that

$$\begin{aligned}0\leq\mathcal{N}_\delta(k\delta)-\mathcal{N}(k\delta)&=\frac{2V(k\delta)-2V_\delta(k\delta)}{\mathcal{N}(k\delta)+\mathcal{N}_\delta(k\delta)}\\&\leq 2e^{\frac{2}{\lambda_1}+\frac{1}{T^*-k\delta}}e^{C_{41}(1+k\delta+\frac{1}{k\delta})}\|y_0\|^{12}r^{-11}\delta^2.\end{aligned}$$

Since $k\delta<T^*$, the above leads to (123) for some $C_4\triangleq C_4(\Omega,\omega)$.

In summary, we end the proof of Theorem 5.3.

□

5.3. *Some properties on minimal time functions*

Some inequalities, as well as properties, related to the minimal time functions $M \to \mathcal{T}_\delta(M)$ and $M \to \mathcal{T}(M)$ will be given in this subsection.

Theorem 5.4 *For each $M > 0$ and $\eta \in (0,1)$, there is a measurable subset $\mathcal{A}_{M,\eta} \subset (0,1)$ (depending also on y_0 and r), with $\lim_{h\to 0^+} \frac{1}{h}|\mathcal{A}_{M,\eta} \cap (0,h)| = \eta$, so that for each $\delta \in \mathcal{A}_{M,\eta}$, there is $a_\delta \in (0,\eta)$ so that*

$$\mathcal{T}_\delta(M) - \mathcal{T}(M) = (1-a_\delta)\delta \ \text{ and } \ M \geq \mathcal{N}_\delta(\mathcal{T}_\delta(M)) + \frac{1}{2}\lambda_1^{3/2} r(1-\eta)\delta. \tag{132}$$

Proof. Arbitrarily fix $M > 0$ and $\eta \in (0,1)$. For each $k \in \mathbb{N}^+$ and $a \in (0,\eta)$, we define a subset of $\mathbb{R}^+$ in the following manner:

$$\mathcal{B}^{k,a}_{M,\eta} \triangleq \{\delta > 0 \ : \ (k+a)\delta = \mathcal{T}(M)\}. \tag{133}$$

We then define another subset of $\mathbb{R}^+$ as follows:

$$\mathcal{B}_{M,\eta} \triangleq \cup_{k\in\mathbb{N}^+} \cup_{a\in(0,\eta)} \mathcal{B}^{k,a}_{M,\eta}. \tag{134}$$

The rest proof is divided into the following two steps:

Step 1. To prove that $\lim_{h\to 0^+} \frac{1}{h}|\mathcal{B}_{M,\eta} \cap (0,h)| = \eta$

From (133), we see that

$$\cup_{a\in(0,\eta)}\mathcal{B}^{k,a}_{M,\eta} = (\mathcal{T}(M)/(k+\eta), \mathcal{T}(M)/k) \ \text{ for each } \ k \in \mathbb{N}^+.$$

From this and (134), it follows that

$$\mathcal{B}_{M,\eta} = \cup_{k\in\mathbb{N}^+} \left(\mathcal{T}(M)/(k+\eta), \mathcal{T}(M)/k\right). \tag{135}$$

For each $j \in \mathbb{N}^+$, we let $h_j \triangleq \mathcal{T}(M)/j$. For each $h \in (0, \mathcal{T}(M))$, we let $j(h)$ be the integer so that $h_{j(h)+1} \leq h < h_{j(h)}$. Then, by (135), one can easily verify that

$$\lim_{h\to 0^+} \frac{|\mathcal{B}_{M,\eta} \cap (0, h_{j(h)})|}{h_{j(h)}} = \eta; \ \ \lim_{h\to 0^+} \frac{h_{j(h)+1}}{h} = \lim_{h\to 0^+} \frac{h_{j(h)}}{h} = 1; \tag{136}$$

$$\frac{h_{j(h)+1}}{h} \frac{|\mathcal{B}_{M,\eta} \cap (0, h_{j(h)+1})|}{h_{j(h)+1}} \leq \frac{|\mathcal{B}_{M,\eta} \cap (0,h)|}{h} \leq \frac{|\mathcal{B}_{M,\eta} \cap (0, h_{j(h)})|}{h_{j(h)}} \frac{h_{j(h)}}{h}. \tag{137}$$

From (136) and (137), we can easily obtain the conclusion in Step 1.

Step 2. To show (132)

We first claim that for each $\delta \in \mathcal{B}_{M,\eta} \cap (0,1)$, there is a unique pair (k_δ, a_δ) so that

$$(k_\delta + a_\delta)\delta = \mathcal{T}(M) \ \text{ with } \ k_\delta \in \mathbb{N}^+ \ \text{ and } \ a_\delta \in (0,\eta). \tag{138}$$

Indeed, the existence of such a pair follows from (134) and (133) at once, while the uniqueness of such pairs can be directly checked.

Thus, for each $\delta \in \mathcal{B}_{M,\eta} \cap (0,1)$, we can define k_δ to be the first component of the unique pair satisfying (138). We next claim that there exists $\delta^1_{M,\eta} \in (0,1)$ so that

$$M \geq \mathcal{N}_\delta((k_\delta+1)\delta) + \frac{1}{2}\lambda_1^{3/2} r(1-\eta)\delta \ \text{ for each } \ \delta \in \mathcal{B}_{M,\eta} \cap (0, \delta^1_{M,\eta}). \tag{139}$$

To this end, we notice that $\mathcal{T}(M) < T^*$ (see (iii) of Theorem 4.1). Arbitrarily fix $\delta \in \mathcal{B}_{M,\eta} \cap (0,1)$ so that

$$0 < \delta < \min\{\mathcal{T}(M)/2, (T^* - \mathcal{T}(M))/2\}. \tag{140}$$

(The existence of such δ is ensured by (135).) Then it follows from (140) and (138) that

$$2\delta < \mathcal{T}(M) < (k_\delta+1)\delta < \mathcal{T}(M) + \delta < (T^* + \mathcal{T}(M))/2 < T^*.$$

This, along with the definition of $\mathcal{P}_{T^*}$ (see (67)), yields that

$$(\delta, k_\delta) \in \mathcal{P}_{T^*} \ \text{ and } \ 2\delta < \mathcal{T}(M) < (k_\delta+1)\delta < T^*.$$

By these, we can apply Theorem 5.3 (with $(\delta, k) = (\delta, k_\delta)$) and Theorem 5.2 (with $T_1 = \mathcal{T}(M)$ and $T_2 = (k_\delta+1)\delta$) to get that

$$\begin{aligned}\mathcal{N}_\delta((k_\delta+1)\delta) \leq& \mathcal{N}((k_\delta+1)\delta) + e^{C_4\left[1+T^*+\frac{1}{(k_\delta+1)\delta}+\frac{1}{T^*-(k_\delta+1)\delta}\right]}\|y_0\|^{12} r^{-11}\delta^2 \\ \leq& \mathcal{N}(\mathcal{T}(M)) - \lambda_1^{3/2} r\big((k_\delta+1)\delta - \mathcal{T}(M)\big) + \\ & e^{C_4\left[1+T^*+\frac{1}{(k_\delta+1)\delta}+\frac{1}{T^*-(k_\delta+1)\delta}\right]}\|y_0\|^{12} r^{-11}\delta^2,\end{aligned} \tag{141}$$

where C_4 is given by (123). Meanwhile, by (138) and (140), we find that

$$(k_\delta+1)\delta - \mathcal{T}(M) \geq (1-\eta)\delta \ \text{ and } \ \mathcal{T}(M) < (k_\delta+1)\delta < (T^* + \mathcal{T}(M))/2.$$

These, along with (141) and (ii) of Theorem 4.1, yield that

$$\begin{aligned}\mathcal{N}_\delta((k_\delta+1)\delta) \leq& \mathcal{N}(\mathcal{T}(M)) - \lambda_1^{3/2} r(1-\eta)\delta + e^{C_4\left[1+T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{12} r^{-11}\delta^2 \\ =& M - \lambda_1^{3/2} r(1-\eta)\delta + e^{C_4\left[1+T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{12} r^{-11}\delta^2.\end{aligned}$$

By this and (140), we obtain (139).

Define a set $\mathcal{A}_{M,\eta}$ in the following manner:

$$\mathcal{A}_{M,\eta} \triangleq \mathcal{B}_{M,\eta} \cap (0, \delta^1_{M,\eta}), \ \text{ with } \ \delta^1_{M,\eta} \ \text{ given by } \ (139). \tag{142}$$

We now show that the second conclusion in (132) holds for each δ in $\mathcal{A}_{M,\eta}$ defined by (142). To this end, we arbitrarily fix $\delta \in \mathcal{A}_{M,\eta}$. We claim that

$$\mathcal{T}_\delta(M) \leq (k_\delta+1)\delta \ \text{ and } \ \mathcal{T}_\delta(M) > k_\delta\delta. \tag{143}$$

To show the first inequality in (143), we let u_δ be an admissible control to $(\mathcal{NP})_\delta^{(k_\delta+1)\delta}$ and let $\tilde{u}_\delta$ be the zero extension of u_δ over $\mathbb{R}^+ \times \Omega$. Since $\mathcal{N}_\delta((k_\delta+1)\delta) \leq M$ (see (139)), one can easily check that $\tilde{u}_\delta$ is an admissible control (to $(\mathcal{TP})_\delta^M$), which drives the solution to $B_r(0)$ at time $(k_\delta+1)\delta$. This, along with the optimality of $\mathcal{T}_\delta(M)$, leads to the first inequality in (143). To prove the second inequality in (143), we notice that $\mathcal{U}_\delta^M \subset \mathcal{U}^M$. This, along with (2) and (6), yields that $\mathcal{T}(M) \leq \mathcal{T}_\delta(M)$. From this and (138), we obtain the second inequality in (143).

Since $\mathcal{T}_\delta(M)$ is a multiple of δ (see (7)), it follows from (143) that

$$\mathcal{T}_\delta(M) = (k_\delta+1)\delta. \tag{144}$$

This, along with (139), implies that the second conclusion in (132).

Finally, from (144) and (138), we see that the first conclusion in (132) is true for each $\delta \in \mathcal{A}_{M,\eta}$ defined by (142).

In summary, we end the proof of Theorem 5.4.

$\square$

6. The proofs of the main theorems

In this section, we will prove Theorems 1.3-1.5. The strategies to show them have been introduced in Subsection 1.3.

6.1. *The proof of Theorem 1.3*

Proof of Theorem 1.3. We will prove the conclusions (i) and (ii) in Theorem 1.3 one by one.

(i) Arbitrarily fix $M > 0$. Recall that T^* is given by (66). From the conclusion (iii) in Theorem 4.1, it follows

$$0 < \mathcal{T}(M) < T^*. \tag{145}$$

We take δ so that

$$0 < \delta < \min\left\{\mathcal{T}(M)/2, \big(T^* - \mathcal{T}(M)\big)/4\right\} \triangleq \delta_1. \tag{146}$$

Let $\hat{k}_\delta \in \mathbb{N}$ satisfy that

$$(\hat{k}_\delta - 1)\delta < \mathcal{T}(M) \le \hat{k}_\delta \delta. \tag{147}$$

We first claim that

$$\mathcal{N}_\delta((\hat{k}_\delta + 1)\delta) \le M + e^{C_4\left[1+T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{12}r^{-11}\delta^2 - \lambda_1^{3/2}r\delta. \tag{148}$$

Indeed, from the definition of $\mathcal{P}_{T^*}$ (given by (67)) and (145)-(147), one can easily check that

$$0 < \mathcal{T}(M) < (\hat{k}_\delta + 1)\delta < T^* \ \text{ and } \ (\delta, \hat{k}_\delta + 1) \in \mathcal{P}_{T^*}. \tag{149}$$

Three facts are given in order: (a) By the second conclusion in (149), we can apply Theorem 5.3, with $(\delta, k) = (\delta, \hat{k}_\delta + 1)$, to obtain that

$$\mathcal{N}_\delta((\hat{k}_\delta + 1)\delta) \le \mathcal{N}((\hat{k}_\delta + 1)\delta) + e^{C_4\left[1+T^*+\frac{1}{(\hat{k}_\delta+1)\delta}+\frac{1}{T^*-(\hat{k}_\delta+1)\delta}\right]}\|y_0\|^{12}r^{-11}\delta^2,$$

where $C_4 \triangleq C_4(\Omega, \omega)$ is given by (123). (b) By the first conclusion in (149), we can use the first inequality in (106) in Theorem 5.2 (where $T_1 = \mathcal{T}(M)$ and $T_2 = (\hat{k}_\delta + 1)\delta$) to get that

$$\begin{aligned}\mathcal{N}((\hat{k}_\delta + 1)\delta) &\le \mathcal{N}(\mathcal{T}(M)) - \lambda_1^{3/2}r|(\hat{k}_\delta + 1)\delta - \mathcal{T}(M)| \\ &\le \mathcal{N}(\mathcal{T}(M)) - \lambda_1^{3/2}r\delta;\end{aligned}$$

(c) By (ii) of Theorem 4.1, we have that $\mathcal{N}(\mathcal{T}(M)) = M$.

From above three facts (a)-(c), we find that

$$\mathcal{N}_\delta((\hat{k}_\delta+1)\delta) \le M + e^{C_4\left[1+T^*+\frac{1}{(\hat{k}_\delta+1)\delta}+\frac{1}{T^*-(\hat{k}_\delta+1)\delta}\right]}\|y_0\|^{12}r^{-11}\delta^2 - \lambda_1^{3/2}r\delta. \tag{150}$$

Meanwhile, from (147), (146) and (145), one can easily check that

$$\mathcal{T}(M) \le (\hat{k}_\delta - 1)\delta + 2\delta \le \mathcal{T}(M) + \frac{T^* - \mathcal{T}(M)}{2} = \frac{T^* + \mathcal{T}(M)}{2}.$$

This, along with (150), leads to (148).

We next claim that

$$\mathcal{T}_\delta(M) \le (\hat{k}_\delta+1)\delta \ \text{ for each } \ 0<\delta<\delta_0 \triangleq \min\{\delta_1,\delta_2\}), \tag{151}$$

where δ_1 is given by (146) and δ_2 is defined by

$$\delta_2 \triangleq \frac{1}{2}\lambda_1^{3/2}e^{-C_4\left[1+T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{-12}r^{12}. \tag{152}$$

In fact, for an arbitrarily fixed $\delta \in (0,\delta_0)$, by (148) and (152), after some computations, we find that

$$M \ge \mathcal{N}_\delta((\hat{k}_\delta+1)\delta) + \frac{1}{2}\lambda_1^{3/2}r\delta > \mathcal{N}_\delta((\hat{k}_\delta+1)\delta). \tag{153}$$

Let u_δ be the zero extension of an admissible control (to $(\mathcal{NP})_\delta^{(\hat{k}_\delta+1)\delta}$) over $\mathbb{R}^+\times\Omega$. Then by (153), one can easily check that u_δ is an admissible control (to $(\mathcal{TP})_\delta^M$), which drives the solution to $B_r(0)$ at time $(\hat{k}_\delta+1)\delta$. This, along with the optimality of $\mathcal{T}_\delta(M)$, leads to (151).

We now show (8) with δ_0 given by (151). For this purpose, we arbitrarily fix $\delta\in(0,\delta_0)$. Since $\mathcal{U}_\delta^M \subset \mathcal{U}^M$, it follows by (2) and (6) that $\mathcal{T}(M) \le \mathcal{T}_\delta(M)$. This, along with (147) and (151), leads to (8), which ends the proof of the conclusion (i).

(ii) Let $\mathcal{A}_{M,\eta}$, with $M>0$ and $\eta\in(0,1)$, be given by Theorem 5.4. Then the conclusion (ii) of Theorem 1.3 follows from the first conclusion in (132) at once.

In summary, we end the proof of Theorem 1.3.

□

6.2. *The proof of Theorem 1.4*

Proof of Theorem 1.4. For each $M>0$ and $\delta>0$, we let u_M^* and $u_{\delta,M}^*$ be the optimal control and the optimal control with the minimal norm to $(\mathcal{TP})^M$ and $(\mathcal{TP})_\delta^M$ respectively (see Theorem 3.1). We will prove the conclusions (i)-(ii) one by one.

(i) Let $M>0$. Let $\delta_0=\delta_0(M,y_0,r)$ and $C_3=C_3(\Omega,\omega)$ be given by Theorem 1.3 and Theorem 5.2, respectively. Arbitrarily fix $\delta>0$. *In the proof of (i) of Theorem 1.4, we simply write u^* and u_δ^* for u_M^* and $u_{\delta,M}^*$, respectively.*

Since $\mathcal{T}_\delta(M)$ is a multiple of δ (see (6)), we can write

$$\mathcal{T}_\delta(M) \triangleq k_\delta\delta \ \text{ with } \ k_\delta\in\mathbb{N}^+. \tag{154}$$

In the case that

$$\delta \ge \min\Big\{\delta_0, \frac{T^*-\mathcal{T}(M)}{4}, \frac{\mathcal{T}(M)}{3}, \frac{M}{4e^{C_3(1+\frac{1}{\mathcal{T}(M)})}(1+\|y_0\|)}\Big\}, \tag{155}$$

one can easily show (10). In fact, it follows from (155) that

$$\|u^* - u_\delta^*\|_{L^2(\mathbb{R}^+\times\Omega)} \le \|u^*\|_{L^2(\mathbb{R}^+\times\Omega)} + \|u_\delta^*\|_{L^2(\mathbb{R}^+\times\Omega)}$$
$$\le 2M \le \frac{6M}{\mathcal{T}(M)}\delta \triangleq \hat{C}(M, y_0, r)\delta. \tag{156}$$

Thus, we only need to show (10) for the case that

$$0 < \delta < \min\Big\{\delta_0, \frac{T^* - \mathcal{T}(M)}{4}, \frac{\mathcal{T}(M)}{3}, \frac{M}{4e^{C_3(1+\frac{1}{\mathcal{T}(M)})}(1+\|y_0\|)}\Big\}. \tag{157}$$

For this purpose, some preliminaries are needed. First we claim that

$$0 < \mathcal{T}(M) \le \mathcal{T}_\delta(M) \le (\mathcal{T}(M) + T^*)/2 < T^*. \tag{158}$$

Indeed, (158) follows from the next three facts at once. Fact one: From Theorem 4.1, we have that $0 < \mathcal{T}(M) < T^*$; Fact two: Since $\mathcal{U}_\delta^M \subset \mathcal{U}^M$, we find from (2) and (6) that $\mathcal{T}(M) \le \mathcal{T}_\delta(M)$; Fact three: By Theorem 1.3 and (157), we see that

$$\mathcal{T}_\delta(M) \le \mathcal{T}(M) + 2\delta \le (\mathcal{T}(M) + T^*)/2.$$

Then it follows from (158), (157) and the definition of $\mathcal{P}_{T^*}$ (given by (67)) that

$$3\delta \le \mathcal{T}_\delta(M) < T^*, \text{ i.e., } (\delta, \mathcal{T}_\delta(M)/\delta) \triangleq (\delta, k_\delta) \in \mathcal{P}_{T^*}, \text{ with } k_\delta \text{ given by (154).} \tag{159}$$

Next, we let $z^* \ne 0$ and $z_\delta^* \ne 0$ be the minimizers of $J^{\mathcal{T}(M)}$ and $J_\delta^{\mathcal{T}_\delta(M)}$, respectively (see Theorem 4.2 and Theorem 4.3). Write

$$\hat{z}^* \triangleq z^*/\|z^*\| \text{ and } \hat{z}_\delta^* \triangleq z_\delta^*/\|z_\delta^*\|. \tag{160}$$

By (iii) of Theorem 4.1, the restriction of u^* over $(0, \mathcal{T}(M)) \times \Omega$ is the optimal control to $(\mathcal{NP})^{\mathcal{T}(M)}$. (It can be treated as a function from $(0, \mathcal{T}(M))$ to $L^2(\Omega)$.) Then by (ii) of Theorem 4.2 (with $T = \mathcal{T}(M) \in (0, T^*)$), we see that $u^*(\cdot) = \chi_\omega \varphi(\cdot; \mathcal{T}(M), z^*)$ over $(0, \mathcal{T}(M))$. Meanwhile, by (ii) of Theorem 4.1, we find that $\mathcal{N}(\mathcal{T}_\delta(M)) = M$. These, along with (160), yield that

$$u^*(t) = \chi_\omega \varphi(t; \mathcal{T}(M), z^*) = M \frac{\chi_\omega \varphi(t; \mathcal{T}(M), \hat{z}^*)}{\|\chi_\omega \varphi(\cdot; \mathcal{T}(M), \hat{z}^*)\|_{L^2((0,\mathcal{T}(M))\times\Omega)}} \text{ a.e. } t \in (0, \mathcal{T}(M)). \tag{161}$$

Finally, it follows from (iii) of Theorem 3.1 that the restriction of u_δ^* over $(0, \mathcal{T}_\delta(M)) \times \Omega$ is an optimal control to $(\mathcal{NP})_\delta^{\mathcal{T}_\delta(M)}$. (It can be treated as a function from $(0, \mathcal{T}_\delta(M))$ to $L^2(\Omega)$.) This, along with the fact that u_δ^* is an optimal control to $(\mathcal{TP})_\delta^M$, yields that

$$\mathcal{N}_\delta(\mathcal{T}_\delta(M)) = \|u_\delta^*\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \le \|u_\delta^*\|_{L^2(\mathbb{R}^+\times\Omega)} \le M. \tag{162}$$

Meanwhile, by (159), we can apply Theorem 4.3 (with $(\delta, k) = (\delta, \mathcal{T}_\delta(M)/\delta) \triangleq (\delta, k_\delta)$), as well as (160), to obtain that

$$u_\delta^*(t) = \chi_\omega \overline{\varphi}_\delta(t; \mathcal{T}_\delta(M), z_\delta^*) = M_\delta \frac{\chi_\omega \overline{\varphi}_\delta(t; \mathcal{T}_\delta(M), \hat{z}_\delta^*)}{\|\chi_\omega \overline{\varphi}_\delta(\cdot; \mathcal{T}_\delta(M), \hat{z}_\delta^*)\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}} \text{ a.e. } t \in (0, \mathcal{T}_\delta(M)). \tag{163}$$

Here, $\overline{\varphi}_\delta(\cdot; \mathcal{T}_\delta(M), z_\delta^*)$ and $\overline{\varphi}_\delta(\cdot; \mathcal{T}_\delta(M), \hat{z}_\delta^*)$ are given by (20) with $(\delta, k) = (\delta, k_\delta)$ and M_δ is defined by

$$M_\delta \triangleq \mathcal{N}_\delta(\mathcal{T}_\delta(M)). \tag{164}$$

We now prove (10) for an arbitrarily fixed δ (satisfying (157)) by several steps.

Step 1. To show that

$$\|y(\mathcal{T}(M);y_0,u_\delta^*)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \le \|y_0\|/\mathcal{T}(M) + 2\left(\sqrt{\mathcal{T}_\delta(M)}\|z_\delta^*\|_{H_0^1(\Omega)} + \|z_\delta^*\|\right) \tag{165}$$

One can easily check the following two estimates:

$$\|y(\mathcal{T}(M);y_0,u_\delta^*)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \le \|e^{\Delta\mathcal{T}(M)}y_0\|_{H^2(\Omega)\cap H_0^1(\Omega)} + \|y(\mathcal{T}(M);0,u_\delta^*)\|_{H^2(\Omega)\cap H_0^1(\Omega)};$$

$$\|e^{\Delta\mathcal{T}(M)}y_0\|_{H^2(\Omega)\cap H_0^1(\Omega)} = \|\Delta e^{\Delta\mathcal{T}(M)}y_0\| \le \|y_0\|/\mathcal{T}(M).$$

From these, we see that to prove (165), it suffices to show that

$$\|y(\mathcal{T}(M);0,u_\delta^*)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \le 2\big(\sqrt{\mathcal{T}_\delta(M)}\|z_\delta^*\|_{H_0^1(\Omega)} + \|z_\delta^*\|\big). \tag{166}$$

For this purpose, let $\bar{k}$ be the integer so that $\bar{k}\delta < \mathcal{T}(M) \le (\bar{k}+1)\delta$. Because $\mathcal{T}(M) \le \mathcal{T}_\delta(M)$ (see (158)) and $\mathcal{T}_\delta(M)$ is a multiple of δ (see (6)), we find that $\mathcal{T}_\delta(M) \ge (\bar{k}+1)\delta$. Since u_δ^* is a piece-wise constant function over $(0,\mathcal{T}_\delta(M))$, and because

$$\Delta e^{\Delta(\mathcal{T}(M)-t)}f dt = -\frac{d}{dt}(e^{\Delta(\mathcal{T}(M)-t)}f) \text{ for each } f \in L^2(\Omega),$$

one can easily check that

$$\begin{aligned}\Delta y(\mathcal{T}(M);0,u_\delta^*) &= \Delta \int_{\bar{k}\delta}^{\mathcal{T}(M)} e^{\Delta(\mathcal{T}(M)-t)}\chi_\omega u_\delta^*((\bar{k}+1)\delta)\,\mathrm{d}t + \sum_{j=1}^{\bar{k}} \Delta \int_{(j-1)\delta}^{j\delta} e^{\Delta(\mathcal{T}(M)-t)}\chi_\omega u_\delta^*(j\delta)\,\mathrm{d}t \\ &= \sum_{j=1}^{\bar{k}} e^{\Delta(\mathcal{T}(M)-j\delta)}\chi_\omega\big(u_\delta^*((j+1)\delta) - u_\delta^*(j\delta)\big) + e^{\Delta\mathcal{T}(M)}\chi_\omega u_\delta^*(\delta) - \chi_\omega u_\delta^*((\bar{k}+1)\delta).\end{aligned}$$

This yields that

$$\begin{aligned}&\|y(\mathcal{T}(M);0,u_\delta^*)\|_{H^2(\Omega)\cap H_0^1(\Omega)} = \|\Delta y(\mathcal{T}(M);0,u_\delta^*)\| \\ \le &\sum_{j=1}^{\bar{k}} \|u_\delta^*((j+1)\delta) - u_\delta^*(j\delta)\| + \|u_\delta^*(\delta)\| + \|u_\delta^*((\bar{k}+1)\delta)\|.\end{aligned} \tag{167}$$

Meanwhile, from the first equality in (163), one can easily verify that when $j = 1,\dots,\bar{k}$,

$$\begin{aligned}&\|u_\delta^*((j+1)\delta) - u_\delta^*(j\delta)\| \\ =&\Big\|\frac{1}{\delta}\int_{(j-1)\delta}^{j\delta} \chi_\omega\varphi(s+\delta;\mathcal{T}_\delta(M),z_\delta^*)\,\mathrm{d}s - \frac{1}{\delta}\int_{(j-1)\delta}^{j\delta} \chi_\omega\varphi(s;\mathcal{T}_\delta(M),z_\delta^*)\,\mathrm{d}s\Big\| \\ =&\Big\|\chi_\omega\frac{1}{\delta}\int_{(j-1)\delta}^{j\delta}\int_s^{s+\delta} \partial_\tau\varphi(\tau;\mathcal{T}_\delta(M),z_\delta^*)\,\mathrm{d}\tau\mathrm{d}s\Big\| \le \int_{(j-1)\delta}^{(j+1)\delta} \|\partial_\tau\varphi(\tau;\mathcal{T}_\delta(M),z_\delta^*)\|\,\mathrm{d}\tau.\end{aligned}$$

This, along with (167), (163), (164) and the contractivity of $\{e^{\Delta t}\}_{t\ge0}$, yields that

$$\|y(\mathcal{T}(M);0,u_\delta^*)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq 2 \int_0^{\mathcal{T}_\delta(M)} \|\partial_\tau \varphi(\tau;\mathcal{T}_\delta(M),z_\delta^*)\| \,\mathrm{d}\tau + 2\|z_\delta^*\|. \tag{168}$$

Since $\|\partial_\tau \varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \leq \|z_\delta^*\|_{H_0^1(\Omega)}$ (see (85)), by applying the Hölder inequality to (168), we obtain (166). This ends the proof of (165).

Step 2. To show that

$$\|\varphi(\cdot;\mathcal{T}(M),z_\delta^*) - \overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \leq \|z_\delta^*\|_{H_0^1(\Omega)}(|\mathcal{T}_\delta(M)-\mathcal{T}(M)|+\delta) \tag{169}$$

Observe that

$$\begin{aligned}
&\|\varphi(\cdot;\mathcal{T}(M),z_\delta^*) - \overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2((0,\mathcal{T}(M))\times\Omega)}\\
\leq &\|\varphi(\cdot;\mathcal{T}(M),z_\delta^*) - \varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2((0,\mathcal{T}(M))\times\Omega)}\\
&+\|\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*) - \overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{L^2((0,\mathcal{T}(M))\times\Omega)}\\
\triangleq &I_1 + I_2.
\end{aligned}$$

We first claim that

$$I_1 \leq \|z_\delta^*\|_{H_0^1(\Omega)}(\mathcal{T}_\delta(M)-\mathcal{T}(M)). \tag{170}$$

Write $\{\lambda_j\}_{j=1}^\infty$ for the family of all eigenvalues of $-\Delta$ with the zero Dirichlet boundary condition so that $\lambda_1 < \lambda_2 \leq \cdots$. Let $\{e_j\}_{j=1}^\infty$ be the family of the corresponding normalized eigenvectors. Write

$$z_\delta^* = \sum_{j=1}^\infty a_j e_j \quad \text{with} \quad \{a_j\}_{j=1}^\infty \subset \mathbb{R}.$$

From this, it follows that for each $t\in[0,\mathcal{T}(M)]$,

$$\varphi(t;\mathcal{T}(M),z_\delta^*) = \sum_{j=1}^\infty a_j e^{-\lambda_j(\mathcal{T}(M)-t)} e_j \quad \text{and} \quad \varphi(t;\mathcal{T}_\delta(M),z_\delta^*) = \sum_{j=1}^\infty a_j e^{-\lambda_j(\mathcal{T}_\delta(M)-t)} e_j.$$

This yields that

$$\begin{aligned}
I_1 &= \Big\| \sum_{j=1}^\infty a_j\lambda_j(\mathcal{T}_\delta(M)-\mathcal{T}(M))\Big(\int_0^1 e^{-s\lambda_j(\mathcal{T}_\delta(M)-\mathcal{T}(M)}\mathrm{d}s\Big)e^{-\lambda_j(\mathcal{T}(M)-t)}e_j\Big\|_{L^2((0,\mathcal{T}(M))\times\Omega)}\\
&\leq (\mathcal{T}_\delta(M)-\mathcal{T}(M))\Big(\sum_{j=1}^\infty a_j^2\lambda_j^2\Big)^{1/2} = \|z_\delta^*\|_{H_0^1(\Omega)}(\mathcal{T}_\delta(M)-\mathcal{T}(M)),
\end{aligned}$$

which leads to (170).

We next estimate I_2. Since $\mathcal{T}_\delta(M) = k_\delta\delta$ (see (154)) and because $\mathcal{T}_\delta(M)\geq\mathcal{T}(M)$, we see from (20) that

$$I_2^2 \leq \int_0^{\mathcal{T}_\delta(M)} \|\varphi(\cdot;\mathcal{T}_\delta(M),z_\delta^*) - \overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|^2\,\mathrm{d}t$$

$$= \sum_{j=1}^{k_\delta} \int\limits_{(j-1)\delta}^{j\delta} \left\| \varphi(t; \mathcal{T}_\delta(M), z_\delta^*) - \frac{1}{\delta} \int\limits_{(j-1)\delta}^{j\delta} \varphi(s; \mathcal{T}_\delta(M), z_\delta^*) \, \mathrm{d}s \right\|^2 \mathrm{d}t$$

$$\leq \sum_{j=1}^{k_\delta} \int\limits_{(j-1)\delta}^{j\delta} \left(\int\limits_{(j-1)\delta}^{j\delta} \| \partial_\tau \varphi(\tau; \mathcal{T}_\delta(M), z_\delta^*) \| \, \mathrm{d}\tau \right)^2 \mathrm{d}t.$$

By using the Hölder inequality in the above and by (85), we see that

$$I_2 \leq \| \partial_\tau \varphi(\cdot; \mathcal{T}_\delta(M), z_\delta^*) \|_{L^2((0,\mathcal{T}_\delta(M)) \times \Omega)} \delta \leq \| z_\delta^* \|_{H_0^1(\Omega)} \delta.$$

Finally, (169) follows from the above estimates on I_1 and I_2.

Step 3. To prove that

$$|\mathcal{T}(M) - \mathcal{T}_\delta(M)| + |M - M_\delta| \leq 2e^{C_3(1+\frac{1}{\mathcal{T}(M)})}(1 + \|y_0\|)\delta \triangleq C_1(M, y_0)\delta \tag{171}$$

By (159), we can use Theorem 5.3, with $(\delta, k) = (\delta, k_\delta)$ (where k_δ is given by (154)), to see that $\mathcal{N}_\delta(\mathcal{T}_\delta(M)) \geq \mathcal{N}(\mathcal{T}_\delta(M))$. By (162) and (164), we find that $M \leq M_\delta$. These, along with (ii) of Theorem 4.1, yield that

$$0 \leq M - M_\delta = \mathcal{N}(\mathcal{T}(M)) - \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq \mathcal{N}(\mathcal{T}(M)) - \mathcal{N}(\mathcal{T}_\delta(M)). \tag{172}$$

Meanwhile, by (158), we can use Theorem 5.2 (with $T_1 = \mathcal{T}(M)$ and $T_2 = \mathcal{T}_\delta(M)$) to see that

$$\mathcal{N}(\mathcal{T}(M)) - \mathcal{N}(\mathcal{T}_\delta(M)) \leq e^{C_3(\Omega,\omega)(1+1/\mathcal{T}(M))} \|y_0\| (\mathcal{T}_\delta(M) - \mathcal{T}(M)).$$

where $C_3(\Omega, \omega)$ is given by (106). The above, along with (172), yields that

$$|M - M_\delta| \leq e^{C_3(\Omega,\omega)(1+1/\mathcal{T}(M))} \|y_0\| |\mathcal{T}_\delta(M) - \mathcal{T}(M)|.$$

Since $\delta \in (0, \delta_0)$, the above, along with Theorem 1.3, leads to (171).

Step 4. To show that

$$\|\hat{z}^* - \hat{z}_\delta^*\| \leq C_2(M, y_0, r)\delta \tag{173}$$

Define an affiliated control $\hat{u}_\delta$ from $\mathbb{R}^+$ to $L^2(\Omega)$ by

$$\hat{u}_\delta(t) \triangleq M \frac{\chi_\omega \varphi(t; \mathcal{T}(M), \hat{z}_\delta^*)}{\|\chi_\omega \varphi(\cdot; \mathcal{T}(M), \hat{z}_\delta^*)\|_{L^2((0,\mathcal{T}(M)) \times \Omega)}}, \ t \in (0, \mathcal{T}(M)); \ \hat{u}_\delta(t) \triangleq 0, \ t \in [\mathcal{T}(M), \infty). \tag{174}$$

We divide the rest of the proof of Step 4 by several parts.

Part 4.1. To prove that

$$\langle \hat{z}^* - \hat{z}_\delta^*, \hat{z}^* - \hat{z}_\delta^* \rangle \leq -\frac{1}{r} \big\langle \hat{z}^* - \hat{z}_\delta^*, y(\mathcal{T}(M); y_0, \hat{u}_\delta) - y(\mathcal{T}_\delta(M); y_0, u_\delta^*) \big\rangle \tag{175}$$

By (161) and (174), one can directly check that

$$0 \leq \langle \chi_\omega \varphi(t; \mathcal{T}(M), \hat{z}^*) - \chi_\omega \varphi(t; \mathcal{T}(M), \hat{z}_\delta^*), u^*(t, \cdot) - \hat{u}_\delta(t, \cdot) \big\rangle \ \text{for a.e.} \ t \in (0, \mathcal{T}(M)).$$

Hence, we have that

$$0 \le \big\langle \chi_\omega \varphi(\cdot; \mathcal{T}(M), \hat{z}^*) - \chi_\omega \varphi(\cdot; \mathcal{T}(M), \hat{z}^*_\delta), u^* - \hat{u}_\delta \big\rangle_{L^2((0,\mathcal{T}(M))\times\Omega)}$$
$$= \big\langle \hat{z}^* - \hat{z}^*_\delta, y(\mathcal{T}(M); y_0, u^*) - y(\mathcal{T}(M); y_0, \hat{u}_\delta) \big\rangle.$$

This, along with (69), (72) and (160), yields that

$$\langle \hat{z}^* - \hat{z}^*_\delta, \hat{z}^* - \hat{z}^*_\delta \rangle = \Big\langle \hat{z}^* - \hat{z}^*_\delta, -\frac{1}{r}\big(y(\mathcal{T}(M); y_0, u^*) - y(\mathcal{T}_\delta(M); y_0, u^*_\delta)\big) \Big\rangle$$
$$\le -\frac{1}{r}\big\langle \hat{z}^* - \hat{z}^*_\delta, y(\mathcal{T}(M); y_0, \hat{u}_\delta) - y(\mathcal{T}_\delta(M); y_0, u^*_\delta) \big\rangle,$$

which leads to (175).

Part 4.2. To show that there exists $C_{21} \triangleq C_{21}(\Omega) > 0$ so that

$$\|y(\mathcal{T}(M); y_0, \hat{u}_\delta) - y(\mathcal{T}_\delta(M); y_0, u^*_\delta)\| \le C_{21}\Big[\big(1 + \frac{1}{\mathcal{T}(M)} + \sqrt{T^*}\big)(\|y_0\| + \|z^*_\delta\|_{H^1_0(\Omega)})$$
$$\times(\mathcal{T}_\delta(M) - \mathcal{T}(M)) + \|\hat{u}_\delta - u^*_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)}\Big] \tag{176}$$

Three facts are given in order. Fact one: By the Hölder inequality, we find that for some $C_{22} \triangleq C_{22}(\Omega) > 0$,

$$\|y(\mathcal{T}(M); y_0, \hat{u}_\delta) - y(\mathcal{T}(M); y_0, u^*_\delta)\| = \Big\| \int_0^{\mathcal{T}(M)} e^{\Delta(\mathcal{T}(M)-t)} \chi_\omega (\hat{u}_\delta - u^*_\delta)(t, \cdot)\, \mathrm{d}t \Big\|$$
$$\le \int_0^{\mathcal{T}(M)} e^{-\lambda_1(\mathcal{T}(M)-t)} \|(\hat{u}_\delta - u^*_\delta)(t, \cdot)\|\, \mathrm{d}t \le C_{22}\|\hat{u}_\delta - u^*_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)}; \tag{177}$$

Fact two: Since $\|u^*_\delta\|_{L^\infty(0,\mathcal{T}_\delta(M);L^2(\Omega))} \le \|z^*_\delta\|$ (which follows from (163) and (20)), and because $\mathcal{T}(M) \le \mathcal{T}_\delta(M)$ (see (158)), we find that

$$\|y(\mathcal{T}(M); y_0, u^*_\delta) - y(\mathcal{T}_\delta(M); y_0, u^*_\delta)\|$$
$$\le \|y(\mathcal{T}(M); y_0, u^*_\delta) - e^{\Delta(\mathcal{T}_\delta(M)-\mathcal{T}(M))} y(\mathcal{T}(M); y_0, u^*_\delta)\| + \Big\| \int_{\mathcal{T}(M)}^{\mathcal{T}_\delta(M)} e^{\Delta(\mathcal{T}_\delta(M)-t)} \chi_\omega u^*_\delta(t, \cdot)\, \mathrm{d}t \Big\|$$
$$\le \big(\mathcal{T}_\delta(M) - \mathcal{T}(M)\big)\Big[\|y(\mathcal{T}(M); y_0, u^*_\delta)\|_{H^2(\Omega)\cap H^1_0(\Omega)} + \|z^*_\delta\|\Big]. \tag{178}$$

Fact three: By (158), we see that

$$\mathcal{T}_\delta(M) < T^*. \tag{179}$$

Now, by the triangle inequality, (177), (165), (178), (179) and the Poincaré inequality, we obtain (176) for some $C_{21} = C_{21}(\Omega)$.

Part 4.3. To show that there exists $C_{23} \triangleq C_{23}(\Omega) > 1$ so that

$$\|\hat{u}_\delta - u^*_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \le C_{23}\big(1 + \|z^*_\delta\|^4_{H^1_0(\Omega)}\big)\big(1 + \frac{1}{M}\big)\big(|\mathcal{T}_\delta(M) - \mathcal{T}(M)| + \delta + |M - M_\delta|\big) \tag{180}$$

Recall (160) for the definition of $\hat{z}^*_\delta$. *In Part 4.3, we simply write respectively $\varphi(\cdot)$ and $\overline{\varphi}_\delta(\cdot)$ for $\varphi(\cdot; \mathcal{T}(M), \hat{z}^*_\delta)$ and $\overline{\varphi}_\delta(\cdot; \mathcal{T}_\delta(M), \hat{z}^*_\delta)$; simply write $\|\cdot\|_{0,\mathcal{T}(M)}$, $\|\cdot\|_{0,\mathcal{T}_\delta(M)}$ and $\|\cdot\|_{\mathcal{T}(M),\mathcal{T}_\delta(M)}$ for $\|\cdot$*

$\|_{L^2((0,\mathcal{T}(M))\times\Omega)}$, $\|\cdot\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}$ *and* $\|\cdot\|_{L^2((\mathcal{T}(M),\mathcal{T}_\delta(M))\times\Omega)}$, *respectively.* From (174) and (163), using the triangle inequality, we obtain that

$$\|\hat{u}_\delta - u_\delta^*\|_{0,\mathcal{T}(M)} = \Big\| M\frac{\chi_\omega\varphi}{\|\chi_\omega\varphi\|_{0,\mathcal{T}(M)}} - M_\delta\frac{\chi_\omega\overline{\varphi}_\delta}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}}\Big\|_{0,\mathcal{T}(M)}$$
$$\leq M_\delta\Big\|\frac{\chi_\omega\varphi}{\|\chi_\omega\varphi\|_{0,\mathcal{T}(M)}} - \frac{\chi_\omega\overline{\varphi}_\delta}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}}\Big\|_{0,\mathcal{T}(M)} + |M-M_\delta|. \tag{181}$$

By direct computations, we find that

$$\frac{\chi_\omega\varphi}{\|\chi_\omega\varphi\|_{0,\mathcal{T}(M)}} - \frac{\chi_\omega\overline{\varphi}_\delta}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}} \tag{182}$$
$$= \frac{\chi_\omega\varphi\big(\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)} - \|\chi_\omega\varphi\|_{0,\mathcal{T}(M)}\big)}{\|\chi_\omega\varphi\|_{0,\mathcal{T}(M)}\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}} + \frac{\chi_\omega\varphi - \chi_\omega\overline{\varphi}_\delta}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}};$$

$$\big|\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)} - \|\chi_\omega\varphi\|_{0,\mathcal{T}(M)}\big|$$
$$\leq \frac{\big|\|\chi_\omega\overline{\varphi}_\delta\|^2_{0,\mathcal{T}(M)} - \|\chi_\omega\varphi\|^2_{0,\mathcal{T}(M)}\big| + \|\chi_\omega\overline{\varphi}_\delta\|^2_{\mathcal{T}(M),\mathcal{T}_\delta(M)}}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)} + \|\chi_\omega\varphi\|_{0,\mathcal{T}(M)}}$$
$$\leq \|\chi_\omega\varphi - \chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}(M)} + \frac{\|\chi_\omega\overline{\varphi}_\delta\|^2_{\mathcal{T}(M),\mathcal{T}_\delta(M)}}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}}. \tag{183}$$

From (181), (182) and (183), we deduce that

$$\|\hat{u}_\delta - u_\delta^*\|_{0,\mathcal{T}(M)} \leq \frac{M_\delta}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}}\Big[2\|\chi_\omega\varphi - \chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}(M)} + \frac{\|\chi_\omega\overline{\varphi}_\delta\|^2_{\mathcal{T}(M),\mathcal{T}_\delta(M)}}{\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}}\Big]$$
$$+|M-M_\delta|. \tag{184}$$

Meanwhile, by (163) and (160), we see that $M_\delta = \|z_\delta^*\|\|\chi_\omega\overline{\varphi}_\delta\|_{0,\mathcal{T}_\delta(M)}$. This, together with (184) and (160), yields that

$$\|\hat{u}_\delta - u_\delta^*\|_{0,\mathcal{T}(M)} \leq \|z_\delta^*\|\Big[2\|\varphi(\cdot;\mathcal{T}(M),z_\delta^*) - \overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|_{0,\mathcal{T}(M)}$$
$$+\frac{\|z_\delta^*\|}{M_\delta}\|\chi_\omega\overline{\varphi}_\delta(\cdot;\mathcal{T}_\delta(M),z_\delta^*)\|^2_{\mathcal{T}(M),\mathcal{T}_\delta(M)}\Big] + |M-M_\delta|. \tag{185}$$

Since $\|\overline{\varphi}_\delta(t;\mathcal{T}_\delta(M),z_\delta^*)\| \leq \|z_\delta^*\|$ for each $t\in(0,\mathcal{T}_\delta(M))$ (which follows from (20)), we find from (185) and (169) that

$$\|\hat{u}_\delta - u_\delta^*\|_{0,\mathcal{T}(M)} \leq \Big(1 + 2\|z_\delta^*\|_{H_0^1(\Omega)}\|z_\delta^*\| + \frac{\|z_\delta^*\|^4}{M_\delta}\Big)\Big(|\mathcal{T}_\delta(M) - \mathcal{T}(M)| + \delta + |M-M_\delta|\Big).$$

At the same time, it follows from (171) and (157) that

$$M_\delta \geq M - 2e^{C_3(1+\frac{1}{\mathcal{T}(M)})}(1+\|y_0\|)\delta \geq M/2.$$

The above two inequalities, along with the Poincaré inequality, yield (180).

Part 4.4. To show (173)

By (176) and (180), we can easily check that

$$\begin{aligned}&\|y(\mathcal{T}(M);y_0,\hat{u}_\delta)-y(\mathcal{T}_\delta(M);y_0,u^*_\delta)\|\\&\le C_{21}C_{23}\left(1+\frac{1}{\mathcal{T}(M)}+\sqrt{T^*}\right)\left(1+\|y_0\|+\|z^*_\delta\|_{H^1_0(\Omega)}+\|z^*_\delta\|^4_{H^1_0(\Omega)}\right)\\&\quad\times\big(1+\frac{1}{M}\big)\times\big(|\mathcal{T}_\delta(M)-\mathcal{T}(M)|+|M_\delta-M|+\delta\big).\end{aligned}\tag{186}$$

Meanwhile, by (159), we can use Theorem 5.1 (with $(\delta,k)=(\delta,k_\delta)$, where k_δ is given by (154)), as well as (158), to get that

$$1+\|y_0\|+\|z^*_\delta\|_{H^1_0(\Omega)}+\|z^*_\delta\|^4_{H^1_0(\Omega)}\le 4e^{C_1(1+\frac{1}{\mathcal{T}(M)})}\|y_0\|^{24}r^{-20},\tag{187}$$

where $C_1=C_1(\Omega,\omega)$ is given by (85). This, along with (186), leads to that

$$\|y(\mathcal{T}(M);y_0,\hat{u}_\delta)-y(\mathcal{T}_\delta(M);y_0,u^*_\delta)\|\le\hat{C}_3\big(1+\frac{1}{M}\big)\big(|\mathcal{T}_\delta(M)-\mathcal{T}(M)|+|M_\delta-M|+\delta\big),\tag{188}$$

where

$$\hat{C}_3\triangleq\hat{C}_3(M,y_0,r)\triangleq 4C_{21}C_{23}\left(1+\frac{1}{\mathcal{T}(M)}+\sqrt{T^*}\right)e^{C_1(1+\frac{1}{\mathcal{T}(M)})}\|y_0\|^{24}r^{-20}.$$

Now it follows from (175) that

$$\|\hat{z}^*-\hat{z}^*_\delta\|^2\le\frac{1}{2}\|\hat{z}^*-\hat{z}^*_\delta\|^2+\frac{1}{2r^2}\|y(\mathcal{T}(M);y_0,\hat{u}_\delta)-y(\mathcal{T}_\delta(M);y_0,u^*_\delta)\|^2.$$

This, along with (188) and (171), yields (173) with

$$C_2(M,y_0,r)\triangleq\frac{1}{r}\Big[\hat{C}_3(M,y_0,r)\big(1+\frac{1}{M}\big)(C_1(M,y_0)+1)\Big],$$

which ends the proof of Step 4.

Step 5. To show that

$$\|u^*-u^*_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)}\le C_4(M,y_0,r)\delta\tag{189}$$

Recall (160) for the definitions of $\hat{z}^*_\delta$ and $\hat{z}^*$. *In Step 5, we simply write $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ for $\varphi(\cdot;\mathcal{T}(M),\hat{z}^*)$ and $\varphi(\cdot;\mathcal{T}(M),\hat{z}^*_\delta)$, respectively; simply write $\|\cdot\|_{0,\mathcal{T}(M)}$ for $\|\cdot\|_{L^2((0,\mathcal{T}(M))\times\Omega)}$.* By (161) and (174), we see that

$$\begin{aligned}\|u^*-\hat{u}_\delta\|_{0,\mathcal{T}(M)}&\le M\Big\|\frac{\chi_\omega\varphi_1}{\|\chi_\omega\varphi_1\|_{0,\mathcal{T}(M)}}-\frac{\chi_\omega\varphi_2}{\|\chi_\omega\varphi_2\|_{0,\mathcal{T}(M)}}\Big\|_{0,\mathcal{T}(M)}\\&\le\frac{2M}{\|\chi_\omega\varphi_1\|_{0,\mathcal{T}(M)}}\|\chi_\omega\varphi_1-\chi_\omega\varphi_2\|_{0,\mathcal{T}(M)}.\end{aligned}\tag{190}$$

Meanwhile, from (161) and (160), we find that

$$M=\|z^*\|\|\chi_\omega\varphi_1\|_{0,\mathcal{T}(M)}.$$

This, along with (190), yields that

$$\|u^*-\hat{u}_\delta\|_{0,\mathcal{T}(M)}\le 2\|z^*\|\|\varphi_1-\varphi_2\|_{0,\mathcal{T}(M)}\le 2\|z^*\|\|\hat{z}^*-\hat{z}^*_\delta\|.\tag{191}$$

By (191), (180) and the triangle inequality, then using (173), we see that

$$\|u^* - u^*_\delta\|_{0,\mathcal{T}(M)} \leq 2C_2(M, y_0, r)\|z^*\|\delta$$
$$+C_{23}(1 + \|z^*_\delta\|^4_{H^1_0(\Omega)})\big(|\mathcal{T}_\delta(M) - \mathcal{T}(M)| + \delta + |M - M_\delta|\big).$$

From this, (187) and (171), we obtain (189), with

$$C_4(M, y_0, r) \triangleq (1 + e^{C_1(1+T^*+\frac{1}{\mathcal{T}(M)})}\|y_0\|^{24}r^{-20})(8C_2 + C_{23}(1 + C_1)),$$

where $C_1 = C_1(M, y_0)$ is given by (171), $C_2 = C_2(M, y_0, r)$ is given by (173) and $C_{23} = C_{23}(\Omega)$ is given by (180). This ends the proof of Step 5.

In summary, we end the proof of the conclusion (i) in Theorem 1.4.

(ii) Arbitrarily fix $M > 0$ and $\eta > 0$. Let $\mathcal{A}_{M,\eta}$ be given by Theorem 5.4. Then, by Theorem 5.4, we see that

$$\lim_{h\to 0^+} \frac{1}{h}|\mathcal{A}_{M,\eta} \cap (0, h)| = \eta$$

and that for each $\delta \in \mathcal{A}_{M,\eta}$,

$$M - \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \geq \frac{1}{2}\lambda_1^{3/2} r(1 - \eta)\delta. \tag{192}$$

Arbitrarily fix $\delta \in \mathcal{A}_{M,\eta}$. Let u^*_M and $u^*_{M,\delta}$ be the optimal control to $(\mathcal{TP})^M$ and the optimal control optimal with the minimal norm to $(\mathcal{TP})^M_\delta$, respectively. (see Theorem 3.1.) Three facts are given in order: (a) By Theorem 4.1, one can easily check

$$\|u^*_M\|_{L^2((0,\mathcal{T}(M))\times\Omega)} = \mathcal{N}(\mathcal{T}(M)) = M;$$

(b) From (iii) of Theorem 3.1, we see that

$$\|u^*_{M,\delta}\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} = \mathcal{N}_\delta(\mathcal{T}_\delta(M));$$

(c) Since $\mathcal{U}^M_\delta \subset \mathcal{U}^M$, by (2) and (6), we find that $\mathcal{T}(M) \leq \mathcal{T}_\delta(M)$. Combining the above facts (a)-(c) with (192), we find that

$$\|u^*_M - u^*_{M,\delta}\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \geq \|u^*_M\|_{L^2((0,\mathcal{T}(M))\times\Omega)} - \|u^*_{M,\delta}\|_{L^2((0,\mathcal{T}(M))\times\Omega)}$$
$$\geq M - \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \geq \frac{1}{2}\lambda_1^{3/2} r(1 - \eta)\delta,$$

which leads to (11). Thus, the conclusion (ii) in Theorem 1.4 is true.

In summary, we end the proof of Theorem 1.4.

□

6.3. *The proof of Theorem 1.5*

This subsection devotes to the proof of Theorem 1.5. To show (13) in Theorem 1.5, we need the next lemma which gives a lower bound for the diameter of the subset $\mathcal{O}_{M,\delta}$ (in $L^2((0, \mathcal{T}(M)) \times \Omega)$), which is defined by (21).

Lemma 6.1 *Let $M > 0$. Then there is $\delta_M \triangleq \delta_M(y_0, r) > 0$ so that for each $\eta \in (0, 1)$ and $\delta \in \mathcal{A}_{M,\eta} \cap (0, \delta_M)$ (where $\mathcal{A}_{M,\eta}$ is given by Theorem 5.4),*

$$diam\,\mathcal{O}_{M,\delta} \triangleq \sup\{\|u_\delta - v_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \ : \ u_\delta,\, v_\delta \in \mathcal{O}_{M,\delta}\}$$
$$\geq \hat{C}_M\sqrt{(1-\eta)\delta} \quad \textit{for some} \quad \hat{C}_M \triangleq \hat{C}_M(y_0,r). \tag{193}$$

Proof. Arbitrarily fix $M > 0$. Let $\delta_0 > 0$ be given by (i) of Theorem 1.3. From (i) and (ii) of Theorem 4.1, we see that $0 < \mathcal{T}(M) < T^*$. Thus we can take a positive number δ_1 in the following manner:

$$\delta_1 \triangleq \min\{\delta_0, \mathcal{T}(M)/2, (T^* - \mathcal{T}(M))/2\}. \tag{194}$$

Arbitrarily fix $\eta \in (0,1)$ and $\delta \in \mathcal{A}_{M,\eta} \cap (0,\delta_1)$. From (194) and Theorem 1.3, we see that

$$2\delta < \mathcal{T}(M) \leq \mathcal{T}_\delta(M) \leq \mathcal{T}(M) + 2\delta < T^*. \tag{195}$$

Meanwhile, it follows from the second conclusion in (132) in Theorem 5.4 that

$$M_\delta \triangleq \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq M - \frac{1}{2}\lambda_1^{3/2} r(1-\eta)\delta. \tag{196}$$

To show (193), it suffices to find a subset $\mathcal{O}^2_{M,\delta} \subset \mathcal{O}_{M,\delta}$ so that for some $\hat{C}_M \triangleq C_M(y_0,r)$, $\hat{C}_M\sqrt{(1-\eta)\delta}$ is a lower bound for the "diam $\mathcal{O}^2_{M,\delta}$". To this end, we first introduce an affiliated subset $\mathcal{O}^1_{M,\delta} \subset \mathcal{O}_{M,\delta}$ in the following manner: Let u^*_δ be the optimal control with the minimal norm to $(\mathcal{TP})^M_\delta$ (see (iii) of Theorem 3.1). Arbitrarily fix $\hat{v}_\delta \in L^2_\delta((0,\mathcal{T}_\delta(M))\times\Omega)$ so that

$$\begin{cases} \mathtt{supp}\,\hat{v}_\delta \subset (0,\mathcal{T}(M))\times\Omega,\ \langle \hat{v}_\delta, u^*_\delta\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} = 0, \\ \|\hat{v}_\delta\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} = 1,\ \langle y(\mathcal{T}_\delta(M);0,u^*_\delta), y(\mathcal{T}_\delta(M);0,\hat{v}_\delta)\rangle \leq 0. \end{cases} \tag{197}$$

(The existence of such $\hat{v}_\delta$ can be easily verified.) Define $\mathcal{O}^1_{M,\delta}$ to be the set of all solutions u_δ to the following problem:

$$u_\delta = \alpha u^*_\delta + \beta\hat{v}_\delta,\ \alpha,\,\beta \in \mathbb{R};\ \ \|u_\delta\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \leq M;\ \ \|y(\mathcal{T}_\delta(M);y_0,u_\delta)\| \leq r. \tag{198}$$

From (198), we see that $\mathcal{O}^1_{M,\delta} \subset \mathcal{O}_{M,\delta}$.

We next characterize elements in $\mathcal{O}^1_{M,\delta}$ via studying the problem (198). To this end, we first claim

$$\begin{cases} \|u^*_\delta\|_{L^2(0,\mathcal{T}_\delta(M))} = \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \triangleq M_\delta, \\ \|y(\mathcal{T}_\delta(M);y_0,u^*_\delta)\| = r, \\ \langle y(\mathcal{T}_\delta(M);y_0,u^*_\delta), y(\mathcal{T}_\delta(M);0,u^*_\delta)\rangle = -\dfrac{rM_\delta^2}{\|z^*_\delta\|}, \\ \langle y(\mathcal{T}_\delta(M);y_0,u^*_\delta), y(\mathcal{T}_\delta(M);0,\hat{v}_\delta)\rangle = 0, \end{cases} \tag{199}$$

where z^*_δ denotes the minimizer of $(JP)^{\mathcal{T}_\delta(M)}_\delta$. Indeed, the first equality in (199) follows from (iii) of Theorem 3.1; To show the second one, two facts are given in order. Fact one: From (iii) of Theorem 3.1, we see that the restriction of u^*_δ over $(0,\mathcal{T}_\delta(M))$, denoted in the same manner, is an optimal control to $(\mathcal{NP})^{\mathcal{T}_\delta(M)}_\delta$. Fact two: By (195) and the definition of $\mathcal{P}_{T^*}$ (given by (67)), we find that $(\delta, \mathcal{T}_\delta(M)/\delta) \in \mathcal{P}_{T^*}$. By these two facts, we can use (72) in Theorem 4.3 (with $(\delta,k) = (\delta,\mathcal{T}_\delta(M)/\delta)$) to obtain the second equality in (199); To show the third equality in (199), we recall the above two facts. Then we can apply (ii) in Theorem 4.3 (with $(\delta,k) = (\delta,\mathcal{T}_\delta(M)/\delta)$) to get that

$$
\begin{aligned}
&\langle y(\mathcal{T}_\delta(M); y_0, u_\delta^*), y(\mathcal{T}_\delta(M); 0, u_\delta^*)\rangle = \Big\langle -r\frac{z_\delta^*}{\|z_\delta^*\|}, y(\mathcal{T}_\delta(M); 0, u_\delta^*)\Big\rangle \\
= &-\frac{r}{\|z_\delta^*\|}\langle \chi_\omega \varphi(\cdot; \mathcal{T}_\delta(M), z_\delta^*), u_\delta^*\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \\
= &-\frac{r}{\|z_\delta^*\|}\langle \chi_\omega \overline{\varphi}_\delta(\cdot; \mathcal{T}_\delta(M), z_\delta^*), u_\delta^*\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} = -\frac{r}{\|z_\delta^*\|}\|u_\delta^*\|^2_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}.
\end{aligned} \tag{200}
$$

(The first equality on the last line of (200) is obtained by the same way as that used to show (82).) Then the third equality in (199) follows from (200) and the first equality in (199) at once; To show the last equality in (199), we still recall the above two facts (given in the proof of the second equality in (199)). Then we can apply (ii) in Theorem 4.3 (with $(\delta, k) = (\delta, \mathcal{T}_\delta(M)/\delta)$) to see that

$$
\begin{aligned}
&\langle y(\mathcal{T}_\delta(M); y_0, u_\delta^*), y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta)\rangle = \Big\langle -r\frac{z_\delta^*}{\|z_\delta^*\|}, y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta)\Big\rangle \\
= &-\frac{r}{\|z_\delta^*\|}\langle \chi_\omega \varphi(\cdot; \mathcal{T}_\delta(M), z_\delta^*), \hat{v}_\delta\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \\
= &-\frac{r}{\|z_\delta^*\|}\langle \chi_\omega \overline{\varphi}_\delta(\cdot; \mathcal{T}_\delta(M), z_\delta^*), \hat{v}_\delta\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} = -\frac{r}{\|z_\delta^*\|}\langle u_\delta^*, \hat{v}_\delta\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}.
\end{aligned} \tag{201}
$$

(The first equality on the last line in (201) is obtained by the same way as that used to show (82).) From (201) and (197), we are led to the last equality in (199). Hence, (199) has been proved.

With the aid of (199), we can characterize elements u_δ of $\mathcal{O}^1_{M,\delta}$ as follows:

$$
\begin{cases}
u_\delta = \alpha u_\delta^* + \beta \hat{v}_\delta, \ \alpha, \ \beta \in \mathbb{R}, \\
\alpha^2 M_\delta^2 + \beta^2 \le M^2, \\
a_\delta^2\beta^2 + 2(\alpha-1)b_\delta\beta \le 2(\alpha-1)\dfrac{rM_\delta^2}{\|z_\delta^*\|} - (\alpha-1)^2 c_\delta^2,
\end{cases} \tag{202}
$$

where the pair $(a_\delta, b_\delta, c_\delta)$ is given by

$$
\begin{cases}
a_\delta \triangleq \|y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta)\|, \\
b_\delta \triangleq \langle y(\mathcal{T}_\delta(M); 0, u_\delta^*), y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta)\rangle, \\
c_\delta \triangleq \|y(\mathcal{T}_\delta(M); 0, u_\delta^*)\|.
\end{cases} \tag{203}
$$

Indeed, for each $u_\delta = \alpha u_\delta^* + \beta\hat{v}_\delta$, with $\alpha, \beta \in \mathbb{R}$, we have that

$$
y(\mathcal{T}_\delta(M); y_0, u_\delta) = y(\mathcal{T}_\delta(M); y_0, u_\delta^*) + (\alpha-1)y(\mathcal{T}_\delta(M); y_0, u_\delta^*) + \beta y(\mathcal{T}_\delta(M); 0, \hat{v}_\delta).
$$

Thus, from (199), (197) and (198), we can easily verify that u_δ is a solution to the problem (198) if and only if u_δ is a solution to the problem (202).

We now on the position to introduce the desired subset $\mathcal{O}^2_{M,\delta}$. Define a number $\hat{\lambda}$ by

$$
\hat{\lambda} \triangleq \min\left\{\frac{rM_\delta^3}{\|z_\delta^*\|c_\delta^2(M-M_\delta)}, \frac{1}{2}\right\}. \tag{204}
$$

(Notice that since $\delta \in \mathcal{A}_{M,\eta}$, it follows from (132) in Theorem 5.4 that $M > M_\delta$.) Let $\mathcal{O}^2_{M,\delta}$ be the set of solutions u_δ to the following problem:

$$\begin{cases} u_\delta = \alpha u_\delta^* + \beta \hat{v}_\delta,\ \alpha = 1 + \hat{\lambda}\dfrac{M - M_\delta}{M_\delta},\ \beta > 0, \\ \beta^2 \leq M(1-\hat{\lambda})(M - M_\delta), \\ a_\delta^2 \beta^2 \leq \dfrac{\hat{\lambda} r M_\delta (M - M_\delta)}{\|z_\delta^*\|}. \end{cases} \tag{205}$$

We claim that

$$\mathcal{O}_{M,\delta}^2 \subset \mathcal{O}_{M,\delta}^1 \subset \mathcal{O}_{M,\delta}. \tag{206}$$

Since the second conclusion in (206) has been proved, we only need to show the first one. Arbitrarily fix $\hat{u}_\delta \triangleq \hat{\alpha} u_\delta^* + \hat{\beta}\hat{v}_\delta \in \mathcal{O}_{M,\delta}^2$. We will show that $(\hat{u}_\delta, \hat{\alpha}, \hat{\beta})$ satisfies (202). Since $\hat{\lambda} \in (0,1)$ and $M > M_\delta$ (see (204) and (196)), it follows from (205) that

$$\begin{aligned} \hat{\alpha}^2 M_\delta^2 + \hat{\beta}^2 &\leq \big(M_\delta + \hat{\lambda}(M - M_\delta)\big)^2 + M(1-\hat{\lambda})(M - M_\delta) \\ &= M^2 - (1-\hat{\lambda})(M - M_\delta)\big(M - (1-\hat{\lambda})(M - M_\delta)\big) \leq M^2. \end{aligned}$$

Meanwhile, since $b_\delta \leq 0$ (see (203) and (197)), we find from (205) and (204) that

$$\begin{aligned} a_\delta^2 \hat{\beta}^2 + 2(\alpha - 1) b_\delta \hat{\beta} \leq a_\delta^2 \hat{\beta}^2 &\leq \frac{\hat{\lambda} r M_\delta (M - M_\delta)}{\|z_\delta^*\|} \\ &= (\hat{\alpha} - 1)\frac{r M_\delta^2}{\|z_\delta^*\|} \leq 2(\hat{\alpha} - 1)\frac{r M_\delta^2}{\|z_\delta^*\|} - (\hat{\alpha} - 1)^2 c_\delta^2. \end{aligned}$$

From these, we see that $(\hat{u}_\delta, \hat{\alpha}, \hat{\beta})$ verifies (202). Hence, (206) is true. By (206), (205) and (197), we find that

$$\operatorname{diam} \mathcal{O}_{M,\delta} \geq \sup\{\|u_\delta - u_\delta^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} : u_\delta \in \mathcal{O}_{M,\delta}^2\} \geq \beta \|\hat{v}_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)} = \beta, \tag{207}$$

when β satisfies that

$$0 < \beta^2 \leq \min\Big\{M(1-\hat{\lambda})(M - M_\delta), \frac{\hat{\lambda} r M_\delta (M - M_\delta)}{a_\delta^2 \|z_\delta^*\|}\Big\}.$$

(Here, we agree that $\frac{1}{0} \triangleq \infty$.) Then by (207) and (204), we get that

$$\operatorname{diam} \mathcal{O}_{M,\delta} \geq C_{M,\delta} \min\{\sqrt{M - M_\delta}, 1\}, \tag{208}$$

where $C_{M,\delta}$ is defined by

$$C_{M,\delta} \triangleq C_{M,\delta}(y_0, r) \triangleq \min\Big\{\sqrt{\frac{M}{2}}, \frac{r M_\delta^2}{a_\delta c_\delta \|z_\delta^*\|}, \sqrt{\frac{r M_\delta}{2 a_\delta^2 \|z_\delta^*\|}}\Big\}. \tag{209}$$

To get a lower bound of $C_{M,\delta}$ w.r.t. δ, we first present the following inequalities (their proofs will be given at the end of the proof of this lemma):

$$a_\delta \leq \frac{1}{\sqrt{2\lambda_1}};\ c_\delta \leq \frac{M_\delta}{\sqrt{2\lambda_1}};\ M_\delta \geq M - 2e^{C_3(1+\frac{1}{\mathcal{T}(M)})}\|y_0\|\delta;\ \|z_\delta^*\| \leq e^{C_1(1+\frac{1}{\mathcal{T}(M)})}\|y_0\|^4 r^{-3}, \tag{210}$$

where C_3 and C_1 are given by Theorem 5.2 and (i) of Theorem 5.1, respectively. We next define

$$\delta_M \triangleq \delta_M(y_0,r) \triangleq \min\left\{\delta_1, \frac{1}{4}Me^{-C_3(1+\frac{1}{\mathcal{T}(M)})}\|y_0\|^{-1}, \lambda_1^{-3/2}r^{-1}\right\}, \tag{211}$$

where δ_1 is given by (194). From (210) and (211), we have that

$$M_\delta \geq M/2 \;\text{ for each }\; \delta \in \mathcal{A}_{M,\eta} \cap (0, \delta_M).$$

This, along with (209) and (210), yields that for each $\delta \in \mathcal{A}_{M,\eta} \cap (0,\delta_M)$,

$$C_{M,\delta} \geq \min\Big\{\sqrt{\frac{M}{2}}, \frac{2\lambda_1 r M_\delta}{\|z_\delta^*\|}, \sqrt{\frac{\lambda_1 r M_\delta}{\|z_\delta^*\|}}\Big\} \geq \min\Big\{\sqrt{\frac{M}{2}}, \frac{\lambda_1 r M}{\|z_\delta^*\|}, \sqrt{\frac{\lambda_1 r M}{2\|z_\delta^*\|}}\Big\}.$$

By this and the last inequality in (210), we can find $C'_M \triangleq C'_M(y_0,r) > 0$ so that $C_{M,\delta} \geq C'_M$, when $\delta \in \mathcal{A}_{M,\eta} \cap (0,\delta_M)$. (*Hence, C'_M is a lower bound for $C_{M,\delta}$ w.r.t. δ.*) This, along with (208), (196) and (211), yields that for each $\delta \in \mathcal{A}_{M,\eta} \cap (0,\delta_M)$,

$$\operatorname{diam} \mathcal{O}_{M,\delta} \geq C'_M \min\Big\{\sqrt{\frac{1}{2}\lambda_1^{3/2} r(1-\eta)\delta}, 1\Big\} = \Big(\frac{1}{\sqrt{2}}C'_M \lambda_1^{3/4}\sqrt{r}\Big)\sqrt{(1-\eta)\delta}.$$

By the above and (211), we obtain (193), with $\hat{C}_M = \frac{1}{\sqrt{2}}C'_M\lambda_1^{3/4}\sqrt{r}$.

Finally, we show (210). By the Hölder inequality, (203) and (197), we find that

$$a_\delta \leq \int_0^{\mathcal{T}_\delta(M)} \|e^{\Delta(\mathcal{T}_\delta(M)-t)}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\|\hat{v}_\delta(t,\cdot)\|\,\mathrm{d}t \leq \int_0^{\mathcal{T}_\delta(M)} e^{-\lambda_1(\mathcal{T}_\delta(M)-t)}\|\hat{v}_\delta(t,\cdot)\|\,\mathrm{d}t \leq 1/\sqrt{2\lambda_1}.$$

Similarly, from (203) and (199), we can obtain the estimate for c_δ in (210). We now show the third inequality in (210). By (195) and the definition of $\mathcal{P}_{T^*}$ (given by (67)), we have that

$$0 < \mathcal{T}(M) < T^*, \;\; (\delta, \mathcal{T}_\delta(M)/\delta) \in \mathcal{P}_{T^*} \;\text{ and }\; 0 < \mathcal{T}_\delta(M) - \mathcal{T}(M) < 2\delta. \tag{212}$$

From the first two conclusions in (212), we can apply (ii) of Theorem 4.1 and the first inequality in (123) in Theorem 5.3 (with $(\delta,k) = (\delta, \mathcal{T}_\delta(M)/\delta)$), as well as (196), to get that

$$M - M_\delta = \mathcal{N}(\mathcal{T}(M)) - \mathcal{N}_\delta(\mathcal{T}_\delta(M)) \leq \mathcal{N}(\mathcal{T}(M)) - \mathcal{N}(\mathcal{T}_\delta(M)). \tag{213}$$

From (213), the second inequality in (106) in Theorem 5.2, with $T_1 = \mathcal{T}(M)$ and $T_2 = \mathcal{T}_\delta(M)$, (Notice that $\mathcal{T}_\delta(M) > \mathcal{T}(M)$.) and the last inequality in (212), we can easily derive the last inequality in (210). Hence, (210) is true. This ends the proof of Lemma 6.1.

□

We are now on the position to prove Theorem 1.5.

Proof of Theorem 1.5. Let $M > 0$. For each $\delta > 0$, we let u^* and u^*_δ be the optimal control and the optimal control with the minimal norm to $(\mathcal{TP})^M$ and $(\mathcal{TP})^M_\delta$ respectively (see Theorem 3.1). We will prove the conclusions (i)-(ii) of Theorem 1.5 one by one.

(i) For each $\delta > 0$, there are only two possibilities: either (155) or (157) holds. In the case when δ verifies (155), we can obtain (12) by the similar way to that used to show (10). We next consider the case that δ satisfies (157). Recall (21) for the subset $\mathcal{O}_{M,\delta}$ (which consists of all optimal controls to $(\mathcal{TP})^M_\delta$). Then it follows from Definition 1.2 that

$$\|u_\delta^*\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \leq \|v_\delta\|_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \leq M \ \text{ for each } \ v_\delta \in \mathcal{O}_{M,\delta}. \tag{214}$$

Arbitrarily fix $v_\delta \in \mathcal{O}_{M,\delta}$. One can directly check that

$$\lambda v_\delta + (1-\lambda)u_\delta^* \in \mathcal{O}_{M,\delta} \ \text{ for each } \ \lambda \in (0,1).$$

From this and (214), we find that for each $\lambda \in (0,1)$,

$$\begin{aligned}\|u_\delta^*\|^2_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} &\leq \|\lambda(v_\delta - u_\delta^*) + u_\delta^*\|^2_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \\ &= \|u_\delta^*\|^2_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} + 2\lambda\langle v_\delta - u_\delta^*, u_\delta^*\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \\ &\quad + \lambda^2\|v_\delta - u_\delta^*\|^2_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}.\end{aligned}$$

Dividing the above by λ and then sending $\lambda \to \infty$, we obtain that

$$\langle u_\delta^*, u_\delta^*\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \leq \langle v_\delta, u_\delta^*\rangle_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)}.$$

From this, (214) and (163) (as well as (164)), one can directly check that

$$\|v_\delta - u_\delta^*\|^2_{L^2((0,\mathcal{T}_\delta(M))\times\Omega)} \leq 2M(M - \mathcal{N}_\delta(\mathcal{T}_\delta(M))) \triangleq 2M(M - M_\delta). \tag{215}$$

(Here, we used the fact that $M \geq \mathcal{N}_\delta(\mathcal{T}_\delta(M))$, which follows from (iii) of Theorem 3.1.) Hence, from (189), (215) and (171), we find that

$$\begin{aligned}&\|u^* - u_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \leq \|u^* - u_\delta^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} + \|u_\delta^* - u_\delta\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \\ &\leq \left[C_4(M,y_0,r) + 2MC_1(M,y_0)\right]\delta \triangleq C_5(M,y_0,r)\delta,\end{aligned}$$

where $C_1(M,y_0)$ and $C_4(M,y_0,r)$ are respectively given by (171) and (189). This ends the proof of the conclusion (i) of Theorem 1.5.

(ii) We mainly use Lemma 6.1 to prove (13). Arbitrarily fix $\eta \in (0,1)$. Let $\mathcal{A}_{M,\eta}$ be given by Theorem 5.4. Let $\hat{C}_M$ and δ_M be given by Lemma 6.1. Arbitrarily fix $\delta \in \mathcal{A}_{M,\eta} \cap (0,\delta_M)$. We claim that there is $\hat{u}_{M,\delta} \in \mathcal{O}_{M,\delta}$ so that

$$\|\hat{u}_{M,\delta} - u_\delta^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \geq \hat{C}_M\sqrt{(1-\eta)\delta}/3. \tag{216}$$

By contradiction, we suppose that it were not true. Then we would find that

$$\|v_\delta - u_\delta^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \leq \hat{C}_M\sqrt{(1-\eta)\delta}/3, \ \forall\, v_\delta \in \mathcal{O}_{M,\delta}.$$

This, along with the definition of $\mathcal{O}_{M,\delta}$ (see (193)), implies that

$$\begin{aligned}\operatorname{diam}\mathcal{O}_{M,\delta} &\leq \sup\{\|v_\delta^1 - v_\delta^2\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \ : \ v_\delta^1,\, v_\delta^2 \in \mathcal{O}_{M,\delta}\} \\ &\leq \sup\{2\|v_\delta - u_\delta^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \ : \ v_\delta \in \mathcal{O}_{M,\delta}\} \\ &\leq 2\hat{C}_M\sqrt{(1-\eta)\delta}/3,\end{aligned}$$

which contradicts Lemma 6.1. Thus, (216) is true.

Now, we arbitrarily fix $\hat{u}_{M,\delta} \in \mathcal{O}_{M,\delta}$ satisfying (216). Then by (216) and by (i) of Theorem 1.4, there is $C(M,y_0,r) > 0$ so that

$$\begin{aligned}\|\hat{u}_{M,\delta} - u^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} &\geq \|\hat{u}_{M,\delta} - u_\delta^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} - \|u_\delta^* - u^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \\ &\geq \hat{C}_M\sqrt{(1-\eta)\delta}/3 - C(M,y_0,r)\delta.\end{aligned} \tag{217}$$

Write

$$\delta_{M,\eta} \triangleq \min\left\{\delta_M, \left(\hat{C}_M/(6C(M,y_0,r))\right)^2(1-\eta)\right\}; \quad \hat{\mathcal{A}}_{M,\eta} \triangleq \mathcal{A}_{M,\eta} \cap (0, \delta_{M,\eta}). \tag{218}$$

Then, one can easily check that

$$\lim_{h\to 0^+} \frac{1}{h}|\hat{\mathcal{A}}_{M,\eta} \cap (0,h)| = \eta.$$

From (217), and (218), one can easily verify that

$$\|\hat{u}_{M,\delta} - u^*\|_{L^2((0,\mathcal{T}(M))\times\Omega)} \geq \hat{C}_M\sqrt{(1-\eta)\delta}/6 \ \text{ for each } \ \delta \in \hat{\mathcal{A}}_{M,\eta},$$

which leads to (13), with $C_M \triangleq \hat{C}_M/6$. This ends the proof of Theorem 1.5.

□

6.4. *Further discussions on the main results*

From (ii) of Theorem 1.3, we see that when $\delta \in \mathcal{A}_{M,\eta}$, $\mathcal{T}_\delta(M) - \mathcal{T}(M)$ has a lower bound $(1-\eta)\delta$. The next Theorem 6.2 tells us that when $\delta \notin \mathcal{A}_{M,\eta}$, $(1-\eta)\delta$ will not be a lower bound for $\mathcal{T}_\delta(M) - \mathcal{T}(M)$.

Theorem 6.2 *Let $M > 0$. Then there is $k_0 \in \mathbb{N}^+$ and $\{\delta_k\}_{k=k_0}^\infty \subset \mathbb{R}^+$, with $\lim_{k\to\infty}\delta_k = 0$, so that when $k \geq k_0$,*

$$\mathcal{T}_{\delta_k}(M) - \mathcal{T}(M) = C_M\delta_k^2 \ \text{ for some } \ C_M \triangleq C_M(y_0,r). \tag{219}$$

Proof. Arbitrarily fix $\eta \in (0,1)$. Let

$$\hat{k}_0 \triangleq 4a\mathcal{T}(M), \ \text{ with } \ a \triangleq a(M,y_0,r) \triangleq 2\lambda_1^{-3/2}e^{C_4\left[1+T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{12}r^{-12}. \tag{220}$$

We define a sequence $\{\delta_k\}_{k=k_0}^\infty$ of $\mathbb{R}^+$ in the following manner:

$$\delta_k \triangleq \frac{2\mathcal{T}(M)}{(k+1)+\sqrt{(k+1)^2-4a\mathcal{T}(M)}}, \ k \geq \hat{k}_0. \tag{221}$$

One can easily check that

$$\delta_k \in (0,1/a) \ \text{ and } \ (k+1)\delta_k - a\delta_k^2 = \mathcal{T}(M) \ \text{ for all } \ k \geq \hat{k}_0. \tag{222}$$

We now claim that there exists $k_0 \geq \hat{k}_0$ so that

$$M \geq \mathcal{N}_{\delta_k}((k+1)\delta_k) + \frac{1}{2}\lambda_1^{3/2}ra\delta_k^2 \ \text{ for all } \ k \geq k_0. \tag{223}$$

In fact, by (221), we can choose $\hat{k}_1 \geq k_0$ large enough so that

$$0 < \delta_k < \min\{\mathcal{T}(M)/2, (T^*-\mathcal{T}(M))/2\}, \ \text{ when } \ k \geq \hat{k}_1. \tag{224}$$

Arbitrarily fix $k \geq \hat{k}_1$. Since $\mathcal{T}(M) < T^*$ (see (iii) in Theorem 4.1), from (224) and (222), we can easily check that

$$2\delta_k < \mathcal{T}(M) < (k+1)\delta_k < \mathcal{T}(M) + \delta_k < (T^*+\mathcal{T}(M))/2 < T^*.$$

These, along with the definition of $\mathcal{P}_{T^*}$ (given by (67)), yield that

$$2\delta_k < \mathcal{T}(M) < (k+1)\delta_k < T^* \;\text{ and }\; (\delta_k, k+1) \in \mathcal{P}_{T^*}. \tag{225}$$

By (225), we can apply Theorem 5.3 (see the second inequality in (123), where (δ, k) is replaced by $(\delta, k+1)$) and Theorem 5.2 (see the first inequality in (106), with $T_1 = \mathcal{T}(M)$ and $T_2 = (k+1)\delta_k$) to obtain that

$$\begin{aligned}\mathcal{N}_{\delta_k}((k+1)\delta_k) \leq & \mathcal{N}((k+1)\delta_k) + e^{C_4\left[1+T^*+\frac{1}{(k+1)\delta_k}+\frac{1}{T^*-(k+1)\delta_k}\right]}\|y_0\|^{12}r^{-11}\delta_k^2 \\ \leq & \mathcal{N}(\mathcal{T}(M)) - \lambda_1^{3/2}r\big((k+1)\delta_k - \mathcal{T}(M)\big) + \\ & e^{C_4\left[1+T^*+\frac{1}{(k+1)\delta_k}+\frac{1}{T^*-(k+1)\delta_k}\right]}\|y_0\|^{12}r^{-11}\delta_k^2,\end{aligned} \tag{226}$$

where $C_4 \triangleq C_4(\Omega, \omega)$ is given by (123). Meanwhile, by (222) and (224), we find that

$$(k+1)\delta_k - \mathcal{T}(M) = a\delta_k^2 \;\text{ and }\; \mathcal{T}(M) < (k+1)\delta_k < (T^* + \mathcal{T}(M))/2.$$

These, along with (226) and (ii) of Theorem 4.1, yield that

$$\begin{aligned}\mathcal{N}_{\delta_k}((k+1)\delta_k) \leq & \mathcal{N}(\mathcal{T}(M)) - \lambda_1^{3/2}ra\delta_k^2 + e^{C_4\left[1+T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{12}r^{-11}\delta_k^2 \\ = & M - \lambda_1^{3/2}ra\delta_k^2 + e^{C_4\left[1+T^*+\frac{1}{\mathcal{T}(M)}+\frac{2}{T^*-\mathcal{T}(M)}\right]}\|y_0\|^{12}r^{-11}\delta_k^2.\end{aligned}$$

This, together with (220), leads to (223), with $k_0 = \hat{k}_1$.

Next, we arbitrarily fix $k \geq k_0 \triangleq \hat{k}_1$. Let u_{δ_k} be an admissible control to $(\mathcal{NP})_{\delta_k}^{(k+1)\delta_k}$. Let $\tilde{u}_{\delta_k}$ be the zero extension of u_{δ_k} over $\mathbb{R}^+ \times \Omega$. Then by (223), one can easily check that $\tilde{u}_{\delta_k}$ is an admissible control (to $(\mathcal{TP})_{\delta_k}^M$), which drives the solution to $B_r(0)$ at time $(k+1)\delta_k$. This, along with the optimality of $\mathcal{T}_\delta(M)$, yields that

$$\mathcal{T}_{\delta_k}(M) \leq (k+1)\delta_k. \tag{227}$$

Meanwhile, Since $\mathcal{U}_{\delta_k}^M \subset \mathcal{U}^M$, we find from (2) and (6) that $\mathcal{T}(M) \leq \mathcal{T}_{\delta_k}(M)$. From this and (222), we get that

$$\mathcal{T}_{\delta_k}(M) \geq k\delta_k + \delta_k(1 - a\delta_k) > k\delta_k. \tag{228}$$

Since $\mathcal{T}_{\delta_k}(M)$ is a multiple of δ_k (see (7)), from (227) and (228), we obtain that

$$\mathcal{T}_{\delta_k}(M) = (k+1)\delta_k.$$

This, along with (222) and (220), yields (219), with $C_M = a(M, y_0, r)$ and with k_0 given by (223). Thus, we end the proof of Theorem 6.2.

□

Remark 6.3 (i) The above theorem implies that the following conclusion is not true: For each $M > 0$*, there exists* $\delta_1 > 0$ *and* $C > 0$ *so that*

$$|\mathcal{T}_\delta(M) - \mathcal{T}(M)| \geq C\delta \;\text{ for each }\; \delta \in (0, \delta_1).$$

(ii) We think of that the similar result to that in Theorem 1.3 can be obtained for optimal controls. But it seems for us that the corresponding proof will be more complicated.

7. Appendix

The next lemma is a copy of [36, Lemma 5.1] without proof.

Lemma 7.1 ([36], Lemma 5.1) *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let X, Y and Z be three Banach spaces over $\mathbb{K}$, with their dual spaces X^*, Y^* and Z^*. Let $R \in \mathcal{L}(Z,X)$ and $O \in \mathcal{L}(Z,Y)$. Then the following two propositions are equivalent:*
(i) There exists $\widehat{C}_0 > 0$ and $\hat{\varepsilon}_0 > 0$ so that for each $z \in Z$,

$$\|Rz\|_X^2 \leq \widehat{C}_0 \|Oz\|_Y^2 + \hat{\varepsilon}_0 \|z\|_Z^2. \tag{229}$$

(ii) There is $C_0 > 0$ and $\varepsilon_0 > 0$ so that for each $x^* \in X^*$, there is $y^* \in Y^*$ satisfying that

$$\frac{1}{C_0}\|y^*\|_{Y^*}^2 + \frac{1}{\varepsilon_0}\|R^*x^* - O^*y^*\|_{Z^*}^2 \leq \|x^*\|_{X^*}^2. \tag{230}$$

Furthermore, when one of the above two propositions holds, the pairs (C_0, ε_0) and $(\widehat{C}_0, \hat{\varepsilon}_0)$ can be chosen to be the same.

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