MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS SCHRÖDINGER-POISSON EQUATIONS WITH SIGN-CHANGING POTENTIAL

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Abstract. In this paper, we study the following nonhomogeneous Schrödinger-Poisson equations

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = f(x,u) + g(x), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter. Under some suitable assumptions on V, K, f and g, the existence of multiple solutions is proved by using the Ekeland's variational principle and the Mountain Pass Theorem in critical point theory. In particular, the potential V is allowed to be sign-changing.

1. Introduction and main results

In this paper we consider the following Schrödinger-Poisson equations

$$(SP)_\lambda \begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = f(x,u) + g(x), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\lambda \geq 1$ is a parameter, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Such a system, also called Schrödinger-Maxwell equations, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charged particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger's and Poisson's equations (we refer reader to [6, 23] and the references therein for more details on the physical aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve system $(SP)_\lambda$.

Variational methods and critical point theory are powerful tools in studying nonlinear differential equations [19, 25, 33], and in particular Hamiltonian system [26, 27], and also impulsive Hamiltonian systems [21]. In recent years, $(SP)_\lambda$ has been studied widely via modern variational methods under various hypotheses on

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the potential V and the nonlinear term f, see [3, 4, 6, 10, 13, 23, 31, 39] and the references therein. We recall some of them as follows.

The case of $g \equiv 0$, that is the homogeneous case, has been studied widely in [3, 8, 9, 12, 13, 18, 20, 23, 24] when V is a constant or radially symmetric, and in [31, 40] when V is not radially symmetric. Very recently, Azzollni and Pomponio in [4] proved the existence of a ground state solution for the following system

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

with $f(x,u) = |u|^{p-2}u(2 < p < 6)$ and non-constant potential V which may be unbounded below; When $V(x)$ and $f(x,u)$ are 1-periodic in each $x_i, i = 1,2,3$, Zhao et al. [39] obtained the existence of infinitely many geometrically distinct solutions by using the nonlinear superposition principle established in [1]. Zhao et al. [41] considered the existence of nontrivial solution and concentration results of $(SP)_\lambda$ provided that V satisfies:

(V_0) There is $b > 0$ such that $meas\{x \in \mathbb{R}^3 : V(x) \le b\} < +\infty$, where $meas$ denotes the Lebesgue measures;

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and V is bounded below;

(V_2) $\Omega = intV^{-1}(0)$ is nonempty and has smooth boundary and $\overline{\Omega} = V^{-1}(0)$.

This kind of hypotheses was first introduced by Bartsch and Wang [5] in the study of a nonlinear Schrödinger equation and the potential $V(x)$ with V satisfying $(V0)$–$(V2)$ is referred as the steep well potential.

Sun, Su and Zhao [29] got infinitely many solutions under suitable assumptions. Wu [35] studied the combined effect of concave and convex nonlinearities on the number of solutions for a semi-linear elliptic equation. For more results on the effect of concave and convex terms of elliptic equations see [36, 37] and the reference therein. In 2014, Ye and Tang [38] studied the existence and multiplicity of solutions for homogeneous system of $(SP)_\lambda$ when the potential V may change sign and the nonlinear term f is superlinear or sublinear in u as $|u| \to \infty$. For the Schrödinger-Poisson system with sign-changing potential see [30] and sublinear Schrödinger-Poisson system see [28].

Next, we consider the nonhomogeneous case, that is $g \not\equiv 0$. The existence of radially symmetric solutions was obtained for above nonhomogeneous system with $\lambda \equiv 1$ and $K(x) \equiv 1$ in [24]. Chen and Tang [11] obtained two solutions for the nonhomogeneous system with $f(x,u)$ satisfying Amborosetti-Rabinowitz type condition and V being nonradially symmetric. In 2015, Wang and Ma [32] considered the nonhomogeneous Schrödinger-Poisson equation containing concave and convex terms. For more results on the nonhomogeneous case see [14, 16, 17, 42] and the reference therein.

Motivated by the above works, in the present paper we consider system $(SP)_\lambda$ with more general potential $V(x), K(x)$ and $f(x,u)$. Under (V_0)–(V_1) and some more generic 4-superlinear conditions on $f(x,u)$, we prove the existence of multiple solutions of problem $(SP)_\lambda$ when $\lambda > 0$ large by using variation method. Precisely, we make the following assumptions.

(f_1) $F(x,u) = \int_0^u f(x,s)ds \ge 0$ for all (x,u) and $f(x,u) = o(u)$ uniformly in x as $u \to 0$.

(f_2) $F(x,u)/u^4 \to +\infty$ as $|u| \to +\infty$ uniformly in x.
(f_3) $\mathcal{F}(x,u) := \frac{1}{4} f(x,u)u - F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.
(f_4) There exist $a_1, L_1 > 0$ and $\tau \in (3/2, 2)$ such that

$$|f(x,u)|^\tau \leq a_1 \mathcal{F}(x,u)|u|^\tau, \ \textit{for all } x \in \mathbb{R}^3 \textit{ and } |u| \geq L_1.$$

(K) $K(x) \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $K(x) \geq 0$ is not identically zero for a.e. $x \in \mathbb{R}^3$.
(g) $g(x) \in L^2(\mathbb{R}^3)$ and $g(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$.

Remark 1 It follows (f_2) and (f_4) that $|f(x,u)|^\tau \leq \frac{a_1}{4}|f(x,u)||u|^{\tau+1}$ for large u. Thus, by (f_1), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon|u| + C_\varepsilon |u|^{p-1}, \ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R} \tag{1.2}$$

and

$$|F(x,u)| \leq \varepsilon|u|^2 + C_\varepsilon |u|^p, \ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.3}$$

where $p = 2\tau/(\tau - 1) \in (4, 2^*), 2^* = 6$ is the critical exponent for the Sobolev embedding in dimension 3.

Before stating our main results, we give several notations.

Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard product and norm

$$(u,v)_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx; \qquad \|u\|^2_{H^1} = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_D := \|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

For any $1 \leq s \leq +\infty$ and $\Omega \subset \mathbb{R}^3, L^s(\Omega)$ denotes a Lebesgue space; the norm in $L^s(\Omega)$ is denoted by $|u|_{s,\Omega}$, where Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_s$ when $\Omega = \mathbb{R}^3$.
$\bar{S}$ is the best Sobolev constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, that is,

$$\bar{S} = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_D}{|u|_6}.$$

For any $r > 0$ and $z \in \mathbb{R}^3, B_r(z)$ denotes the ball of radius r centered at z.
The letters c_0, d_i, C_0 will be used to denote various positive constants which may vary from line to line and are not essential to the problem. We denote " $\rightharpoonup$ " weak convergence and by " $\to$ " strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$. We use $o(1)$ to denote any quantity which tends to zero when $n \to \infty$.
Now we state our main results:

Theorem 1 *Assume that* (V_0)–(V_1), (K), (g) *and* (f_1)–(f_4) *are satisfied. If* $V(x) < 0$ *for some* $x \in \mathbb{R}^3$, *then for each* $k \in \mathbb{N}$, *there exist* $\lambda_k > k$, $b_k > 0$ *and* $\eta_k > 0$ *such that problem* $(SP)_\lambda$ *has at least two nontrivial solutions for every* $\lambda = \lambda_k, |g|_2 \leq \eta_k$ *and* $|K|_2 < b_k($ *or* $|K|_\infty < b_k)$.

Theorem 2 *Assume that* (V_0)–(V_1), (K), (g) *and* (f_1)–(f_4) *are satisfied. If* $V^{-1}(0)$

has nonempty interior, then there exist $\Lambda > 0, b_\lambda > 0$ *and* $\eta_\lambda > 0$ *such that problem* $(SP)_\lambda$ *has at least two nontrivial solutions for every* $\lambda > \Lambda, |g|_2 \leq \eta_\lambda$ *and* $|K|_2 < b_\lambda$ *(* *or* $|K|_\infty < b_\lambda$*).*

If $V \geq 0$, the restriction on the norm of K can be removed and we have the following theorem.

Theorem 3 *Assume that* $V \geq 0$, (V_0)–(V_1), (K), (g) *and* (f_1)–(f_4) *are satisfied. If* $V^{-1}(0)$ *has nonempty interior* Ω, *then there exist* $\Lambda_* > 0$ *and* $\eta > 0$ *such that problem* $(SP)_\lambda$ *has at least two nontrivial solutions for every* $\lambda > \Lambda_*$ *and* $|g|_2 \leq \eta$.

To obtain our main results, we have to overcome several difficulties in using variational method. The main difficulty consists in the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p \in (2, 6)$. Since we assume that the potential is no radially symmetric, we cannot use the usual way to recover compactness, for example, restricting in the subspace $H^1_r(\mathbb{R}^3)$ of radially symmetric functions or using concentration compactness methods. To recover the compactness, we borrow some ideas used in [5, 15] and establish the parameter dependent compactness conditions. Let us point out that the adaptation of the ideas to the procedure of our problem is not trivial at all, because of the presence of the nonlocal term $K(x)\phi u$..

Remark 2 (a) It is not difficult to find out functions f satisfying (f_1)–(f_4), for example,

$$f(x,t) = h(x)t^3\left(2ln(1+t^2) + \frac{t^2}{1+t^2}\right), \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

where h is a is a continuous bounded function with $\inf_{x\in\mathbb{R}^3} h(x) > 0$.

(b) To the best of our knowledge, it seems that our theorems are the first results about the existence of multiple solutions for the nonhomogeneous Schrödinger-Poisson equations on $\mathbb{R}^3$ with sign-changing potential and general nonlinear term. Although the methods are used before, we need to study carefully some properties of the term $K(x)\phi u$ and the effect of the sign-changing potential V.

The paper is organized as follows. In Section 2, we will introduce the variational setting for the problem and establish the compactness conditions. In Section 3, we give the proofs of main results.

2. **Variational setting and compactness condition**

In this section, we give the variational setting of the problem $(SP)_\lambda$ and establish the compactness conditions.

Let $V(x) = V^+(x) - V^-(x)$, where $V^\pm = \max\{\pm V(x), 0\}$. Let

$$E = \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V^+(x)u^2 dx < \infty\right\}$$

be equipped with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V^+(x)uv)dx, \quad \|u\| = (u,u)^{1/2}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$(u,v)_\lambda = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V^+(x) uv) dx, \quad \|u\|_\lambda = (u,u)_\lambda^{1/2}.$$

It is clear $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Set $E_\lambda = (E, \|\cdot\|_\lambda)$. It follows from (V_0)–(V_1) and the Poincaré inequality that the embedding $E_\lambda \hookrightarrow H^1(\mathbb{R}^3)$ is continuous, and hence, for $s \in [2,6]$, there exists $d_s > 0$(independent of $\lambda \ge 1$) such that

$$|u|_s \le d_s \|u\|_\lambda, \ \forall u \in E_\lambda. \tag{2.1}$$

Let

$$F_\lambda = \{u \in E_\lambda : supp u \subset V^{-1}([0,\infty))\},$$

and $F_\lambda^\perp$ denote the orthogonal complement of F_λ in E_λ. Clearly, $F_\lambda = E_\lambda$ if $V \ge 0$, otherwise $F_\lambda^\perp \ne \{0\}$. Define

$$A_\lambda := -\Delta + \lambda V,$$

then A_λ is formally self-adjoint in $L^2(\mathbb{R}^3)$ and the associated bilinear form

$$a_\lambda(u,v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x) uv) dx$$

is continuous in E_λ. As in [15], for fixed $\lambda > 0$, we consider the eigenvalue problem

$$-\Delta u + \lambda V^+(x) u = \mu \lambda V^-(x) u, \quad u \in F_\lambda^\perp. \tag{2.2}$$

Since (V_0)–(V_1), we see that the quadratic form $u \mapsto \int_{\mathbb{R}^3} \lambda V^-(x) u^2 dx$ is weakly continuous. Hence following Theorem 4.45 and Theorem 4.46 in [34] , we deduce the following proposition, which is the spectral theorem for compact self-adjoint operators jointly with the Courant-Fischer minimax characterization of eigenvalues.

Proposition 2.1 *Assume that (V_0)–(V_1) hold, then for any fixed $\lambda > 0$, problem (2.2) has a sequence of positive eigenvalues $\{\mu_j(\lambda)\}$, which may be characterized by*

$$\mu_j(\lambda) = \inf_{dim M \ge j, M \subset F_\lambda^\perp} \sup \left\{ \|u\|_\lambda^2 : u \in M, \int_{\mathbb{R}^3} \lambda V^-(x) u^2 dx = 1 \right\}, \quad j = 1,2,3,....$$

Furthermore, $\mu_1(\lambda) \le \mu_2(\lambda) \le \cdots \le \mu_j(\lambda) \to +\infty$ as $j \to +\infty$, and the corresponding eigenfunctions $\{e_j(\lambda)\}$, which may be be chosen so that $(e_i(\lambda), e_j(\lambda))_\lambda = \delta_{ij}$, are a basis of $F_\lambda^\perp$.

Now we give the properties for the eigenvalues $\{\mu_j(\lambda)\}$ defined above.

Proposition 2.2 ([15]) *Assume that (V_0)–(V_1) hold and $V^- \not\equiv 0$. Then, for each fixed $j \in \mathbb{N}$,*

(i) $\mu_j(\lambda) \to 0$ as $\lambda \to +\infty$.

(ii) $\mu_j(\lambda)$ is a non-increasing continuous function of λ.

Remark 3 By Proposition 2.2 (i), there exists $\Lambda_0 > 0$ such that $\mu_1(\lambda) \le 1$ for all $\lambda > \Lambda_0$.

Let

$$E_\lambda^- := span\{e_j(\lambda) : \mu_j(\lambda) \le 1\} \quad and \quad E_\lambda^+ := span\{e_j(\lambda) : \mu_j(\lambda) > 1\}.$$

Then $E_\lambda = E_\lambda^- \bigoplus E_\lambda^+ \bigoplus F_\lambda$ is an orthogonal decomposition. The quadratic form a_λ is negative semidefinite on E_λ^-, positive definite on $E_\lambda^+ \bigoplus F_\lambda$ and it is easy to see that $a_\lambda(u,v) = 0$ if u, v are in different subspaces of the above decomposition of E_λ.

From Remark 3, we have that $\dim E_\lambda^- \geq 1$ when $\lambda > \Lambda_0$. Moreover, since $\mu_j(\lambda) \to +\infty$ as $j \to +\infty$, $\dim E_\lambda^- < +\infty$ for every fixed $\lambda > 0$.

It is well known that problem $(SP)_\lambda$ can be reduced to a single equation with a nonlocal term (see [23]). In fact, for every $u \in E_\lambda$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = K(x)u^2 \tag{2.3}$$

with

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}dy.$$

If $K \in L^\infty(\mathbb{R}^3)$, by (2.3), the Höder inequality and the Sobolev inequality, we get

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leq \bar{S}^{-2} d_{12/5}^4 |K|_\infty^2 \|u\|_\lambda^4.$$

Similarly, if $K \in L^2(\mathbb{R}^3)$,

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leq \bar{S}^{-2} d_6^4 |K|_2^2 \|u\|_\lambda^4.$$

Thus, there exists $C_0 > 0$, such that

$$\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leq C_0\|u\|_\lambda^4, \quad \forall K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3). \tag{2.4}$$

Therefore, problem $(SP)_\lambda$ can be reduced to the following equation:

$$-\Delta u + \lambda V(x)u + K(x)\phi_u(x)u = f(x,u) + g(x),\ x \in \mathbb{R}^3.$$

$(SP)_\lambda$ is variational and its solutions are the critical points of the functional defined in E_λ by

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \\ &- \int_{\mathbb{R}^3} g(x)udx. \end{aligned}$$

Furthermore, it is easy to prove that the functional I_λ is of class C^1 in E_λ and that

$$\langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv + K(x)\phi_u uv - f(x,u)v - g(x)v)dx$$

for all $u, v \in E_\lambda$. Hence, if $u \in E_\lambda$ is a critical point of I_λ, then $(u, \phi_u) \in E_\lambda \times D^{1,2}(\mathbb{R}^3)$ is a solution of problem $(SP)_\lambda$. We refer the readers to [6] and [13] for the details.

Set

$$N(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx = \frac{1}{4\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{K(x)K(y)u^2(x)u^2(y)}{|x-y|}dxdy.$$

Now we give some properties about the functional N and its derivative N' possess BL-splitting property, which is similar to Brezis-Lieb Lemma [7].

Proposition 2.3 ([38], Lemma 2.1) *Let $K \in L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$, then*
(i) $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and $N(u) \le \liminf_{n\to\infty} N(u_n)$;
(ii) $N(u_n - u) = N(u_n) - N(u) + o(1)$;
(iii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$ in $H^{-1}(\mathbb{R}^3)$.

Next, we investigate the compactness conditions for the functional I_λ. Recall that a C^1 functional J satisfies Cerami condition at level c ($(C)_c$ condition for short) if any sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and $(1 + \|u_n\|)J'(u_n) \to 0$ has a convergent subsequence; and such sequence is called a $(C)_c$ sequence.

We only consider the case $K \in L^\infty(\mathbb{R}^3)$, the other case $K \in L^2(\mathbb{R}^3)$ is similar.

Lemma 2.1 *Suppose that (V_0)–(V_1), (K), (f_1)–(f_4) and (g) are satisfied. Then every $(C)_c$ sequence of I_λ is bounded in E_λ for each $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(C)_c$ sequence of I_λ. Arguing indirectly, we can assume that

$$I_\lambda(u_n) \to c, \quad (1 + \|u_n\|_\lambda) I'_\lambda(u_n) \to 0, \quad \|u_n\|_\lambda \to \infty \tag{2.5}$$

as $n \to \infty$ after passing to a subsequence. Take $w_n := u_n/\|u_n\|_\lambda$. Then $\|w_n\|_\lambda = 1, w_n \rightharpoonup w$ in E_λ and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^3$.

We first consider the case $w = 0$. Combining this with (2.5), (f_3) and the fact $w_n \to 0$ in $L^2(\{x \in \mathbb{R}^3 : V(x) < 0\})$, we have

$$\begin{aligned}
o(1) &= \frac{1}{\|u_n\|_\lambda^2}\left(I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle\right) \\
&\ge \frac{1}{4}\|w_n\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x) w_n^2 dx + \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} \mathcal{F}(x, u_n) dx - \frac{3}{4\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} g(x) u_n dx \\
&\ge \frac{1}{4} - \frac{\lambda}{4}|V^-|_\infty \int_{suppV^-} w_n^2 dx - \frac{3}{4}|g|_2 d_2 \frac{1}{\|u_n\|_\lambda} \\
&= \frac{1}{4} + o(1),
\end{aligned}$$

a contradiction.

If $w \ne 0$, then the set $\Omega_1 = \{x \in \mathbb{R}^3 : w(x) \ne 0\}$ has positive Lebesgue measure. For $x \in \Omega_1$, one has $|u_n(x)| \to \infty$ as $n \to \infty$, and then, by (f_2),

$$\frac{F(x, u_n(x))}{u_n^4(x)} w_n^4(x) \to +\infty \quad as\ n \to \infty,$$

which, jointly with Fatou's lemma, shows that

$$\int_{\Omega_1} \frac{F(x, u_n)}{u_n^4} w_n^4 dx \to +\infty \quad as\ n \to \infty. \tag{2.6}$$

We see from (f_1), (2.4), the first limit of (2.5), (2.6) and (g) that

$$\frac{C_0}{4} \ge \limsup_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(x, u_n)}{\|u_n\|_\lambda^4} dx \ge \limsup_{n\to\infty} \int_{\Omega_1} \frac{F(x, u_n)}{u_n^4} w_n^4 dx = +\infty.$$

This is impossible.

Hence $\{u_n\}$ is bounded in E_λ. □

Lemma 2.2 *Suppose that* (V_0), (V_1) , (K) , (g) *and* (1.2) *hold. If* $u_n \rightharpoonup u$ *in* E_λ, $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^3$, *and we denote* $w_n := u_n - u$, *then*

$$I_\lambda(u_n) = I_\lambda(w_n) + I_\lambda(u) + o(1) \tag{2.7}$$

and

$$\langle I'_\lambda(u_n), \varphi\rangle = \langle I'_\lambda(w_n), \varphi\rangle + \langle I'_\lambda(u), \varphi\rangle - \int_{\mathbb{R}^3} g\varphi dx + o(1), \ \text{for all } \varphi \in E_\lambda \tag{2.8}$$

as $n \to \infty$. *In particular, if* $I_\lambda(u_n) \to d$ *and* $I'_\lambda(u_n) \to 0$ *in* E^*_λ(*the dual space of* E_λ), *then* $I'_\lambda(u) = 0$ *and*

$$\begin{aligned} &I_\lambda(w_n) \to d - I_\lambda(u), \\ &\langle I'_\lambda(w_n), \varphi\rangle \to -\int_{\mathbb{R}^3} g\varphi dx, \ \text{for all } \varphi \in E_\lambda \end{aligned} \tag{2.9}$$

after passing to a subsequence.

Proof. Since $u_n \rightharpoonup u$ in E_λ, we have $(u_n - u, u)_\lambda \to 0$ as $n \to \infty$, which implies that

$$\|u_n\|^2_\lambda = (w_n + u, w_n + u)_\lambda = \|w_n\|^2_\lambda + \|u\|^2_\lambda + o(1). \tag{2.10}$$

By (V_0), $w_n \rightharpoonup 0$ and the Hölder inequality, we have

$$\left|\int_{\mathbb{R}^3} V^-(x) w_n u dx\right| = \left|\int_{suppV^-} V^- w_n u dx\right| \le |V^-|_\infty \left(\int_{suppV^-} w_n^2 dx\right)^{1/2} |u|_2 \to 0$$

as $n \to \infty$. Thus

$$\int_{\mathbb{R}^3} V^-(x) u_n^2 dx = \int_{\mathbb{R}^3} V^-(x) w_n^2 dx + \int_{\mathbb{R}^3} V^-(x) u^2 dx + o(1).$$

Consequently, this together with Proposition 2.3 (ii) and (2.10), we obtain

$$\frac{1}{2}a_\lambda(u_n, u_n) + \frac{1}{4}N(u_n) = \left(\frac{1}{2}a_\lambda(w_n, w_n) + \frac{1}{4}N(w_n)\right) + \left(\frac{1}{2}a_\lambda(u, u) + \frac{1}{4}N(u)\right) + o(1).$$

Similarly, by Proposition 2.3 (iii), we have

$$\begin{aligned} a_\lambda(u_n, h) + \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n h dx &= \left(a_\lambda(w_n, h) + \int_{\mathbb{R}^3} K(x)\phi_{w_n} w_n h dx\right) \\ &\quad + \left(a_\lambda(u, h) + \int_{\mathbb{R}^3} K(x)\phi_u u h dx\right) + o(1), \quad \forall h \in E_\lambda. \end{aligned}$$

Since

$$\int_{\mathbb{R}^3} g(x) u_n dx = \int_{\mathbb{R}^3} g(x) w_n dx + \int_{\mathbb{R}^3} g(x) u dx,$$

therefore, to obtain (2.7) and (2.8), it suffices to check that

$$\int_{\mathbb{R}^3} (F(x, u_n) - F(x, w_n) - F(x, u)) dx = o(1) \tag{2.11}$$

and

$$\sup_{\|h\|_\lambda = 1} \int_{\mathbb{R}^3} (f(x, u_n) - f(x, w_n) - f(x, u)) h dx = o(1). \tag{2.12}$$

Here, we only prove (2.11), the verification of (2.12) is similar. Inspired by [2], we observe that

$$F(x,u_n)-F(x,u_n-u)=-\int_0^1\left(\frac{d}{dt}F(x,u_n-tu)\right)dt=\int_0^1 f(x,u_n-tu)udt.$$

and hence, by (1.2), we have

$$|F(x,u_n)-F(x,u_n-u)|\le\varepsilon_1|u_n||u|+\varepsilon_1|u|^2+C_{\varepsilon_1}|u_n|^{p-1}|u|+C_{\varepsilon_1}|u|^p,$$

where $\varepsilon_1, C_{\varepsilon_1}>0$ and $p\in(4,6)$. Hence, for each $\varepsilon>0$, and the Young inequality, we have

$$|F(x,u_n)-F(x,w_n)-F(x,u)|\le C[\varepsilon|u_n|^2+C_\varepsilon|u|^2+\varepsilon|u_n|^p+C_\varepsilon|u|^p].$$

Next, we consider the function g_n given by

$$g_n(x):=\max\left\{|F(x,u_n)-F(x,w_n)-F(x,u)|-C\varepsilon(|u_n|^2+|u_n|^p),0\right\}.$$

Then $0\le g_n(x)\le CC_\varepsilon(|u|^2+|u|^p)\in L^1(\mathbb{R}^3)$. Moreover, by the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^3} g_n(x)dx\to 0\quad as\ n\to\infty, \tag{2.13}$$

since $u_n\to u$ a.e. in $\mathbb{R}^3$. By the definition of g_n, it follows that

$$|F(x,u_n)-F(x,w_n)-F(x,u)|\le g_n(x)+C\varepsilon(|u_n|^2+|u_n|^p),$$

which, together with (2.13) and (2.1) shows that

$$|F(x,u_n)-F(x,w_n)-F(x,u)|\le C\varepsilon$$

for n sufficiently large. Which implies that

$$\int_{\mathbb{R}^3}[F(x,u_n)-F(x,w_n)-F(x,u)]dx=o(1).$$

Next, we check that $I'_\lambda(u)=0$. Indeed, for each $\psi\in C_0^\infty(\mathbb{R}^3)$, we have

$$(u_n-u,\psi)_\lambda\to 0\ as\ n\to\infty. \tag{2.14}$$

and

$$\left|\int_{\mathbb{R}^3}V^-(x)(u_n-u)\psi dx\right|\le|V^-|_\infty\left(\int_{supp\psi}(u_n-u)^2dx\right)^{1/2}|\psi|_2\to 0\ as\ n\to\infty, \tag{2.15}$$

since $u_n\to u$ in $L^2_{loc}(\mathbb{R}^3)$. By Proposition 2.3 (i), $u_n\rightharpoonup u$ in E_λ yields $\phi_{u_n}\rightharpoonup\phi_u$ in $D^{1,2}(\mathbb{R}^3)$. So

$$\phi_{u_n}\rightharpoonup\phi_u\ in\ L^6(\mathbb{R}^3).$$

For every $\psi\in C_0^\infty(\mathbb{R}^3)$, by the Hölder inequality we obtain

$$\int_{\mathbb{R}^3}|K(x)u\psi|^{6/5}dx\le|K|_\infty^{6/5}|\psi|_{12/5}^{6/5}|u|_{12/5}^{6/5},$$

that is $K(x)u\psi\in L^{6/5}(\mathbb{R}^3)$, and hence

$$\int_{\mathbb{R}^3}K(x)(\phi_{u_n}-\phi_u)u\psi dx\to 0.$$

By $u_n \to u$ in $L^3_{loc}(\mathbb{R}^3)$ and the Hölder inequality, we have

$$\begin{aligned}&|\int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n-u)\psi dx|\\&\le |\psi|_2|K|_\infty|\phi_{u_n}|_6|u_n-u|_{3,\Omega_\psi}\\&\le C|u_n-u|_{3,\Omega_\psi}\to 0 \quad as\ n\to\infty,\end{aligned}$$

here Ω_ψ is the support set of ψ. Consequently,

$$\begin{aligned}&|\int_{\mathbb{R}^3}[K(x)\phi_{u_n}u_n\psi - K(x)\phi_u u\psi]dx|\\&\le \int_{\mathbb{R}^3}|K(x)\phi_{u_n}(u_n-u)\psi|dx + \int_{\mathbb{R}^3}|K(x)(\phi_{u_n}-\phi_u)u\psi|dx\\&= o(1).\end{aligned} \tag{2.16}$$

Furthermore, by (1.2) and the dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^3}[f(x,u_n)-f(x,u)]\psi dx = \int_{\Omega_\psi}[f(x,u_n)-f(x,u)]\psi dx = o(1).$$

Since $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and $g\in L^2(\mathbb{R}^3)$, we obtain $\int_{\mathbb{R}^3} g(u_n-u)dx = o(1)$. This jointly with (2.14), (2.15), (2.16) and the dominated convergence theorem, shows that

$$\langle I'_\lambda(u),\psi\rangle = \lim_{n\to\infty}\langle I'_\lambda(u_n),\psi\rangle = 0, \quad \forall\psi\in C_0^\infty(\mathbb{R}^3).$$

Hence $I'_\lambda(u)=0$. (2.9) follows from (2.7)-(2.8) and Proposition 2.3(iii). The proof is complete. □

Lemma 2.3 *Suppose $V\ge 0, (V_0), (V_1), (K), (g)$ and (f_1)–(f_4) hold. Then, for any $M>0$, there is $\Lambda=\Lambda(M)>0$ such that I_λ satisfies $(C)_c$ condition for all $c<M$ and $\lambda>\Lambda$.*

Proof. Let $\{u_n\}\subset E_\lambda$ be a $(C)_c$ sequence with $c<M$. According to Lemma 2.1, $\{u_n\}$ is bounded in E_λ, and there exists C_λ such that $\|u_n\|_\lambda \le C_\lambda$. Therefore, up to a subsequence, we can assume that

$$\begin{aligned}&u_n \rightharpoonup u\ in\ E_\lambda;\\&u_n\to u\ in\ L^s_{loc}(\mathbb{R}^3)(2\le s<2^*);\\&u_n(x)\to u(x)\ a.e.\ x\in\mathbb{R}^3.\end{aligned} \tag{2.17}$$

Now we can prove that $u_n\to u$ in E_λ for $\lambda>0$ large. Denote $w_n := u_n-u$, then $w_n\rightharpoonup 0$ in E_λ. By Lemma 2.2, we have $I'_\lambda(u)=0$, and

$$I_\lambda(w_n)\to c-I_\lambda(u), \quad I'_\lambda(w_n)\to 0 \quad as\ n\to\infty. \tag{2.18}$$

Noting $V\ge 0$ and using (f_3), we have

$$\begin{aligned}I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u),u\rangle\\&= \frac{1}{4}\|u\|^2_\lambda + \int_{\mathbb{R}^3}\mathcal{F}(x,u)dx - \frac{3}{4}\int_{\mathbb{R}^3} gudx = \Phi_\lambda(u) - \frac{3}{4}\int_{\mathbb{R}^3} gudx,\end{aligned}$$

here $\Phi_\lambda(u) = \frac{1}{4}\|u\|^2_\lambda + \int_{\mathbb{R}^3}\mathcal{F}(x,u)dx \ge 0$.

Again by (2.9) and (2.17), we obtain

$$\begin{aligned}\frac{1}{4}\|w_n\|_\lambda^2+\int_{\mathbb{R}^3}\mathcal{F}(x,w_n)dx+o(1)&=I_\lambda(w_n)-\frac{1}{4}\langle I'_\lambda(w_n),w_n\rangle\\&=c-I_\lambda(u)+\frac{1}{4}\int_{\mathbb{R}^3}gw_ndx+o(1)\\&=c-\left[\Phi_\lambda(u)-\frac{3}{4}\int_{\mathbb{R}^3}gudx\right]+\frac{1}{4}\int_{\mathbb{R}^3}gw_ndx+o(1)\\&=c-\Phi_\lambda(u)+\frac{3}{4}\int_{\mathbb{R}^3}gudx+o(1)\\&\le M+\tilde{M}+o(1).\end{aligned}\tag{2.19}$$

Here we use the fact $c<M$ and

$$\frac{3}{4}|g|_2|u|_2\le\frac{3}{4}|g|_2d_2\|u\|_\lambda\le\frac{3}{4}|g|_2d_2\liminf_{n\to\infty}\|u_n\|_\lambda\le|g|_2d_2C\le\tilde{M},$$

where $\tilde{M}$ is a positive constant independent of λ. Hence

$$\int_{\mathbb{R}^3}\mathcal{F}(x,w_n)dx\le M+\tilde{M}+o(1).\tag{2.20}$$

Since $V(x)<b$ on a set of finite measure and $w_n\rightharpoonup 0$,

$$|w_n|_2^2\le\frac{1}{\lambda b}\int_{V\ge b}\lambda V^+(x)w_n^2dx+\int_{V<b}w_n^2dx\le\frac{1}{\lambda b}\|w_n\|_\lambda^2+o(1).\tag{2.21}$$

For $2<s<2^*$, by (2.21) and the Hölder and Sobolev inequality, we obtain

$$\begin{aligned}|w_n|_s^s=\int_{\mathbb{R}^3}|w_n|^sdx&\le\left(\int_{\mathbb{R}^3}|w_n|^2dx\right)^{\frac{6-s}{4}}\left(\int_{\mathbb{R}^3}|w_n|^6dx\right)^{\frac{s-2}{4}}\\&\le\left[\frac{1}{\lambda b}\int_{\mathbb{R}^3}\left(|\nabla w_n|^2+\lambda V^+w_n^2\right)dx\right]^{\frac{6-s}{4}}\left(\bar{S}^{-6}\left[\int_{\mathbb{R}^3}|\nabla w_n|^2dx\right]^3\right)^{\frac{s-2}{4}}+o(1)\\&\le\left(\frac{1}{\lambda b}\right)^{\frac{6-s}{4}}\bar{S}^{-\frac{3(s-2)}{2}}\|w_n\|_\lambda^s+o(1).\end{aligned}\tag{2.22}$$

By (f_1), for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that $|f(x,t)|\le\varepsilon|t|$ for all $x\in\mathbb{R}^3$ and $|t|\le\delta$, and (f_4) is satisfied for $|t|\ge\delta$ (with the same τ but possibly larger a_1). Hence we get

$$\int_{|w_n|\le\delta}f(x,w_n)w_ndx\le\varepsilon\int_{|w_n|\le\delta}w_n^2dx\le\frac{\varepsilon}{\lambda b}\|w_n\|_\lambda^2+o(1),\tag{2.23}$$

and

$$\begin{aligned}\int_{|w_n|\ge\delta}f(x,w_n)w_ndx&\le\left(\int_{|w_n|\ge\delta}\left|\frac{f(x,w_n)}{w_n}\right|^\tau dx\right)^{1/\tau}|w_n|_s^2\\&\le\left(\int_{|w_n|\ge\delta}a_1\mathcal{F}(x,w_n)dx\right)^{1/\tau}|w_n|_s^2\\&\le[a_1(M+\tilde{M})]^{1/\tau}\bar{S}^{-\frac{3(2s-4)}{2s}}\left(\frac{1}{\lambda b}\right)^\theta\|w_n\|_\lambda^2+o(1)\end{aligned}\tag{2.24}$$

by (f_4), (2.20), (2.22) with $s = 2\tau/(\tau-1)$ and the Hölder inequality, where $\theta = \frac{6-s}{2s} > 0$.
Since $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and $g \in L^2(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} g(u_n-u)dx \to 0 \quad \text{as } n\to\infty. \tag{2.25}$$

Therefore, by (2.23), (2.24) and (2.25) we have

$$\begin{aligned} o(1) &= \langle I'_\lambda(w_n), w_n\rangle \\ &\geq \|w_n\|_\lambda^2 - \int_{\mathbb{R}^3} f(x,w_n)w_n dx - \int_{\mathbb{R}^3} g w_n dx \\ &\geq \left[1-\frac{\varepsilon}{\lambda b} - [a_1(M+\tilde{M})]^{1/\tau}\bar{S}^{-\frac{3(2s-4)}{2s}}\left(\frac{1}{\lambda b}\right)^{\theta}\right]\|w_n\|_\lambda^2 + o(1). \end{aligned} \tag{2.26}$$

So, there exists $\Lambda = \Lambda(M) > 0$ such that $w_n \to 0$ in E_λ when $\lambda > \Lambda$. Since $w_n = u_n - u$, it follows that $u_n \to u$ in E_λ. This completes the proof. □

Lemma 2.4 *Let $(V_0), (V_1), (K), (g)$ and (f_1)–(f_4) be satisfied. Let $\{u_n\}$ be a $(C)_c$ sequence of I_λ with level $c > 0$. Then for any $M > 0$, there is $\Lambda = \Lambda(M) > 0$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E_λ with u being a nontrivial critical point of I_λ and satisfying $I_\lambda(u) \leq c$ for all $c < M$ and $\lambda > \Lambda$.*

Proof. We modify the proof of Lemma 2.3. By Lemma 2.2, we obtain

$$I'_\lambda(u) = 0, \quad I_\lambda(w_n) \to c - I_\lambda(u), \quad I'_\lambda(w_n) \to 0 \quad \text{as } n\to\infty. \tag{2.27}$$

However, since V is allowed to be sign-changing and the appearance of nonlinear term g, from

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\ &= \frac{1}{4}\|u\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x)u^2 dx + \int_{\mathbb{R}^3}\mathcal{F}(x,u)dx - \frac{3}{4}\int_{\mathbb{R}^3} gudx, \end{aligned}$$

we cannot deduce that $I_\lambda(u) \geq 0$. We consider two possibilities:

(i) $I_\lambda(u) < 0$;
(ii) $I_\lambda(u) \geq 0$.

If $I_\lambda(u) < 0$, then $u \neq 0$ and then u is nontrivial and the proof is done. If $I_\lambda(u) \geq 0$, following the argument in the proof of Lemma 2.3 step by step, we can get $u_n \to u$ in E_λ. In fact, by (V_0) and $w_n \to 0$ in $L^2(\{x\in\mathbb{R}^3 : V(x) < b\})$, we have

$$\left|\int_{\mathbb{R}^3} V^-(x)w_n^2(x)dx\right| \leq |V^-|_\infty \int_{suppV^-} w_n^2 dx = o(1).$$

Combining this with (2.27), we have

$$\begin{aligned} &\int_{\mathbb{R}^3}\mathcal{F}(x,w_n)dx \\ &= I_\lambda(w_n) - \frac{1}{4}\langle I'_\lambda(w_n), w_n\rangle - \frac{1}{4}\|w_n\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3}\lambda V^-(x)w_n^2 dx + \frac{3}{4}\int_{\mathbb{R}^3} g w_n dx \\ &\leq c - I_\lambda(u) + o(1) \leq M + o(1). \end{aligned}$$

It follows that (2.24), (2.25) and (2.26) remain valid. Hence $u_n \to u$ in E_λ and $I_\lambda(u) = c(> 0)$. The proof is complete. □

3. Proofs of main results

If V is sign-changing, we first verify that the functional I_λ have the linking geometry to apply the following linking theorem [22].

Proposition 3.1 *Let $E = E_1 \bigoplus E_2$ be a Banach space with $\dim E_2 < \infty, \Phi \in C^1(E, \mathbb{R}^3)$. If there exist $R > \rho > 0, \alpha > 0$ and $e_0 \in E_1$ such that*

$$\alpha := \inf \Phi(E_1 \cap S_\rho) > \sup \Phi(\partial Q)$$

where $S_\rho = \{u \in E : \|u\| = \rho\}, Q = \{u = v + te_0 : v \in E_2, t \geq 0, \|u\| \leq R\}$. Then Φ has a $(C)_c$ sequence with $c \in [\alpha, \sup \Phi(Q)]$.

In our context, we use Proposition 3.1 with $E_1 = E_\lambda^+ \bigoplus F_\lambda$ and $E_2 = E_\lambda^-$. By Proposition 2.2, $\mu_j(\lambda) \to 0$ as $\lambda \to \infty$ for every fixed j. By Remark 3, there is $\Lambda_0 > 0$ such that $E_\lambda^- \neq \emptyset$ and E_λ^- is finite dimensional for $\lambda > \Lambda_0$. Now we investigate the linking structure of the functional I_λ.

Lemma 3.1 *Suppose that $(V_0), (V_1), (K), (g)$ and (1.2) with $p \in (4, 2^*)$ are satisfied. Then, for each $\lambda > \Lambda_0($ is the constant given in Remark 2$)$, there exist $\alpha_\lambda, \rho_\lambda$ and $\eta_\lambda > 0$ such that*

$$I_\lambda(u) \geq \alpha_\lambda \text{ for all } u \in E_\lambda^+ \bigoplus F_\lambda \text{ with } \|u\|_\lambda = \rho_\lambda \text{ and } |g|_2 < \eta_\lambda. \tag{3.1}$$

Furthermore, if $V \geq 0$, we can choose $\alpha, \rho, \eta > 0$ independent of λ.

Proof. For any $u \in E_\lambda^+ \bigoplus F_\lambda$, writing $u = u_1 + u_2$ with $u_1 \in E_\lambda^+$ and $u_2 \in F_\lambda$. Clearly, $(u_1, u_2)_\lambda = 0$, and

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x) u^2) dx = \int_{\mathbb{R}^3} (|\nabla u_1|^2 + \lambda V(x) u_1^2) dx + \|u_2\|_\lambda^2. \tag{3.2}$$

By Proposition 2.1 we have $\mu_j(\lambda) \to +\infty$ as $j \to +\infty$ for each fixed $\lambda > \Lambda_0$. So there is a positive integer n_λ such that $\mu_j(\lambda) \leq 1$ for $j \leq n_\lambda$ and $\mu_j(\lambda) > 1$ for $j > n_\lambda + 1$. For $u_1 \in E_\lambda^+$, we set $u_1 = \Sigma_{j=n_\lambda+1}^\infty \mu_j(\lambda) e_j(\lambda)$. Thus

$$\int_{\mathbb{R}^3} (|\nabla u_1|^2 + \lambda V(x) u_1^2) dx = \|u_1\|_\lambda^2 - \int_{\mathbb{R}^3} \lambda V^-(x) u_1^2 dx \geq \left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right) \|u_1\|_\lambda^2. \tag{3.3}$$

Now, using (2.1), (3.2) and (3.3), we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right) \|u\|_\lambda^2 - \varepsilon |u|_2^2 - C_\varepsilon |u|_p^p - |g|_2 |u|_2 \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right) \|u\|_\lambda^2 - \varepsilon d_2^2 \|u\|_\lambda^2 - C_\varepsilon d_p^p \|u\|_\lambda^p - d_2 |g|_2 \|u\|_\lambda \\ &\geq \|u\|_\lambda \left\{ \left[\frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right) - \varepsilon d_2^2\right] \|u\|_\lambda - C_\varepsilon d_p^p \|u\|_\lambda^{p-1} - d_2 |g|_2 \right\}. \end{aligned}$$

Let $h(t) = \left[\frac{1}{2}(1-\frac{1}{\mu_{n_\lambda+1}(\lambda)}-\varepsilon d_2^2\right]t - C_\varepsilon d_p^p t^{p-1}$, for $t>0, p\in(4,6)$ there exists $\rho(\lambda) = \left[\frac{\frac{1}{2}(1-\frac{1}{\mu_{n_\lambda+1}(\lambda)})-\varepsilon d_2^2}{C_\varepsilon d_p^p(p-1)}\right]^{\frac{1}{p-2}}$ such that $\max_{t\geq 0} h(t) = h(\rho(\lambda)) > 0$. It follows from above inequality, $I_\lambda(u)\mid_{\|u\|_\lambda=\rho(\lambda)} > 0$ for all $|g|_2 < \eta_\lambda := \frac{h(\rho(\lambda))}{2d_2}$. Of course, $\rho(\lambda)$ can be chosen small enough, we can obtain the same result: there exists $\alpha_\lambda > 0$, such that $I_\lambda(u)\geq\alpha_\lambda$, here $\|u\|_\lambda = \rho_\lambda$.

If $V\geq 0$, since $E_\lambda = F_\lambda$, and

$$\int_{\mathbb{R}^3}(|\nabla u|^2+\lambda V(x)u^2)dx = \|u\|_\lambda^2,$$

we can choose $\alpha,\rho,\eta>0$ (independent of λ) such that (3.1) holds. □

Lemma 3.2 *Suppose that $(V_0),(V_1),(K),(g)$ and (f_1)–(f_2) are satisfied. Then, for any finite dimensional subspace $\tilde{E}_\lambda\subset E_\lambda$, there holds*

$$I_\lambda(u)\to-\infty \quad as \quad \|u\|_\lambda\to\infty,\ u\in\tilde{E}_\lambda.$$

Proof. Assuming the contrary, there is a sequence $(u_n)\subset\tilde{E}_\lambda$ with $\|u_n\|_\lambda\to\infty$ such that

$$-\infty < \inf_n I_\lambda(u_n). \tag{3.4}$$

Take $v_n := u_n/\|u_n\|_\lambda$. Since $\dim\tilde{E}_\lambda < +\infty$, there exists $v\in\tilde{E}_\lambda\setminus\{0\}$ such that

$$v_n\to v \quad in \quad \tilde{E}_\lambda, \qquad v_n(x)\to v(x)\ a.e.\ x\in\mathbb{R}^3$$

after passing to a subsequence. If $v(x)\neq 0$, then $|u_n(x)|\to+\infty$ as $n\to\infty$, and hence by (f_2),

$$\frac{F(x,u_n(x))}{u_n^4(x)}v_n^4(x)\to+\infty \quad as\ n\to\infty.$$

Combining this with (f_1), (2.4) and Fatou's lemma, we have

$$\begin{aligned}\frac{I_\lambda(u_n)}{\|u_n\|_\lambda^4} &\leq \frac{1}{2\|u_n\|_\lambda^2}+\frac{C_0}{4}-\int_{\mathbb{R}^3}\frac{F(x,u_n)}{\|u_n\|_\lambda^4}dx-\int_{\mathbb{R}^3}g(x)\frac{u_n}{\|u_n\|_\lambda^4}dx\\ &\leq \frac{1}{2\|u_n\|_\lambda^2}+\frac{C_0}{4}-\left(\int_{v=0}+\int_{v\neq0}\right)\frac{F(x,u_n)}{u_n^4}v_n^4dx+\frac{|g|_2d_2}{\|u_n\|_\lambda^3}\\ &\leq \frac{1}{2\|u_n\|_\lambda^2}+\frac{C_0}{4}-\int_{v\neq0}\frac{F(x,u_n)}{u_n^4}v_n^4dx+\frac{|g|_2d_2}{\|u_n\|_\lambda^3}\\ &\to-\infty,\end{aligned}$$

a contradiction with (3.4). □

Lemma 3.3 *Suppose that $(V_0),(V_1),(K),(g)$ and (f_1)–(f_2) are satisfied. If $V(x)<0$ for some x, then , for each $k\in\mathbb{N}$, there exist $\lambda_k>k, w_k\in E_{\lambda_k}^+\bigoplus F_{\lambda_k}, R_{\lambda_k}>\rho_{\lambda_k}(\rho_{\lambda_k}$ is the constant given in Lemma 3.1) and $\eta_k, b_k>0$ such that, for $|g|_2<\eta_k$ and $|K|_\infty<b_k$ (or $|K|_2<b_k$),*
(a) $\sup I_{\lambda_k}(\partial Q_k)\leq 0$,
(b) $\sup I_{\lambda_k}(Q_k)$ is bounded above by a constant independent of λ_k,
where $Q_k := \{u=v+tw_k : v\in E_{\lambda_k}^-, t\geq0, \|u\|_{\lambda_k}\leq R_{\lambda_k}\}$.

Proof. We adapt an argument in Ding and Szulkin [15]. For each $k \in \mathbb{N}$, since $\mu_j(k) \to +\infty$ as $j \to \infty$, there exists $j_k \in \mathbb{N}$ such that $\mu_{j_k}(k) > 1$. By Proposition 2.2, there is $\lambda_k > k$ such that

$$1 < \mu_{j_k}(\lambda_k) < 1 + \frac{1}{\lambda_k}.$$

Taking $w_k := e_{j_k}(\lambda_k)$ be an eigenvalue of $\mu_{j_k}(\lambda_k)$, then $w_k \in E^+_{\lambda_k}$ as $\mu_{j_k}(\lambda_k) > 1$. Since $\dim E^-_{\lambda_k} \bigoplus \mathbb{R}w_k < +\infty$, it follows directly from Lemma 3.2 that (a) holds with $R_{\lambda_k} > 0$ large enough.

By (f_2), for each $\tilde{\eta} > |V^-|_\infty$, there is $r_{\tilde{\eta}} > 0$ such that $F(x,t) \geq \frac{1}{2}\tilde{\eta}t^2$ if $|t| \geq r_{\tilde{\eta}}$. For $u = v + w \in E^-_{\lambda_k} \bigoplus \mathbb{R}w_k$, we obtain

$$\int_{\mathbb{R}^3} V^-(x)u^2dx = \int_{\mathbb{R}^3} V^-(x)v^2dx + \int_{\mathbb{R}^3} V^-(x)w^2dx$$

by the orthogonality of $E^-_{\lambda_k}$ and $\mathbb{R}w_k$. Hence we get

$$\begin{aligned}
I_{\lambda_k}(u) \leq\ & \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla w|^2 + \lambda_k V(x)w^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2dx - \int_{suppV^-} F(x,u)dx \\
& - \int_{\mathbb{R}^3} gudx \\
\leq\ & \frac{1}{2}[\mu_{j_k}(\lambda_k) - 1]\lambda_k \int_{\mathbb{R}^3} V^-(x)w^2dx - \frac{1}{2}\int_{suppV^-} \tilde{\eta}u^2dx + \\
& \frac{1}{4}\bar{S}^{-2}d_{12/5}^4|K|_\infty^2\|u\|_{\lambda_k}^4 + d_2|g|_2\|u\|_{\lambda_k} - \int_{suppV^-,|u|\leq r_{\tilde{\eta}}} \left(F(x,u) - \frac{1}{2}\tilde{\eta}u^2\right)dx \\
\leq\ & \frac{1}{2}\int_{\mathbb{R}^3} V^-(x)w^2dx - \frac{\tilde{\eta}}{2|V^-|_\infty}\int_{\mathbb{R}^3} V^-(x)w^2dx + C_{\tilde{\eta}} + \frac{1}{4}\bar{S}^{-2}d_{12/5}^4|K|_\infty^2 R_{\lambda_k}^4 \\
& + d_2|g|_2 R_{\lambda_k} \\
\leq\ & C_{\tilde{\eta}} + 1
\end{aligned}$$

for $u = v + w \in E^-_{\lambda_k} \bigoplus \mathbb{R}w_k$ with $\|u\|_{\lambda_k} \leq R_{\lambda_k}$, $|K|_\infty < b_k := \bar{S}(d_{12/5}R_{\lambda_k})^{-2}$ and $|g|_2 < \eta_k := \frac{1}{2d_2R_{\lambda_k}}$, where $C_{\tilde{\eta}}$ depends on $\tilde{\eta}$ but not λ. □

Lemma 3.4 *Suppose that* $(V_0), (V_1), (K), (g)$ *and* (f_1)–(f_2) *are satisfied. If* $\Omega := intV^{-1}(0)$ *is nonempty, then, for each* $\lambda > \Lambda_0($ *is the constant given in Remark 2), there exist* $w \in E^+_\lambda \bigoplus F_\lambda, R_\lambda > 0, \eta_\lambda > 0$ *and* $b_\lambda > 0$ *such that for* $|g|_2 < \eta_\lambda, |K|_\infty < b_\lambda($ *or* $|K|_2 < b_\lambda)$*,*
$(a) \sup I_\lambda(\partial Q) \leq 0$,
$(b) \sup I_\lambda(Q)$ *is bounded above by a constant independent of* λ*,*
where $Q := \{u = v + tw : v \in E^-_\lambda, t \geq 0, \|u\|_\lambda \leq R_\lambda\}$.

Proof. Choose $e_0 \in C_0^\infty(\Omega) \setminus \{0\}$, then $e_0 \in F_\lambda$. By Lemma 3.2, there is $R_\lambda > 0$ large such that $I_\lambda(u) \leq 0$ where $u \in E^-_\lambda \bigoplus \mathbb{R}e_0$ and $\|u\|_\lambda \geq R_\lambda$.

For $u = v + w \in E_\lambda^- \bigoplus \mathbb{R}e_0$, we have

$$
\begin{aligned}
I_\lambda(u) &\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_\Omega F(x,u)dx - \int_{\mathbb{R}^3} gudx \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx - \frac{\tilde{\eta}}{2}\int_\Omega u^2 dx - \int_{\Omega, |u|\le r_{\tilde{\eta}}} \left(F(x,u) - \frac{\tilde{\eta}}{2}u^2\right) dx \\
&+ \frac{1}{4}\bar{S}^{-2} d_{12/5}^4 |K|_\infty^2 \|u\|_\lambda^4 + |g|_2 d_2 \|u\|_\lambda \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx - \frac{\tilde{\eta}}{2}\int_\Omega u^2 dx + C_{\tilde{\eta}} + \frac{1}{4}\bar{S}^{-2} d_{12/5}^4 |K|_\infty^2 \|u\|_\lambda^4 + |g|_2 d_2 \|u\|_\lambda. \quad (3.5)
\end{aligned}
$$

Observing $w \in C_0^\infty(\Omega)$, we have

$$
\int_{\mathbb{R}^3} |\nabla w|^2 dx = \int_\Omega (-\Delta w) u dx \le |\Delta w|_2 |u|_{2,\Omega} \le c_0 |\nabla w|_2 |u|_{2,\Omega} \le \frac{c_0^2}{2\tilde{\eta}} |\nabla w|_2^2 + \frac{\tilde{\eta}}{2} |u|_{2,\Omega}^2, \quad (3.6)
$$

where c_0 is a constant depending on e_0. Choosing $\tilde{\eta} > c_0^2$, we have $|\nabla w|_2^2 \le \tilde{\eta} |u|_{2,\Omega}^2$, and it follows from (3.5) that

$$
I_\lambda(u) \le C_{\tilde{\eta}} + \frac{1}{4}\bar{S}^{-2} d_{12/5}^4 |K|_\infty^2 R_\lambda^4 + |g|_2 d_2 R_\lambda \le C_{\tilde{\eta}} + 1
$$

for all $u \in E_\lambda^- \bigoplus \mathbb{R}e_0$ with $\|u\|_\lambda \le R_\lambda$ and $|K|_\infty < b_\lambda := \bar{S}(d_{12/5} R_\lambda)^{-2}$ and $|g|_2 < \eta_\lambda := \frac{1}{2d_2 R_\lambda}$, where $C_{\tilde{\eta}}$ depends on $\tilde{\eta}$ but not λ. □

Now we are in a position to prove our main results.

Proof of Theorem 1. The proof of this theorem is divided in two steps.

Step 1 There exists a function $u_\lambda \in E_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) < 0$. Since $g \in L^2(\mathbb{R}^3)$ and $g \ge 0 (\not\equiv 0)$, we can choose a function $\psi \in E_\lambda$ such that

$$
\int_{\mathbb{R}^3} g(x)\psi(x) dx > 0.
$$

Hence, we have

$$
\begin{aligned}
I(t\psi) &= \frac{t^2}{2}\|\psi\|_\lambda^2 - \frac{\lambda t^2}{2}\int_{\mathbb{R}^3} V^-(x)\psi^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_\psi \psi^2 dx \\
&- \int_{\mathbb{R}^3} F(x, t\psi) dx - t\int_{\mathbb{R}^3} g(x)\psi dx \\
&\le \frac{t^2}{2}\|\psi\|_\lambda^2 + \frac{t^4}{4} C_0 \|\psi\|_\lambda^4 - t\int_{\mathbb{R}^3} g(x)\psi dx \\
&< 0 \quad \textit{for } t > 0 \textit{ small enough.}
\end{aligned}
$$

Thus, there exists u_λ small enough such that $I_\lambda(u_\lambda) < 0$. By Lemma 3.1, we have

$$
c_{0,\lambda} = \inf\{I_\lambda(u) : u \in \overline{B}_{\rho_\lambda}\} < 0,
$$

where $\rho_\lambda > 0$ is given by Lemma 3.1, $B_{\rho_\lambda} = \{u \in E_\lambda : \|u\|_\lambda < \rho_\lambda\}$. By the Ekeland's variational principle, there exists a minimizing sequence $\{u_{n,\lambda}\} \subset \overline{B}_{\rho_\lambda}$ such that

$$
c_{0,\lambda} \le I_\lambda(u_{n,\lambda}) < c_{0,\lambda} + \frac{1}{n_\lambda},
$$

and

$$I_\lambda(w_\lambda) \geq I_\lambda(u_{n,\lambda}) - \frac{1}{n_\lambda}\|w_\lambda - u_{n,\lambda}\|_\lambda$$

for all $w_\lambda \in \overline{B}_{\rho_\lambda}$. Clearly, $\{u_{n,\lambda}\}$ is a bounded Palais-Smale sequence of I_λ. Then, by a standard procedure, Lemma 2.3 and Lemma 2.2 imply that there exists a function $u_\lambda \in E_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_{0,\lambda} < 0$.

If $V \geq 0$, we can get $\rho_\lambda, c_{0,\lambda}, u_{0,\lambda}$ are independent of λ.

Step 2 There exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) > 0$.

It follows from Lemmas 3.1, 3.3 and Proposition 3.1 that, for each $k \in \mathbb{N}, \lambda = \lambda_k, |g|_2 < \eta_k$ and $0 < |K|_\infty < b_k$ (or $0 < |K|_2 < b_k$), I_{λ_k} has a $(C)c$ sequence with $c \in [\alpha_{\lambda_k}, \sup I_{\lambda_k}(Q_k)]$. Setting $M := \sup I_{\lambda_k}(Q_k)$, then I_{λ_k} has a nontrivial critical point according to Lemmas 2.1, 2.4 and Proposition 3.1. That is, there exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) = c \geq \alpha_{\lambda_k} > 0$. The proof is complete. □

Proof of Theorem 2. The first solution is similar to the first solution of Theorem 1. The second solution follows from Lemmas 2.1, 2.4, 3.1, 3.4 and Proposition 3.1. The proof is complete. □

Proof of Theorem 3. The proof of this theorem is divided in two steps.

Step 1 There exists a function $u_0 \in E_\lambda$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) < 0$.

In the proof of Theorem 1, we can choose $c_0 = c_{0,\lambda}, B_\rho = B_{\rho,\lambda}$, then by the By the Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq I_\lambda(u_n) < c_0 + \frac{1}{n},$$

and

$$I_\lambda(w) \geq I_\lambda(u_n) - \frac{1}{n}\|w - u_n\|_\lambda$$

for all $w \in \overline{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of I_λ. Therefore Lemmas 2.3 and 2.2 imply that there exists a function $u_0 \in E_\lambda$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) = c_0 < 0$.

Step 2 There exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) > 0$.

Since we suppose $V \geq 0$, the functional I_λ has mountain pass geometry and the existence of nontrivial solutions can be obtained by mountain pass theorem [22]. In fact, by Lemma 3.1, there exist constants $\alpha, \rho, \eta > 0$ (independent of λ) such that, for each $\lambda > \Lambda_0$,

$$I_\lambda(u) \geq \alpha \ for\ u \in E_\lambda \ with\ \|u\|_\lambda = \rho \ and\ |g|_2 < \eta.$$

Take $e \in C_0^\infty(\Omega) \setminus \{0\}$, by $(f_1), (f_2)$ and Fatou's lemma, we get

$$\frac{I_\lambda(te)}{t^4} \leq \frac{1}{2t^2}\int_\Omega |\nabla e|^2 dx + \frac{1}{4}N(e) - \int_{\{x\in\Omega: e(x)\neq 0\}} \frac{F(x,te)}{(te)^4} e^4 dx - t\int_\Omega ge dx \to -\infty$$

as $t \to +\infty$, which yields that $I_\lambda(te) < 0$ for $t > 0$ large. Clearly, there is $C_1 > 0$ (independent of λ) such that

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \le \sup_{t\ge 0} I_\lambda(te_0) \le C_1$$

where $\Gamma = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \|\gamma(1)\|_\lambda \ge \rho, I_\lambda(\gamma(1)) < 0\}$. By Mountain pass theorem and Lemma 2.3, we obtain a nontrivial critical point $\tilde{u}_\lambda$ of I_λ with $I_\lambda(\tilde{u}_\lambda) \in [\alpha, C_1]$ for λ large. The proof is complete. □

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