The Sorting Index and Permutation Codes

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Abstract

In the combinatorial study of the coefficients of a bivariate polynomial that generalizes both the length and the reflection length generating functions for finite Coxeter groups, Petersen introduced a new Mahonian statistic sor, called the sorting index. Petersen proved that the pairs of statistics $(\mathrm{sor}, \mathrm{cyc})$ and $(\mathrm{inv}, \mathrm{rl\text{-}min})$ have the same joint distribution over the symmetric group, and asked for a combinatorial proof of this fact. In answer to this question, we observe a connection between the sorting index and the B-code of a permutation defined by Foata and Han, and we show that the bijection of Foata and Han serves the purpose of mapping $(\mathrm{inv}, \mathrm{rl\text{-}min})$ to $(\mathrm{sor}, \mathrm{cyc})$. We also give a type B analogue of the bijection of Foata and Han, and derive the equidistribution of $(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})$ and $(\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})$ over signed permutations. So we get a combinatorial interpretation of Petersen's equidistribution of $(\mathrm{inv_B}, \mathrm{nmin_B})$ and $(\mathrm{sor_B}, \mathrm{l}'_\mathrm{B})$. Moreover, we show that the six pairs of set-valued statistics $(\mathrm{Cyc_B}, \mathrm{Rmil_B})$, $(\mathrm{Cyc_B}, \mathrm{Lmap_B})$, $(\mathrm{Rmil_B}, \mathrm{Lmap_B})$, $(\mathrm{Lmap_B}, \mathrm{Rmil_B})$, $(\mathrm{Lmap_B}, \mathrm{Cyc_B})$ and $(\mathrm{Rmil_B}, \mathrm{Cyc_B})$ are equidistributed over signed permutations. For Coxeter groups of type D, Petersen showed that the two statistics $\mathrm{inv_D}$ and $\mathrm{sor_D}$ are equidistributed. We introduce two statistics $\mathrm{nmin_D}$ and $\tilde{\mathrm{l}}'_\mathrm{D}$ for elements of D_n and we prove that the two pairs of statistics $(\mathrm{inv_D}, \mathrm{nmin_D})$ and $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_\mathrm{D})$ are equidistributed.

Keywords: permutation statistic, Mahonian statistic, Coxeter group, set-valued statistic, bijection

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1 Introduction

This paper is concerned with a combinatorial study of the Mahonian statistic sor, introduced by Petersen [10]. This statistic is also interpreted by Wilson [11, 12] as the total distance moved rightward in the random generation of a permutation based on the Fisher-Yates shuffle algorithm.

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Let $[n] = \{1, 2, \ldots, n\}$. The set of permutations of $[n]$ is denoted by S_n. Let us recall the definition of the sorting index of a permutation σ in S_n. Notice that σ has a unique decomposition into transpositions

$$\sigma = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k)$$

such that

$$j_1 < j_2 < \cdots < j_k$$

and

$$i_1 < j_1, i_2 < j_2, \ldots, i_k < j_k.$$

The sorting index is defined by

$$\mathrm{sor}(\sigma) = \sum_{r=1}^{k} (j_r - i_r).$$

Based on the cycle decomposition of a permutation, Foata and Han [6] introduced the B-code of a permutation. We observe that the sorting index of a permutation can be easily expressed in terms of its B-code. Given a permutation $\sigma \in S_n$ with B-code $b = (b_1, b_2, \ldots, b_n)$, it can be seen that the sorting index of σ is given by

$$\mathrm{sor}(\sigma) = \sum_{i=1}^{n} (i - b_i).$$

Petersen [10] has shown that the sorting index sor is a Mahonian statistic, that is, it has the same distribution as the number of inversions. He also introduced the sorting indices for Coxeter groups of type B and type D and showed that they are Mahonian as well.

Let us recall some notation and terminology. For $n \geq 1$, given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, a pair (σ_i, σ_j) is called an inversion if $i < j$ and $\sigma_i > \sigma_j$. Let $\mathrm{inv}(\sigma)$ denote the number of inversions of σ. An element σ_i is said to be a right-to-left minimum of σ if $\sigma_i < \sigma_j$ for all $j > i$. The number of right-to-left minima of σ is denoted by $\mathrm{rl\text{-}min}(\sigma)$. The number of elements of σ that are not right-to-left minima is denoted by $\mathrm{nmin}(\sigma)$. Similarly, one can define a left-to-right maximum. The number of left-to-right maxima of σ is denoted by $\mathrm{lr\text{-}max}(\sigma)$. The number of cycles of σ is denoted by $\mathrm{cyc}(\sigma)$. The reflection length of σ, denoted $\mathrm{l}'(\sigma)$, is the minimal number of transpositions needed to express σ.

By using two factorizations of the diagonal sum, i.e., $\sum_{\sigma \in S_n} \sigma$, in the group algebra $\mathbb{Z}[S_n]$, Petersen has shown that $(\mathrm{sor}, \mathrm{cyc})$ and $(\mathrm{inv}, \mathrm{rl\text{-}min})$ have the same joint distribution by deriving the following generating function formulas:

$$\sum_{\sigma \in S_n} q^{\mathrm{sor}(\sigma)} t^{\mathrm{cyc}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{rl\text{-}min}(\sigma)} = t(t+q) \cdots (t + q + q^2 + \cdots + q^{n-1}).$$

He raised the question of finding a bijection that maps a permutation with inversion number k to a permutation with sorting index k. We find that a bijection constructed by Foata and Han [6] on S_n serves the purpose of mapping (inv, rl-min) to (sor, cyc).

The bijection of Foata and Han is devised to derive the equidistribution of the six pairs of set-valued statistics (Cyc, Rmil), (Cyc, Lmap), (Rmil, Lmap), (Lmap, Rmil), (Lmap, Cyc) and (Rmil, Cyc) over S_n. It should be mentioned that the equidistribution of the three pairs of set-valued statistics (Lmap, Cyc), (Cyc, Lmap), (Lmap, Rmil) reduces to the equidistribution of the three pairs of integer-valued statistics (lr-max, cyc), (cyc, lr-max) and (lr-max, lr-min) established by Cori [4] by employing labeled Dyck paths and the algorithm of Ossona de Mendez and Rosenstiehl [5] on hypermaps.

As for Coxeter groups of type B, the sorting index can be analogously defined and it is Mahonian, see Petersen [10]. Let $\mathrm{sor_B}$, $\mathrm{inv_B}$, $\mathrm{nmin_B}$ and l'_B denote the statistics on signed permutations analogous to sor, inv, nmin and l′ for permutations. Petersen obtained the following formulas for the joint distributions of $(\mathrm{inv_B}, \mathrm{nmin_B})$ and $(\mathrm{sor_B}, \mathrm{l}'_\mathrm{B})$:

$$\sum_{\sigma\in B_n} q^{\mathrm{sor_B}(\sigma)} t^{\mathrm{l}'_\mathrm{B}(\sigma)} = \sum_{\sigma\in B_n} q^{\mathrm{inv_B}(\sigma)} t^{\mathrm{nmin_B}(\sigma)} = \prod_{i=1}^{n} (1 + t[2i]_q - t).$$

We shall present a bijection on B_n which implies the equidistribution of $(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})$ and $(\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})$, where $\mathrm{Lmap_B}$, $\mathrm{Rmil_B}$ and $\mathrm{Cyc_B}$ are set-valued statistics. In particular, this bijection transforms $(\mathrm{inv_B}, \mathrm{nmin_B})$ to $(\mathrm{sor_B}, \mathrm{l}'_\mathrm{B})$. We introduce the A-code and the B-code of a signed permutation, which are analogous to the A-code and the B-code of a permutation. We show that the triple of statistics $(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})$ of a signed permutation can be computed from its A-code, whereas the triple of statistics $(\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})$ can be computed from its B-code. To be more specific, let σ be a signed permutation in B_n with A-code c. Let σ' be a signed permutation in B_n with B-code c. Then the triple of statistics $(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})$ of σ coincides with the triple of statistics $(\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})$ of σ'. We also show that the six pairs of set-valued statistics $(\mathrm{Cyc_B}, \mathrm{Rmil_B})$, $(\mathrm{Cyc_B}, \mathrm{Lmap_B})$, $(\mathrm{Rmil_B}, \mathrm{Lmap_B})$, $(\mathrm{Lmap_B}, \mathrm{Rmil_B})$, $(\mathrm{Lmap_B}, \mathrm{Cyc_B})$ and $(\mathrm{Rmil_B}, \mathrm{Cyc_B})$ are equidistributed over B_n. As a consequence, we see that the four pairs of statistics $(\mathrm{sor_B}, \mathrm{l}'_\mathrm{B})$, $(\mathrm{inv_B}, \mathrm{nmin_B})$, $(\mathrm{inv_B}, \mathrm{nmax_B})$ and $(\mathrm{sor_B}, \mathrm{nmax_B})$ are equidistributed over B_n.

For Coxeter groups of type D, let $\mathrm{sor_D}$ and $\mathrm{inv_D}$ denote the statistics analogous to sor and inv. Let D_n denote the subgroup of B_n consisting of signed permutations with an even number of minus signs. In this case, Petersen has shown that $\mathrm{sor_D}$ and $\mathrm{inv_D}$ have the same generating function, that is,

$$\sum_{\sigma\in D_n} q^{\mathrm{sor_D}(\sigma)} = \sum_{\sigma\in D_n} q^{\mathrm{inv_D}(\sigma)} = [n]_q \prod_{r=1}^{n-1} [2r]_q.$$

We introduce two statistics $\mathrm{nmin_D}$ and $\tilde{\mathrm{l}}'_\mathrm{D}$ analogous to nmin and l′, and we construct a bijection in order to show that the pairs of statistics $(\mathrm{inv_D}, \mathrm{nmin_D})$ and $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_\mathrm{D})$ are equidistributed over D_n. Moreover, we prove that the bivariate generating functions for $(\mathrm{inv_D}, \mathrm{nmin_D})$ and $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})$ are both equal to

$$D_n(q,t) = \prod_{r=1}^{n-1}\left(1 + q^r t + qt \cdot [2r]_q\right) .$$

2 The bijection of Foata and Han

In this section, we give a brief description of Foata and Han's bijection [6] on permutations. Then we show that this bijection transforms (inv, rl-min) to (sor, cyc).

The group of permutations of $[n]$ is also known as a Coxeter group of type A. The length of a permutation $\sigma \in S_n$, denoted by $\mathrm{l}(\sigma)$, is defined to be the minimal number of adjacent transpositions needed to express σ. It is not difficult to see that $\mathrm{inv}(\sigma) = \mathrm{l}(\sigma)$.

We adopt the notation of Foata and Han [6]. They have investigated several set-valued statistics defined as follows. Given a permutation $\sigma \in S_n$, it can be decomposed as a product of disjoint cycles whose minimum elements are $c_1, c_2, \ldots, c_r$. Define Cyc σ to be the set

$$\mathrm{Cyc}\ \sigma = \{c_1, c_2, \ldots, c_r\}.$$

Let $\omega = x_1 x_2 \cdots x_n$ be a word in which the letters are positive integers. The left to right maximum place set of ω, denoted by Lmap ω, is the set of places i such that $x_j < x_i$ for all $j < i$, while the right to left minimum letter set of ω, denoted by Rmil ω, is the set of letters x_i such that $x_j > x_i$ for all $j > i$. For a permutation σ of $[n]$, recall that lr-max(σ) is the number of left-to-right maxima of σ, rl-min(σ) is the number of right-to-left minima of σ, and cyc(σ) is the number of cycles of σ. It is easy to see that the cardinalities of Lmap σ, Rmil σ and Cyc σ are given by lr-max(σ), rl-min(σ) and cyc(σ), respectively.

The Lehmer code [9] of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$ is defined to be a sequence Leh $\sigma = (a_1, a_2, \ldots, a_n)$, where

$$a_i = |\{j \colon 1 \le j \le i, \sigma_j \le \sigma_i\}|.$$

Let SE_n denote the set of integer sequences $(a_1, a_2, \ldots, a_n)$ such that $1 \le a_i \le i$ for all i. It can be seen that Leh: $S_n \longrightarrow \mathrm{SE}_n$ is a bijection. Foata and Han [6] defined the A-code of a permutation σ to be a sequence

$$\text{A-code}\ \sigma = \mathrm{Leh}\ \mathbf{i}\sigma$$

where $\mathbf{i}\colon \sigma \mapsto \sigma^{-1}$ denotes the inverse operation on S_n with respect to product of permutations. For example, let $\sigma = 3\,1\,5\,2\,4$. Then $\mathbf{i}\sigma = 2\,4\,1\,5\,3$. Here a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ standards for a one-to-one function on $[n]$ which maps i to σ_i for

$1 \leq i \leq n$. We multiply permutations from right to left, that is, for $\pi, \sigma \in S_n$, we have $\pi\sigma(i) = \pi(\sigma(i))$ for $1 \leq i \leq n$.

For an integer sequence $a = (a_1, a_2, \ldots, a_n) \in \mathrm{SE}_n$, define Max a to be the set $\{i \colon a_i = i\}$. Given a permutation $\sigma \in S_n$, Foata and Han [6] have shown that the A-code leads to a bijection from S_n to SE_n and the two set-valued statistics Rmil and Lmap of σ are determined by its A-code. Precisely,

$$\mathrm{Rmil}\ \sigma = \mathrm{Max}\ (\text{A-code}\ \sigma), \tag{2.1}$$

$$\mathrm{Lmap}\ \sigma = \mathrm{Rmil}\ (\text{A-code}\ \sigma). \tag{2.2}$$

Following the notation in [6], we rewrite (2.1) and (2.2) as

$$(\mathrm{Rmil}, \mathrm{Lmap})\ \sigma = (\mathrm{Max}, \mathrm{Rmil})\ \text{A-code}\ \sigma. \tag{2.3}$$

Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, the B-code can be defined as follows. For $1 \leq i \leq n$, let k_i be the smallest integer $k \geq 1$ such that $(\sigma^{-k})(i) \leq i$, where σ is considered as a function on $[n]$. Then $b_i = (\sigma^{-k_i})(i)$. In fact, the B-code of a permutation can be easily determined by the cycle decomposition. To compute b_i, we assume that i appears in a cycle C. If i is the smallest element of C, then we set $b_i = i$. Otherwise, we choose b_i to be the element j of C such that $j < i$ and j is the closest to i. Notice that C is viewed as a directed cycle and the distance from j to i is meant to be the number of steps to reach i from j along the cycle. For example, let $\sigma = 2\,4\,5\,1\,3$. Using the cycle decomposition $\sigma = (1\ 2\ 4)(3\ 5)$, we get the B-code $(1, 1, 3, 2, 3)$.

Foata and Han have shown that the B-code is a bijection from S_n to SE_n and the pair of set-valued statistics (Cyc, Lmap) of σ can be determined by the B-code of σ, that is,

$$(\mathrm{Cyc}, \mathrm{Lmap})\ \sigma = (\mathrm{Max}, \mathrm{Rmil})\ \text{B-code}\ \sigma. \tag{2.4}$$

Combining the A-code and the B-code, Foata and Han [6] established a bijection ϕ on S_n as given by

$$\phi = (\text{B-code})^{-1} \circ \text{A-code}.$$

The bijection ϕ implies the following equidistribution.

Theorem 2.1 (Foata and Han [6]) *The six pairs of set-valued statistics* (Cyc, Rmil)*,* (Cyc, Lmap)*,* (Rmil, Lmap)*,* (Lmap, Rmil)*,* (Lmap, Cyc)*,* (Rmil, Cyc) *are equidistributed over* S_n*:*

$$\begin{array}{ccccccccccc} S_n & \xrightarrow{\mathbf{i}} & S_n & \xrightarrow{\phi^{-1}} & S_n & \xrightarrow{\mathbf{i}} & S_n & \xrightarrow{\phi} & S_n & \xrightarrow{\mathbf{i}} & S_n \\ \binom{\mathrm{Cyc}}{\mathrm{Rmil}} & & \binom{\mathrm{Cyc}}{\mathrm{Lmap}} & & \binom{\mathrm{Rmil}}{\mathrm{Lmap}} & & \binom{\mathrm{Lmap}}{\mathrm{Rmil}} & & \binom{\mathrm{Lmap}}{\mathrm{Cyc}} & & \binom{\mathrm{Rmil}}{\mathrm{Cyc}}. \end{array}$$

We now turn to the sorting index. Petersen has shown that the pairs of statistics (sor, cyc) and (inv, rl-min) have the same joint distribution over permutations and asked for a combinatorial interpretation of this fact. We shall show that the map ϕ transforms the pair of statistics (inv, rl-min) of a permutation σ to the pair of statistics (sor, cyc) of the permutation $\phi(\sigma)$. The following lemma indicates that the pair of statistics (inv, rl-min) of σ can be computed from the A-code of σ.

Lemma 2.2 *Let σ be a permutation in S_n with* A-code $a = (a_1, a_2, \ldots, a_n)$. *Then we have*

$$\mathrm{inv}(\sigma) = \sum_{i=1}^{n} (i - a_i) \tag{2.5}$$

and

$$\text{rl-min}(\sigma) = |\mathrm{Max}\ a|. \tag{2.6}$$

Proof. By the definition of the A-code, we find

$$\mathrm{inv}(\sigma) = \binom{n}{2} - \sum_{i=1}^{n} (a_i - 1),$$

which can be rewritten as

$$\sum_{i=1}^{n} (i - a_i).$$

From (2.3) it follows that $\text{rl-min}(\sigma) = |\mathrm{Rmil}\ \sigma| = |\mathrm{Max}\ a|$. This completes the proof. ∎

The following lemma shows that the pair of statistics (sor, cyc) of σ can be recovered from the B-code.

Lemma 2.3 *Let σ be a permutation in S_n with* B-code $b = (b_1, b_2, \ldots, b_n)$. *Then we have*

$$\mathrm{sor}(\sigma) = \sum_{i=1}^{n} (i - b_i) \tag{2.7}$$

and

$$\mathrm{cyc}(\sigma) = |\mathrm{Max}\ b|. \tag{2.8}$$

Proof. Let us examine the algorithm of Foata and Han for recovering a permutation σ from its B-code $b = (b_1, b_2, \ldots, b_n) \in \mathrm{SE}_n$. Start with the identity permutation $\sigma^{(0)} = 1\,2 \cdots n$. For $1 \le i \le n$, the permutation $\sigma^{(i)}$ is obtained by exchanging i and the letter at the b_i-th place in $\sigma^{(i-1)}$. Notice that it may happen that $i = b_i$. Then the resulting permutation $\sigma^{(n)}$ is precisely the permutation with B-code b, that is, $\sigma = \sigma^{(n)}$. So we may write $\sigma^{(i)} = \sigma^{(i-1)}(b_i, i)$, where (b_i, i) is called a transposition even when $b_i = i$. Thus we obtain a decomposition of σ into transpositions

$$\sigma = (b_1, 1)(b_2, 2) \cdots (b_n, n).$$

By the definition of the sorting index, we see that

$$\mathrm{sor}(\sigma) = \sum_{i=1}^{n} (i - b_i).$$

It follows from (2.4) that $\mathrm{cyc}(\sigma) = |\mathrm{Cyc}\ \sigma| = |\mathrm{Max}\ b|$. This completes the proof. ∎

Combining Lemma 2.2 and Lemma 2.3, we conclude that the bijection $\phi = (\text{B-code})^{-1} \circ$ A-code transforms (inv, rl-min) to (sor, cyc), that is, for any $\sigma \in S_n$,

$$(\mathrm{inv}, \text{rl-min})\ \sigma = (\mathrm{sor}, \mathrm{cyc})\ \phi(\sigma).$$

By Theorem 2.1, the bijection ϕ preserves the set-valued statistic Lmap. Since

$$\text{lr-max}(\sigma) = |\mathrm{Lmap}\ \sigma|,$$

ϕ preserves the statistic lr-max. Observing that

$$\text{rl-min}(\sigma) = \text{lr-max}(\mathbf{i}\sigma),$$

we arrive at the following equidistribution.

Theorem 2.4 *The four pairs of statistics* (sor, cyc)*,* (inv, rl-min)*,* (inv, lr-max) *and* (sor, lr-max) *are equidistributed over S_n:*

$$\begin{array}{ccccccc} S_n & \xrightarrow{\phi^{-1}} & S_n & \xrightarrow{\mathbf{i}} & S_n & \xrightarrow{\phi} & S_n \\ \binom{\mathrm{sor}}{\mathrm{cyc}} & & \binom{\mathrm{inv}}{\text{rl-min}} & & \binom{\mathrm{inv}}{\text{lr-max}} & & \binom{\mathrm{sor}}{\text{lr-max}}. \end{array}$$

3 A bijection on signed permutations

In this section, we construct a bijection which serves as a combinatorial interpretation of the equidistribution of the pairs of statistics $(\mathrm{inv_B}, \mathrm{nmin_B})$ and $(\mathrm{sor_B}, \mathrm{l}'_\mathrm{B})$ over signed permutations. In fact, this bijection implies the equidistribution of $(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})$ and $(\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})$ over B_n. Moreover, we show that the six pairs of set-valued statistics $(\mathrm{Cyc_B}, \mathrm{Rmil_B})$, $(\mathrm{Cyc_B}, \mathrm{Lmap_B})$, $(\mathrm{Rmil_B}, \mathrm{Lmap_B})$, $(\mathrm{Lmap_B}, \mathrm{Rmil_B})$, $(\mathrm{Lmap_B}, \mathrm{Cyc_B})$ and $(\mathrm{Rmil_B}, \mathrm{Cyc_B})$ are equidistributed over B_n.

Let us recall some definitions. The hyperoctahedral group B_n is the group of bijections σ on $\{1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}\}$ such that $\sigma(\bar{i}) = \overline{\sigma(i)}$ for $i = 1, 2, \ldots, n$, where $\bar{i}$ denotes $-i$. Clearly, one can represent an element $\sigma \in B_n$ by a signed permutation $a_1 a_2 \cdots a_n$ of $[n]$, that is, a permutation of $[n]$ with some elements associated with a minus sign.

The group B_n has the following Coxeter generators

$$S^B = \{(\bar{1}, 1), (1, 2), (2, 3), \ldots, (n-1, n)\}.$$

The set of reflections of B_n is

$$T^B = \{(i, j) : 1 \leq i < j \leq n\} \cup \{(\bar{i}, j) : 1 \leq i \leq j \leq n\},$$

where the transposition (i, j) means to exchange i and j and exchange $\bar{i}$ with $\bar{j}$ provided that $i \neq \bar{j}$, and $(\bar{i}, i)$ means to exchange i and $\bar{i}$. For $\sigma \in B_n$, let $N(\sigma)$ denote the number of negative elements in the signed permutation notation.

Petersen [10] defined the sorting index for a singed permutation. Let σ be a signed permutation in B_n. He gave a type B analogue of the straight selection sort algorithm of Knuth [8], and proved that σ has a unique factorization into a product of signed transpositions in T^B:

$$\sigma = (i_1, j_1)(i_2, j_2) \cdots (i_m, j_m), \tag{3.1}$$

where $0 < j_1 < j_2 < \cdots < j_m \leq n$. Then the sorting index of σ is defined by

$$\mathrm{sor_B}(\sigma) = \sum_{r=1}^{m} (j_r - i_r - \chi(i_r < 0)).$$

For example, let $\sigma = 5\,\bar{4}\,\bar{3}\,1\,\bar{2}$. Then we have

$$\sigma = (\bar{1}, 2)(\bar{3}, 3)(\bar{2}, 4)(1, 5)$$

and $\mathrm{sor_B}(\sigma) = 2 - (-1) - 1 + 3 - (-3) - 1 + 4 - (-2) - 1 + 5 - 1 = 16$.

For a signed permutation $\sigma \in B_n$, the length of σ, denoted $l_B(\sigma)$, is defined to be the minimal number of transpositions in S^B needed to express σ, see Björner and Brenti [1]. The reflection length of σ, denoted $\mathrm{l}'_{\mathrm{B}}(\sigma)$, is the minimal number of transpositions in T^B needed to express σ. The type B inversion number of σ, denoted $\mathrm{inv_B}(\sigma)$, also denoted finv by Foata and Han [7], is defined as

$$\mathrm{inv_B}(\sigma) = |\{(i, j) : 1 \leq i < j \leq n, \sigma_i > \sigma_j\}| + |\{(i, j) : 1 \leq i \leq j \leq n, \overline{\sigma_i} > \sigma_j\}|.$$

Like the case of type A, we have $\mathrm{inv_B}(\sigma) = \mathrm{l_B}(\sigma)$, see Björner and Brenti [1, Section 8.1].

Recall that for a permutation $\pi \in S_n$, we have $\mathrm{l}'(\pi) = n - \mathrm{cyc}(\pi)$. Similarly, the reflection length of a signed permutation can be determined from its cycle decomposition. A signed permutation σ can be expressed as a product of disjoint signed cycles, see, Brenti [2], Chen and Stanley [3]. For example, let $\sigma = \bar{6}\,\bar{7}\,4\,\bar{3}\,5\,1\,\bar{2}$. Then σ can be written as $\sigma = (1\ \bar{6})(5)(\bar{7}\ \bar{2})(4\ \bar{3})$. A signed cycle is said to be balanced if it contains an even number of minus signs, see [3]. Let $\mathrm{cyc_B}(\sigma)$ denote the number of balanced cycles of σ. It is not difficult to see that $\mathrm{l}'_{\mathrm{B}}(\sigma) = n - \mathrm{cyc_B}(\sigma)$.

We introduce some set-valued statistics for signed permutations which are analogous to those for permutations. For a signed permutation σ, let $C_1, C_2, \ldots, C_r$ be the balanced signed cycles of σ. Let c_i be the smallest absolute value of elements of C_i. Define $\mathrm{Cyc_B}$ to be the set $\{c_1, c_2, \ldots, c_r\}$.

Let $\omega = \omega_1\omega_2\cdots\omega_n$ be a word of length n, where ω_i is an integer. The left to right maximum place set of ω, denoted $\mathrm{Lmap_B}\ \omega$, and the right to left minimum letter set of ω, denoted $\mathrm{Rmil_B}\ \omega$, are defined as follows,

$$\mathrm{Lmap_B}\ \omega = \{i\colon \omega_i > |\omega_j| \text{ for any } j < i\},$$

$$\mathrm{Rmil_B}\ \omega = \{\omega_i\colon 0 < \omega_i < |\omega_j| \text{ for any } j > i\}.$$

When σ is a signed permutation, the cardinality of $\mathrm{Lmap_B}\ \sigma$ is denoted by $\mathrm{lr\text{-}max_B}(\sigma)$ and the cardinality of $\mathrm{Rmil_B}\ \sigma$ is denoted by $\mathrm{rl\text{-}min_B}(\sigma)$. Let

$$\mathrm{nmin_B}(\sigma) = |\{i : \sigma_i > |\sigma_j| \text{ for some } j > i\}| + N(\sigma)$$

and

$$\mathrm{nmax_B}(\sigma) = |\{i : 0 < \sigma_i < |\sigma_j| \text{ for some } j < i\}| + N(\sigma).$$

Evidently, $\mathrm{nmin_B}(\sigma) = n - \mathrm{rl\text{-}min_B}(\sigma)$ and $\mathrm{nmax_B}(\sigma) = n - \mathrm{lr\text{-}max_B}(\sigma)$.

The following theorem is due to Petersen [10].

Theorem 3.1 *The pairs of statistics* $(\mathrm{inv_B}, \mathrm{nmin_B})$ *and* $(\mathrm{sor_B}, \mathrm{l}'_{\mathrm{B}})$ *are equidistributed over* B_n.

Petersen presented two different factorizations of the diagonal sum $\sum_{\sigma\in B_n} \sigma$ and showed that

$$\sum_{\sigma\in B_n} q^{\mathrm{sor_B}(\sigma)} t^{\mathrm{l}'_{\mathrm{B}}(\sigma)} = \sum_{\sigma\in B_n} q^{\mathrm{inv_B}(\sigma)} t^{\mathrm{nmin_B}(\sigma)} = \prod_{i=1}^{n} (1 + t[2i]_q - t).$$

We shall construct a bijection $\psi\colon B_n \longrightarrow B_n$ which transforms $(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})$ to $(\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})$. This bijection can be described in terms of two codes, the A-code and the B-code of a signed permutation. For a signed permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in B_n$, let $\mathbf{i}\colon \sigma \mapsto \sigma^{-1}$ denote the inverse operation on B_n with respect to product of signed permutations. We define the Lehmer code of σ to be an integer sequence $\mathrm{Leh}\ \sigma = (a_1, a_2, \ldots, a_n)$, where for each i,

$$a_i = \mathrm{sign}\ \sigma_i \cdot |\{j\colon 1 \le j \le i, |\sigma_j| \le |\sigma_i|\}|.$$

Then the A-code of a signed permutation σ is defined to be an integer sequence

$$\text{A-code}\ \sigma = \mathrm{Leh}\ \mathbf{i}\sigma.$$

Let $\mathrm{SE}_n^{\mathrm{B}}$ be the set of integer sequences $(a_1, a_2, \ldots, a_n)$ such that $a_i \in [-i, i] \setminus \{0\}$. For an integer sequence $a = (a_1, a_2, \ldots, a_n) \in \mathrm{SE}_n^{\mathrm{B}}$, $\mathrm{Max}\ a$ stands for the set $\{i\colon a_i = i\}$.

The following proposition says that the two set-valued statistics $\mathrm{Rmil_B}$ and $\mathrm{Lmap_B}$ for a signed permutation σ can be recovered from the Lehmer code of σ. The proof is straightforward, and hence it is omitted.

Proposition 3.2 *The Lehmer code* Leh: $B_n \longrightarrow \mathrm{SE}_n^{\mathrm{B}}$ *is a bijection. For each* $\sigma \in B_n$, *we have*

$$\mathrm{Rmil_B}\ \mathrm{Leh}\ \sigma = \mathrm{Rmil_B}\ \sigma \tag{3.2}$$

and

$$\mathrm{Max}\ \mathrm{Leh}\ \sigma = \mathrm{Lmap_B}\ \sigma. \tag{3.3}$$

For example, let $\sigma = 5\,\bar{7}\,1\,\bar{4}\,9\,\bar{2}\,\bar{6}\,3\,8$. Then we have

$$\mathrm{Leh}\ \sigma = (1, -2, 1, -2, 5, -2, -5, 3, 8)$$

and

$$\mathrm{Rmil_B}\ \mathrm{Leh}\ \sigma = \mathrm{Rmil_B}\ \sigma = \{1, 3, 8\},$$
$$\mathrm{Max}\ \mathrm{Leh}\ \sigma = \mathrm{Lmap_B}\ \sigma = \{1, 5\}.$$

The above proposition implies that the A-code is a bijection from B_n to $\mathrm{SE}_n^{\mathrm{B}}$. It is easy to see that $\mathrm{Rmil_B}\ \mathbf{i}\sigma = \mathrm{Lmap_B}\ \sigma$ and $\mathrm{Rmil_B}\ \sigma = \mathrm{Lmap_B}\ \mathbf{i}\sigma$. So we are led to the following theorem which asserts that the two set-valued statistics $\mathrm{Rmil_B}$ and $\mathrm{Lmap_B}$ for a signed permutation σ can be determined by the A-code of σ.

Theorem 3.3 *For any* $\sigma \in B_n$, *we have*

$$(\mathrm{Rmil_B}, \mathrm{Lmap_B})\ \sigma = (\mathrm{Max}, \mathrm{Rmil_B})\ \text{A-code}\ \sigma. \tag{3.4}$$

Next we define the B-code for a signed permutation. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in B_n$. For $1 \leq i \leq n$, let k_i be the smallest integer $k \geq 1$ such that $|\sigma^{-k}(i)| \leq i$. We define the B-code of σ to be the integer sequence $(b_1, b_2, \ldots, b_n)$ with $b_i = (\sigma^{-k_i})(i)$. For example, the B-code of the signed permutation $\sigma = 3\,\bar{1}\,\bar{6}\,\bar{5}\,4\,2$ is $(1, -1, 1, -4, -4, -3)$.

The B-code of a signed permutation can be also defined recursively as follows. First, the B-codes of the two signed permutations of B_1 are defined as B-code $1 = (1)$ and B-code $\bar{1} = (-1)$. For $n \geq 2$, we write a signed permutation $\sigma \in B_n$ as a product of disjoint signed cycles. There are two cases.

Case 1. Assume that n has a positive sign in σ or $\sigma_n = \bar{n}$. Let $\sigma' \in B_{n-1}$ be the signed permutation obtained from σ by deleting n (or $\bar{n}$) in its cycle decomposition. In the case that n (or $\bar{n}$) is in a cycle of length 1, we just delete this cycle. Let $b' = (b_1, b_2, \ldots, b_{n-1})$ be the B-code of σ'. Then we define the B-code of σ to be $b = (b_1, b_2, \ldots, b_{n-1}, \sigma^{-1}(n))$.

Case 2. Assume that n has a minus sign in σ and $\sigma_n \neq \bar{n}$. Changing the sign of σ_n and deleting $\bar{n}$ in the cycle decomposition of σ, we obtain a signed permutation in B_{n-1}, denoted σ'. Let $b' = (b_1, b_2, \ldots, b_{n-1})$ be the B-code of σ'. Then we define the B-code of σ to be $b = (b_1, b_2, \ldots, b_{n-1}, \sigma^{-1}(n))$.

The following theorem shows that the set-valued statistics $\mathrm{Lmap_B}$ and $\mathrm{Cyc_B}$ of a signed permutation can be computed from the B-code.

Theorem 3.4 *The* B-code *is a bijection from* B_n *to* $\mathrm{SE}_n^{\mathrm{B}}$. *Furthermore, for any* $\sigma \in B_n$, *we have*

$$(\mathrm{Cyc_B}, \mathrm{Lmap_B})\ \sigma = (\mathrm{Max}, \mathrm{Rmil_B})\ \text{B-code}\ \sigma. \tag{3.5}$$

Proof. From the recursive definition, it is readily seen that the B-code is a bijection from B_n to $\mathrm{SE}_n^{\mathrm{B}}$. We shall use induction on n to prove (3.5). Clearly, the statement holds for $n = 1$. Assume that (3.5) holds for $n-1$, where $n \geq 2$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a signed permutation of B_n with B-code b. Assume that σ' is the signed permutation of B_{n-1} given in the recursive definition of the B-code. Let $b' = (b_1, b_2, \ldots, b_{n-1})$ be the B-code of σ'.

Now we claim that $\mathrm{Cyc_B}\ \sigma = \mathrm{Max}\ b$. There are two cases according to the sign of n in σ.

First, we consider the case when n has a positive sign in σ. If $\sigma_n \neq n$, let $t = \sigma^{-1}(n)$. Since σ' is obtained from σ by deleting n in its cycle form, the B-code of σ is $b = (b_1, b_2, \ldots, b_{n-1}, t)$. Since $0 < t < n$, we have $\mathrm{Cyc_B}\ \sigma = \mathrm{Cyc_B}\ \sigma'$ and $\mathrm{Max}\ b' = \mathrm{Max}\ b$. By the induction hypothesis, we have $\mathrm{Cyc_B}\ \sigma' = \mathrm{Max}\ b'$. Hence $\mathrm{Cyc_B}\ \sigma = \mathrm{Max}\ b$. If $\sigma_n = n$, it can be easily checked that

$$\mathrm{Cyc_B}\ \sigma = \mathrm{Cyc_B}\ \sigma' \cup \{n\} = \mathrm{Max}\ b' \cup \{n\} = \mathrm{Max}\ b.$$

Then we consider the case when n has a minus sign in σ. If $\sigma_n = \bar{n}$, it is easy to see that

$$\mathrm{Cyc_B}\ \sigma = \mathrm{Cyc_B}\ \sigma' = \mathrm{Max}\ b' = \mathrm{Max}\ b.$$

If $\sigma_n \neq \bar{n}$, we let $t = \sigma^{-1}(n)$. Since n has a minus sign in σ, we have $t < 0$. Since $b' = (b_1, b_2, \ldots, b_{n-1})$ is the B-code of σ', we find that the B-code of σ is $b = (b_1, b_2, \ldots, b_{n-1}, t)$. Since $-n < t < 0$, we see that $\mathrm{Cyc_B}\ \sigma = \mathrm{Cyc_B}\ \sigma'$ and $\mathrm{Max}\ b' = \mathrm{Max}\ b$. By the induction hypothesis, we get $\mathrm{Cyc_B}\ \sigma' = \mathrm{Max}\ b'$. Thus we obtain $\mathrm{Cyc_B}\ \sigma = \mathrm{Max}\ b$.

We now turn to the proof of the relation $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$. There are four cases.

Case 1: $\sigma_n = n - 1$. By the recursive definition of the B-code, we express σ and σ' in the one-line notation as follows. For convenience, we display the identity permutation on the top,

$$\begin{array}{rccccccc} & 1 & \cdots & |\sigma^{-1}(n)| & \cdots & n-1 & n \\ \sigma = & \sigma_1 & \cdots & \epsilon n & \cdots & \sigma_{n-1} & n-1 \\ \sigma' = & \sigma_1 & \cdots & \epsilon(n-1) & \cdots & \sigma_{n-1}. & \end{array}$$

Here $\epsilon = 1$ if n has a positive sign in σ and $\epsilon = -1$ if n has a minus sign in σ. It can be easily checked that $\mathrm{Lmap_B}\ \sigma = \mathrm{Lmap_B}\ \sigma'$. Since $b' = (b_1, b_2, \ldots, b_{n-1})$ is the B-code of σ', we have $b_{n-1} = \sigma^{-1}(n)$ and the B-code of σ is $b = (b_1, b_2, \ldots, b_{n-1}, \sigma^{-1}(n))$. It follows that $\mathrm{Rmil_B}\ b = \mathrm{Rmil_B}\ b'$. By the induction hypothesis, we get $\mathrm{Lmap_B}\ \sigma' = \mathrm{Rmil_B}\ b'$. Hence we deduce that $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$.

Case 2: $\sigma_n = \overline{n-1}$. If n has a minus sign in σ, let t be the positive integer such that $\sigma_t = \bar{n}$. As in Case 1, we express σ and σ' as follows

$$\begin{array}{rcccccc} & 1 & \cdots & t & \cdots & n-1 & n \\ \sigma = & \sigma_1 & \cdots & \bar{n} & \cdots & \sigma_{n-1} & \overline{n-1} \\ \sigma' = & \sigma_1 & \cdots & n-1 & \cdots & \sigma_{n-1}. & \end{array}$$

Clearly, $\mathrm{Lmap_B}\ \sigma = \mathrm{Lmap_B}\ \sigma' \setminus \{t\}$. Since $b' = (b_1, b_2, \ldots, b_{n-1})$ is the B-code of σ', we have $b_{n-1} = \sigma'^{-1}(n-1) = t$. From the recursive construction of the B-code, it follows that the B-code of σ is $b = (b_1, b_2, \ldots, b_{n-1}, -t\,)$. This implies that $\mathrm{Rmil_B}\ b = \mathrm{Rmil_B}\ b' \setminus \{t\}$. By the induction hypothesis, we obtain $\mathrm{Lmap_B}\ \sigma' = \mathrm{Rmil_B}\ b'$. Therefore $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$. If n has a positive sign in σ, let t be the positive integer such that $\sigma_t = n$. Then σ and σ' can be expressed as follows

$$\begin{array}{rcccccc} & 1 & \cdots & t & \cdots & n-1 & n \\ \sigma = & \sigma_1 & \cdots & n & \cdots & \sigma_{n-1} & \overline{n-1} \\ \sigma' = & \sigma_1 & \cdots & \overline{n-1} & \cdots & \sigma_{n-1}. & \end{array}$$

In this case, we have $\mathrm{Lmap_B}\ \sigma = \mathrm{Lmap_B}\ \sigma' \cup \{t\}$. Since $b' = (b_1, b_2, \ldots, b_{n-1})$ is the B-code of σ', then $b_{n-1} = -t$ and the B-code of σ is $b = (b_1, b_2, \ldots, b_{n-1}, t\,)$. It follows that $\mathrm{Rmil_B}\ b = \mathrm{Rmil_B}\ b' \cup \{t\}$. By the induction hypothesis, we deduce that $\mathrm{Lmap_B}\ \sigma' = \mathrm{Rmil_B}\ b'$. So we arrive at $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$.

Case 3: $\sigma_n \neq n-1$, $\sigma_n \neq \overline{n-1}$ and $|\sigma^{-1}(n-1)| < |\sigma^{-1}(n)|$. If n has a positive sign in σ, let $\sigma_t = n$. By the same argument as in Case 2, we find $\mathrm{Lmap_B}\ \sigma = \mathrm{Lmap_B}\ \sigma' \cup \{t\}$ and $\mathrm{Rmil_B}\ b = \mathrm{Rmil_B}\ b' \cup \{t\}$. By the induction hypothesis, we deduce that $\mathrm{Lmap_B}\ \sigma' = \mathrm{Rmil_B}\ b'$. Hence $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$. If n has a minus sign in σ, it can be verified that $\mathrm{Lmap_B}\ \sigma = \mathrm{Lmap_B}\ \sigma'$ and $\mathrm{Rmil_B}\ b = \mathrm{Rmil_B}\ b'$. Therefore, we obtain $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$.

Case 4: $\sigma_n \neq n-1$, $\sigma_n \neq \overline{n-1}$ and $|\sigma^{-1}(n-1)| > |\sigma^{-1}(n)|$. If n has a positive sign in σ, let $\sigma_t = n$. We write σ and σ' as follows

$$\begin{array}{rcccccccc} & 1 & \cdots & t & \cdots & |\sigma^{-1}(n-1)| & \cdots & n-1 & n \\ \sigma = & \sigma_1 & \cdots & n & \cdots & \epsilon(n-1) & \cdots & \sigma_{n-1} & \sigma_n \\ \sigma' = & \sigma_1 & \cdots & \sigma_n & \cdots & \epsilon(n-1) & \cdots & \sigma_{n-1}, & \end{array}$$

where $\epsilon = 1$ if $n-1$ appears as an element in σ and $\epsilon = -1$ if $\overline{n-1}$ appears as an element in σ. It can be seen that

$$\mathrm{Lmap_B}\ \sigma = (\mathrm{Lmap_B}\ \sigma' \cap [1, t-1]) \cup \{t\}.$$

Since $b' = (b_1, b_2, \ldots, b_{n-1})$ is the B-code of σ', we have $b_{n-1} = \sigma^{-1}(n-1)$ and the B-code of σ is $b = (b_1, b_2, \ldots, b_{n-1}, t)$. Hence we get

$$\mathrm{Rmil_B}\ b = (\mathrm{Rmil_B}\ b' \cap [1, t-1]) \cup \{t\}.$$

By the induction hypothesis, we obtain $\mathrm{Lmap_B}\ \sigma' = \mathrm{Rmil_B}\ b'$. Thus we get $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$. If n has a minus sign in σ, it can be checked that

$$\mathrm{Lmap_B}\ \sigma = \mathrm{Lmap_B}\ \sigma' \cap [1, -\sigma^{-1}(n) - 1]$$

and

$$\mathrm{Rmil_B}\ b = \mathrm{Rmil_B}\ b' \cap [1, -\sigma^{-1}(n) - 1].$$

By the induction hypothesis, we conclude that $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ b$. This completes the proof. ∎

In fact, it can be shown that the pair of statistics $(\mathrm{inv_B}, \mathrm{nmin_B})$ of a signed permutation σ can be recovered from its A-code and the pair of statistics $(\mathrm{sor_B}, \mathrm{l'_B})$ can be recovered from its B-code.

We now describe how to recover a signed permutation σ from its A-code $a = (a_1, a_2, \ldots, a_n) \in \mathrm{SE}_n^{\mathrm{B}}$. It is essentially the same as the procedure to recover a permutation from the Lehmer code.

We start with the empty word $\sigma^{(0)}$. It will take n steps to construct a signed permutation σ with A-code a. At the first step, if $a_1 = 1$, then set $\sigma^{(1)} = 1$. If $a_1 = -1$, then set $\sigma^{(1)} = \bar{1}$. For $1 < i \leq n$, assume that at step i, we have constructed a signed permutation $\sigma^{(i-1)} \in B_{i-1}$. If $|a_i| = 1$, the signed permutation $\sigma^{(i)}$ is obtained from $\sigma^{(i-1)}$ by inserting the element i with the sign of a_i before the first element of $\sigma^{(i-1)}$. If $|a_i| > 1$, then the signed permutation $\sigma^{(i)}$ is obtained from $\sigma^{(i-1)}$ by inserting the element i with the sign of a_i after the $(|a_i| - 1)$-th element in $\sigma^{(i-1)}$. Eventually, the signed permutation $\sigma^{(n)}$ is a signed permutation σ with A-code a. For example, let $a = (1, 1, -3, -2, 3)$. Then we have

$$\begin{array}{lrl} & \sigma^{(0)} = & \emptyset, \\ a_1 = 1, & \sigma^{(1)} = & 1, \\ a_2 = 1, & \sigma^{(2)} = & 2\ 1, \\ a_3 = -3, & \sigma^{(3)} = & 2\ 1\ \bar{3}, \\ a_4 = -2, & \sigma^{(4)} = & 2\ \bar{4}\ 1\ \bar{3}, \\ a_5 = 3, & \sigma^{(5)} = & 2\ \bar{4}\ 5\ 1\ \bar{3}. \end{array}$$

So $2\,\bar{4}\,5\,1\,\bar{3}$ is the signed permutation with A-code $(1, 1, -3, -2, 3)$.

The relationship between a signed permutation σ and its B-code $b = (b_1, b_2, \ldots, b_n)$ can be described as follows. Let σ' be the signed permutation obtained from σ as in the recursive construction of the B-code. So the B-code of σ' is $b' = (b_1, b_2, \ldots, b_{n-1})$. If n has a positive sign in σ or $\sigma_n = \bar{n}$, then σ' is obtained from σ by deleting n in its cycle decomposition. Let (i, i) denote the identity permutation for $1 \leq i \leq n$. Since $b_n = \sigma^{-1}(n)$, we have $\sigma = \sigma'(b_n, n)$. Note that σ' is considered as a signed permutation of B_n which maps n to n. If n has a minus sign in σ and $\sigma_n \neq \bar{n}$, then σ' is obtained from σ by changing the sign of σ_n and deleting $\bar{n}$ in its cycle decomposition. Since $b_n = \sigma^{-1}(n)$, we find that $\sigma = \sigma'(b_n, n)$. Again, σ' is considered as a signed permutation of B_n which maps n to n. Hence we obtain that $\sigma = (b_1, 1)(b_2, 2)\cdots(b_n, n)$.

The following lemma gives expressions of $\mathrm{inv_B}(\sigma)$ and $\mathrm{nmin_B}(\sigma)$ in terms of the A-code of σ.

Lemma 3.5 *For a signed permutation $\sigma \in B_n$ with* A-code $a = (a_1, a_2, \ldots, a_n)$, *we have*

$$\mathrm{inv_B}(\sigma) = \sum_{i=1}^{n}(i - a_i - \chi(a_i < 0)) \tag{3.6}$$

and

$$\mathrm{nmin_B}(\sigma) = n - |\mathrm{Max}\ a|. \tag{3.7}$$

Proof. Consider the procedure to recover a signed permutation from the A-code a. It is easily seen that after the i-th step, the type B inversion number increases by $i - a_i$ when $a_i > 0$ and by $i - a_i - 1$ when $a_i < 0$. Hence we have

$$\mathrm{inv_B}(\sigma^{(i)}) - \mathrm{inv_B}(\sigma^{(i-1)}) = i - a_i - \chi(a_i < 0).$$

Since $\mathrm{inv_B}(\sigma^{(0)}) = 0$, we find

$$\mathrm{inv_B}(\sigma) = \sum_{i=1}^{n}(i - a_i - \chi(a_i < 0)).$$

In view of (3.4), it is easy to see that

$$\mathrm{nmin_B}(\sigma) = n - \text{rl-min}_\mathrm{B}(\sigma) = n - |\mathrm{Rmil_B}\ \sigma| = n - |\mathrm{Max}\ a|.$$

This completes the proof. ∎

The following lemma shows that $\mathrm{sor_B}(\sigma)$ and $\mathrm{l'_B}(\sigma)$ can be expressed in terms of the B-code of σ.

Lemma 3.6 *For a signed permutation $\sigma \in B_n$ with* B-code $b = (b_1, b_2, \ldots, b_n)$, *we have*

$$\mathrm{sor_B}(\sigma) = \sum_{i=1}^{n}(i - b_i - \chi(b_i < 0)) \tag{3.8}$$

and

$$\mathrm{l'_B}(\sigma) = n - |\mathrm{Max}\ b|. \tag{3.9}$$

Proof. Since $b = (b_1, b_2, \ldots, b_n)$ is the B-code of σ, it is known that

$$\sigma = (b_1, 1)(b_2, 2)\cdots(b_n, n).$$

By the definition of the sorting index of σ, we see that

$$\mathrm{sor_B}(\sigma) = \sum_{i=1}^{n} (i - b_i - \chi(b_i < 0)).$$

From (3.5) it follows that

$$\mathrm{l'_B}(\sigma) = n - \mathrm{cyc_B}(\sigma) = n - |\mathrm{Cyc_B}\ \sigma| = n - |\mathrm{Max}\ b|.$$

This completes the proof. ∎

Combining Theorem 3.3, Theorem 3.4, Lemma 3.5 and Lemma 3.6, we obtain the equidistribution of $(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})$ and $(\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})$ over B_n.

Theorem 3.7 *The map* $\psi\colon B_n \longrightarrow B_n$ *defined by* $\psi = (\text{B-code})^{-1} \circ \text{A-code}$ *is a bijection. For any* $\sigma \in B_n$*, we have*

$$(\mathrm{inv_B}, \mathrm{Lmap_B}, \mathrm{Rmil_B})\ \sigma = (\mathrm{sor_B}, \mathrm{Lmap_B}, \mathrm{Cyc_B})\ \psi(\sigma). \tag{3.10}$$

In particular,

$$(\mathrm{inv_B}, \mathrm{nmin_B})\ \sigma = (\mathrm{sor_B}, \mathrm{l'_B})\ \psi(\sigma). \tag{3.11}$$

Notice that $\mathrm{Cyc_B}\ \sigma = \mathrm{Cyc_B}\ \mathbf{i}\sigma$ and $\mathrm{Lmap_B}\ \sigma = \mathrm{Rmil_B}\ \mathbf{i}\sigma$. Thus Theorem 3.7 implies the following equidistribution which can be viewed as a type B analogue of the equidistribution given in Theorem 2.1.

Theorem 3.8 *The six pairs of set-valued statistics* $(\mathrm{Cyc_B}, \mathrm{Rmil_B})$*,* $(\mathrm{Cyc_B}, \mathrm{Lmap_B})$*,* $(\mathrm{Rmil_B}, \mathrm{Lmap_B})$*,* $(\mathrm{Lmap_B}, \mathrm{Rmil_B})$*,* $(\mathrm{Lmap_B}, \mathrm{Cyc_B})$ *and* $(\mathrm{Rmil_B}, \mathrm{Cyc_B})$ *are equidistributed over* B_n*:*

$$\begin{array}{ccccccccccccc} B_n & \overset{\mathbf{i}}{\longrightarrow} & B_n & \overset{\psi^{-1}}{\longrightarrow} & B_n & \overset{\mathbf{i}}{\longrightarrow} & B_n & \overset{\psi}{\longrightarrow} & B_n & \overset{\mathbf{i}}{\longrightarrow} & B_n \\ \binom{\mathrm{Cyc_B}}{\mathrm{Rmil_B}} & & \binom{\mathrm{Cyc_B}}{\mathrm{Lmap_B}} & & \binom{\mathrm{Rmil_B}}{\mathrm{Lmap_B}} & & \binom{\mathrm{Lmap_B}}{\mathrm{Rmil_B}} & & \binom{\mathrm{Lmap_B}}{\mathrm{Cyc_B}} & & \binom{\mathrm{Rmil_B}}{\mathrm{Cyc_B}}. \end{array}$$

The above theorem for set-valued statistics reduces to the following equidistribution of pairs of statistics of signed permutations. It is clear that $\mathrm{nmin_B}(\sigma) = \mathrm{nmax_B}(\mathbf{i}\sigma)$. Since the bijection ψ preserves $\mathrm{Lmap_B}$, it is easy to see that ψ also preserves the statistic $\mathrm{nmax_B}$. Hence we are led to the following equidistribution.

Corollary 3.9 *The four pairs of statistics* $(\mathrm{sor_B}, \mathrm{l'_B})$*,* $(\mathrm{inv_B}, \mathrm{nmin_B})$*,* $(\mathrm{inv_B}, \mathrm{nmax_B})$ *and* $(\mathrm{sor_B}, \mathrm{nmax_B})$ *are equidistributed over* B_n*:*

$$\begin{array}{ccccccc} B_n & \overset{\psi^{-1}}{\longrightarrow} & B_n & \overset{\mathbf{i}}{\longrightarrow} & B_n & \overset{\psi}{\longrightarrow} & B_n \\ \binom{\mathrm{sor_B}}{\mathrm{l'_B}} & & \binom{\mathrm{inv_B}}{\mathrm{nmin_B}} & & \binom{\mathrm{inv_B}}{\mathrm{nmax_B}} & & \binom{\mathrm{sor_B}}{\mathrm{nmax_B}}. \end{array}$$

4 A bijection on D_n

In this section, we define two statistics $\mathrm{nmin_D}$ and $\tilde{\mathrm{l}}'_{\mathrm{D}}$ for elements of a Coxeter group of type D and we construct a bijection to derive the equidistribution of the pairs of statistics $(\mathrm{inv_D}, \mathrm{nmin_D})$ and $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})$. This yields a refinement of Petersen's equidistribution of $\mathrm{inv_D}$ and $\mathrm{sor_D}$.

The type D Coxeter group D_n is the subgroup of B_n consisting of signed permutations with an even number of minus signs in the signed permutation notation. As a set of generators for D_n, we take

$$S^D = \{(\bar{1}, 2), (1, 2), (2, 3), \ldots, (n-1, n)\}.$$

For simplicity, let $s_i = (i, i+1)$ for $1 \le i < n$ and $s_{\bar{1}} = (\bar{1}, 2)$. The set of reflections of D_n is

$$R^D = \{(i, j) : 1 \le |i| < j \le n\}.$$

For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in D_n$, the type D inversion number of σ is given by

$$\mathrm{inv_D}(\sigma) = |\{(i, j) : 1 \le i < j \le n, \sigma_i > \sigma_j\}| + |\{(i, j) : 1 \le i < j \le n, \overline{\sigma_i} > \sigma_j\}|.$$

The length of σ, denoted $\mathrm{l_D}(\sigma)$, is the minimal number of transpositions in S^D needed to express σ. It is known that $\mathrm{l_D}(\sigma) = \mathrm{inv_D}(\sigma)$, see Björner and Brenti [1, Section 8.2]. The generating function of $\mathrm{l_D}$ is

$$\sum_{\sigma \in D_n} q^{\mathrm{l_D}(\sigma)} = [n]_q \prod_{r=1}^{n-1} [2r]_q, \tag{4.1}$$

see also [1].

Recall that the set of reflections of B_n is

$$T^B = \{(i, j) : 1 \le i < j \le n\} \cup \{(\bar{i}, j) : 1 \le i \le j \le n\}.$$

For $\sigma \in D_n$, it has a unique factorization into a product of signed transpositions in T^B:

$$\sigma = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k), \tag{4.2}$$

where $0 < j_1 < j_2 < \cdots < j_k \le n$. Petersen defined the type D sorting index of σ as

$$\mathrm{sor_D}(\sigma) = \sum_{r=1}^{k} (j_r - i_r - 2\chi(i_r < 0)).$$

It has been shown by Petersen that $\mathrm{sor_D}$ has the same generating function as $\mathrm{inv_D}$.

Theorem 4.1 *For $n \ge 4$,*

$$\sum_{\sigma \in D_n} q^{\mathrm{sor_D}(\sigma)} = [n]_q \prod_{r=1}^{n-1} [2r]_q. \tag{4.3}$$

Thus, $\mathrm{sor_D}$ *is Mahonian.*

Next we define two statistics $\tilde{\mathrm{l}}'_{\mathrm{D}}$ and $\mathrm{nmin}_{\mathrm{D}}$ for a signed permutation $\sigma \in D_n$. For $1 \leq |i| < j \leq n$, we adopt the notation t_{ij} for the transposition (i, j). For $1 < i \leq n$, we define $t_{\bar{i}i} = (\bar{i}, i)(\bar{1}, 1)$. Then we set

$$T^D = \{t_{ij} : 1 \leq |i| < j \leq n\} \cup \{t_{\bar{i}i} : 1 < i \leq n\}.$$

We denote by $\tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma)$ the minimal number of elements in T^D that are needed to express σ. Define the statistic $\mathrm{nmin}_{\mathrm{D}}$ as

$$\mathrm{nmin}_{\mathrm{D}}(\sigma) = |\{i : \sigma_i > |\sigma_j| \text{ for some } j > i\}| + N(\sigma\backslash\{\bar{1}\}),$$

where $N(\sigma\backslash\{\bar{1}\})$ is the number of minus signs associated with elements greater than 1 in the signed permutation notation of σ.

The following theorem is a refinement of the equidistribution of $\mathrm{inv}_{\mathrm{D}}$ and $\mathrm{sor}_{\mathrm{D}}$. We shall give a combinatorial proof and an algebraic proof.

Theorem 4.2 *For $n \geq 2$, the two pairs of statistics $(\mathrm{inv}_{\mathrm{D}}, \mathrm{nmin}_{\mathrm{D}})$ and $(\mathrm{sor}_{\mathrm{D}}, \tilde{\mathrm{l}}'_{\mathrm{D}})$ are equidistributed over D_n. Moreover,*

$$\sum_{\sigma \in D_n} q^{\mathrm{inv}_{\mathrm{D}}(\sigma)} t^{\mathrm{nmin}_{\mathrm{D}}(\sigma)} = \prod_{r=1}^{n-1} (1 + q^r t + qt \cdot [2r]_q), \tag{4.4}$$

$$\sum_{\sigma \in D_n} q^{\mathrm{sor}_{\mathrm{D}}(\sigma)} t^{\tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma)} = \prod_{r=1}^{n-1} (1 + q^r t + qt \cdot [2r]_q). \tag{4.5}$$

To give a combinatorial proof of the equidistribution of $(\mathrm{inv}_{\mathrm{D}}, \mathrm{nmin}_{\mathrm{D}})$ and $(\mathrm{sor}_{\mathrm{D}}, \tilde{\mathrm{l}}'_{\mathrm{D}})$ in Theorem 4.2, we introduce the co-sorting index $\mathrm{sor}'_{\mathrm{D}}$ which turns out to be equivalent to the sorting index $\mathrm{sor}_{\mathrm{D}}$. To define the co-sorting index, we need the factorization of an element $\sigma \in D_n$ into elements in T^D. More precisely, we can express $\sigma \in D_n$ uniquely in the following form

$$\sigma = t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_m j_m},$$

where $1 < j_1 < j_2 < \cdots < j_m \leq n$. For example, let $\sigma = \bar{2}\,\bar{4}\,5\,\bar{1}\,\bar{3}$. Then we have $\sigma = t_{12} t_{\bar{3}3} t_{\bar{2}4} t_{35}$. The co-sorting index of σ is defined by

$$\mathrm{sor}'_{\mathrm{D}}(\sigma) = \sum_{r=1}^{m} (j_r - i_r - 2\chi(i_r < 0)).$$

Lemma 4.3 *For any $\sigma \in D_n$, we have $\mathrm{sor}_{\mathrm{D}}(\sigma) = \mathrm{sor}'_{\mathrm{D}}(\sigma)$.*

Proof. Recall that σ can be written as

$$\sigma = t_{i_1 j_1} t_{i_2 j_2} \cdots t_{i_m j_m}, \tag{4.6}$$

where $t_{i_1j_1}, t_{i_2j_2}, \ldots, t_{i_mj_m} \in T^D$ and $1 < j_1 < j_2 < \cdots < j_m \leq n$. Since the co-sorting index of σ can be expressed in terms of the factorization (4.6), to prove the the equivalence of the sorting index and the co-sorting index of σ, we proceed to rewrite (4.6) as a product of transpositions in T^B from which the sorting index of σ can be determined.

In fact, it can be shown that σ can be written as a product of transpositions in T^B which is either of the form

$$(p_1, j_1)(p_2, j_2)\cdots(p_m, j_m), \tag{4.7}$$

or of the form

$$(\bar{1}, 1)(p_1, j_1)(p_2, j_2)\cdots(p_m, j_m), \tag{4.8}$$

where for $1 \leq k \leq m$,

$$p_k = \begin{cases} 1 \text{ or } \bar{1}, & \text{if } i_k = 1, \\ 1 \text{ or } \bar{1}, & \text{if } i_k = \bar{1}, \\ i_k, & \text{otherwise.} \end{cases} \tag{4.9}$$

We claim that for $1 \leq r \leq m$, $t_{i_rj_r}t_{i_{r+1}j_{r+1}}\cdots t_{i_mj_m}$ can be expressed as a product of transpositions in T^B which is either of the form

$$(p_r, j_r)(p_{r+1}, j_{r+1})\cdots(p_m, j_m) \tag{4.10}$$

or of the form

$$(\bar{1}, 1)(p_r, j_r)(p_{r+1}, j_{r+1})\cdots(p_m, j_m), \tag{4.11}$$

where p_k is given as in (4.9). Let us first consider the case $r = m$. In this case, if $i_m \neq \overline{j_m}$, then $t_{i_mj_m}$ equals (i_m, j_m), which is of the form (4.10). If $i_m = \overline{j_m}$, then $t_{i_mj_m}$ equals $(\bar{1}, 1)(i_m, j_m)$, which is of the form (4.11).

Assume that the claim holds for r, where $1 < r \leq m$. We wish to show that it holds for $r - 1$. If $t_{i_rj_r}t_{i_{r+1}j_{r+1}}\cdots t_{i_mj_m}$ can be expressed in the form (4.10), then we have

$$t_{i_{r-1}j_{r-1}}t_{i_rj_r}\cdots t_{i_mj_m} = \begin{cases} (\bar{1}, 1)(i_{r-1}, j_{r-1})(p_r, j_r)\cdots(p_m, j_m), & \text{if } i_{r-1} = \overline{j_{r-1}}, \\ (i_{r-1}, j_{r-1})(p_r, j_r)\cdots(p_m, j_m), & \text{otherwise,} \end{cases}$$

which is either of the form (4.11) or of the form (4.10). We now assume that $t_{i_rj_r}t_{i_{r+1}j_{r+1}}\cdots t_{i_mj_m}$ can be expressed in the form (4.11). It follows that

$$t_{i_{r-1}j_{r-1}}t_{i_rj_r}\cdots t_{i_mj_m} = \begin{cases} (i_{r-1}, j_{r-1})(p_r, j_r)\cdots(p_m, j_m), & \text{if } i_{r-1} = \overline{j_{r-1}}, \\ (\bar{1}, 1)(\overline{i_{r-1}}, j_{r-1})(p_r, j_r)\cdots(p_m, j_m), & \text{if } i_{r-1} = 1 \text{ or } \bar{1}, \\ (\bar{1}, 1)(i_{r-1}, j_{r-1})(p_r, j_r)\cdots(p_m, j_m), & \text{otherwise,} \end{cases}$$

which is either of the form (4.10) or of the form (4.11). Thus the claim holds for $1 \leq r \leq m$.

So we have shown that σ can be expressed as (4.7) or (4.8). Hence the sorting index $\mathrm{sor_D}(\sigma)$ can be determined by this factorization, namely,

$$\mathrm{sor_D}(\sigma) = \sum_{r=1}^{m} (j_r - p_r - 2\chi(p_r < 0)).$$

By (4.9), we find that

$$j_r - p_r - 2\chi(p_r < 0) = j_r - i_r - 2\chi(i_r < 0)$$

for $1 \leq r \leq m$. In view of (4.6), we see that

$$\mathrm{sor}'_\mathrm{D}(\sigma) = \sum_{r=1}^{m} (j_r - i_r - 2\chi(i_r < 0)).$$

It follows that $\mathrm{sor_D}(\sigma) = \mathrm{sor}'_\mathrm{D}(\sigma)$. This completes the proof. ∎

To justify the equidistribution of $(\mathrm{inv_D}, \mathrm{nmin_D})$ and $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_\mathrm{D})$, we shall give a bijection which transforms $(\mathrm{inv_D}, \mathrm{nmin_D})$ to $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_\mathrm{D})$. This bijection can be described in terms of two codes, called the E-code and the F-code of an element of D_n. It can be shown that the pair of statistics $(\mathrm{inv_D}, \mathrm{nmin_D})$ can be computed from the E-code, whereas the pair of statistics $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_\mathrm{D})$ can be computed from the F-code.

Given an element $\sigma \in D_n$, the E-code of σ is an integer sequence $e = (e_1, e_2, \ldots, e_n)$ generated by the following procedure. We wish to construct a sequence of signed permutations $\sigma^{(n)}, \sigma^{(n-1)}, \ldots, \sigma^{(1)}$, where $\sigma^{(i)} \in D_i$ for $1 \leq i \leq n$. First, we set $\sigma^{(n)} = \sigma$. For i from n to 2, we construct $\sigma^{(i-1)}$ from $\sigma^{(i)}$. Consider the letter i in $\sigma^{(i)}$. If i has a positive sign in $\sigma^{(i)}$, say, i appears at the p-th position in $\sigma^{(i)}$, then we set $e_i = p$ and let $\sigma^{(i-1)}$ be the signed permutation obtained from $\sigma^{(i)}$ by deleting the element i. If i has a minus sign in $\sigma^{(i)}$, say, $\bar{i}$ appears at the p-th position in $\sigma^{(i)}$, then set $e_i = -p$. Let σ' be the signed permutation obtained from $\sigma^{(i)}$ by deleting $\bar{i}$, and let $\sigma^{(i-1)}$ be the signed permutation obtained from σ' by changing the sign of the element at the first position.

It can be checked that the resulting signed permutation $\sigma^{(1)}$ is the identity permutation 1. Finally, we set $e_1 = 1$. For example, let $\sigma = 2\,\bar{4}\,5\,1\,\bar{3}$. Then we have

$$\begin{aligned}
\sigma^{(5)} &= \; 2\;\bar{4}\;\mathbf{5}\;1\;\bar{3}, \quad e_5 = 3,\\
\sigma^{(4)} &= \; 2\;\bar{\mathbf{4}}\;1\;\bar{3}, \quad e_4 = -2,\\
\sigma^{(3)} &= \; \bar{2}\;1\;\bar{\mathbf{3}}, \quad e_3 = -3,\\
\sigma^{(2)} &= \; \mathbf{2}\;1, \quad e_2 = 1,\\
\sigma^{(1)} &= \; 1, \quad e_1 = 1.
\end{aligned}$$

Hence the E-code of $\sigma = 2\,\bar{4}\,5\,1\,\bar{3}$ is $(1, 1, -3, -2, 3)$.

It can be seen that the above procedure is reversible. In other words, one can recover an element $\sigma \in D_n$ from an E-code $e = (e_1, e_2, \ldots, e_n)$. For $1 < r \leq n$, it is routine to verify that

$$\mathrm{inv_D}(\sigma^{(r)}) - \mathrm{inv_D}(\sigma^{(r-1)}) = r - e_r - 2\chi(e_r < 0) \tag{4.12}$$

and

$$\mathrm{nmin_D}(\sigma^{(r)}) - \mathrm{nmin_D}(\sigma^{(r-1)}) = 1 - \chi(e_r = r). \tag{4.13}$$

So we are led to the following formulas for $\mathrm{inv_D}(\sigma)$ and $\mathrm{nmin_D}(\sigma)$.

Proposition 4.4 *Given an element $\sigma \in D_n$, let $e = (e_1, e_2, \ldots, e_n)$ be its* E-code. *Then*

$$\mathrm{inv_D}(\sigma) = \sum_{r=1}^{n} (r - e_r - 2\chi(e_r < 0)) \tag{4.14}$$

and

$$\mathrm{nmin_D}(\sigma) = n - \sum_{r=1}^{n} \chi(e_r = r). \tag{4.15}$$

We now define the F-code of an element $\sigma \in D_n$ as an integer sequence $f = (f_1, f_2, \ldots, f_n)$ given by the following procedure. To compute the F-code f, we shall generate a sequence of signed permutations $\sigma^{(n)}, \sigma^{(n-1)}, \ldots, \sigma^{(1)} \in D_n$. Let us begin with $\sigma^{(n)} = \sigma$. For i from n to 2, we construct $\sigma^{(i-1)}$ from $\sigma^{(i)}$. Consider the letter i in $\sigma^{(i)}$. If i has a positive sign in $\sigma^{(i)}$, say, $\sigma^{(i)}(p) = i$, then let $f_i = p$ and let $\sigma^{(i-1)}$ be the signed permutation obtained from $\sigma^{(i)}$ by exchanging the letter i and the letter at the i-th position. If i has a minus sign in $\sigma^{(i)}$ and $\sigma^{(i)}(i) = \bar{i}$, then let $f_i = -i$ and let $\sigma^{(i-1)}$ be the signed permutation obtained from $\sigma^{(i)}$ by changing both the signs of the element at the i-th position and the element at the first position. If i has a minus sign in $\sigma^{(i)}$ and $\sigma^{(i)}(i) \neq \bar{i}$, say, $\sigma^{(i)}(p) = \bar{i}$, then let $f_i = -p$ and let $\sigma^{(i-1)} = \sigma^{(i)}(\bar{p}, i)$. It can be readily seen that the resulting signed permutation $\sigma^{(1)}$ is the identity permutation $1\,2 \cdots n$. Finally, we set $f_1 = 1$.

For example, let $\sigma = \bar{2}\,\bar{4}\,5\,\bar{1}\,\bar{3}$. Then we have

$$\begin{aligned}
\sigma^{(5)} &= \bar{2}\ \bar{4}\ \mathbf{5}\ \bar{1}\ \bar{3}, & f_5 &= 3,\\
\sigma^{(4)} &= \bar{2}\ \bar{\mathbf{4}}\ \bar{3}\ \bar{1}\ 5, & f_4 &= -2,\\
\sigma^{(3)} &= \bar{2}\ 1\ \bar{\mathbf{3}}\ 4\ 5, & f_3 &= -3,\\
\sigma^{(2)} &= \mathbf{2}\ 1\ 3\ 4\ 5, & f_2 &= 1,\\
\sigma^{(1)} &= 1\ 2\ 3\ 4\ 5, & f_1 &= 1.
\end{aligned}$$

Hence the F-code of $\sigma = \bar{2}\,\bar{4}\,5\,\bar{1}\,\bar{3}$ is $(1, 1, -3, -2, 3)$. It is easily seen that the above procedure is reversible. So we can recover σ from its F-code.

The following proposition gives expressions of $\mathrm{sor_D}(\sigma)$ and $\tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma)$ in terms of the F-code of σ.

Proposition 4.5 *Given an element $\sigma \in D_n$, let $f = (f_1, f_2, \ldots, f_n)$ be its* F-code. *Then*

$$\mathrm{sor_D}(\sigma) = \sum_{r=1}^{n} (r - f_r - 2\chi(f_r < 0)) \tag{4.16}$$

and

$$\tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma) = n - \sum_{r=1}^{n} \chi(f_r = r). \tag{4.17}$$

Proof. For $1 \leq i \leq n$, we let t_{ii} denote the identity permutation. Examining the procedure to construct the F-code of σ, we see that for $1 < r \leq n$, we have

$$\sigma^{(r)} = \sigma^{(r-1)} t_{f_r r}. \tag{4.18}$$

It follows that

$$\sigma^{(r)} = t_{f_1 1} t_{f_2 2} \ \cdots \ t_{f_r r}. \tag{4.19}$$

By the definition of the co-sorting index, we find

$$\mathrm{sor}'_{\mathrm{D}}(\sigma^{(r)}) - \mathrm{sor}'_{\mathrm{D}}(\sigma^{(r-1)}) = r - f_r - 2\chi(f_r < 0). \tag{4.20}$$

Applying Lemma 4.3, we get

$$\mathrm{sor}_{\mathrm{D}}(\sigma^{(r)}) - \mathrm{sor}_{\mathrm{D}}(\sigma^{(r-1)}) = r - f_r - 2\chi(f_r < 0). \tag{4.21}$$

Summing (4.21) over r gives (4.16).

To prove (4.17), it suffices to show that

$$\tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma^{(r)}) - \tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma^{(r-1)}) = 1 - \chi(f_r = r) \tag{4.22}$$

for $1 < r \leq n$. If $f_r = r$, then it is clear that $\sigma^{(r)} = \sigma^{(r-1)}$. So (4.22) holds in this case. If $f_r \neq r$, let $\tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma^{(r)}) = l$. Then $\sigma^{(r)}$ can be decomposed as follows

$$\sigma^{(r)} = t_{i_1 j_1} t_{i_2 j_2} \ \cdots \ t_{i_l j_l}, \tag{4.23}$$

where $t_{i_1 j_1}, t_{i_2 j_2}, \cdots, t_{i_l j_l} \in T^D$. For $t = t_{ij} \in T^D$ and $1 < k \leq n$, we say that t fixes k if and only if $k \neq i, \bar{i}, j$ or $\bar{j}$ in the sense that if $k \neq i, \bar{i}, j$ or $\bar{j}$, then t_{ij} maps k to k when we consider t_{ij} as a map on $\{1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}\}$. It can be verified that for any $1 < k \leq n$ and $t_1, t_2 \in T^D$, there exist $t_3, t_4 \in T^D$ such that $t_1 t_2 = t_3 t_4$ and t_3 fixes k. Thus we can use (4.23) to express $\sigma^{(r)}$ in the following form

$$\sigma^{(r)} = t_{i'_1 j'_1} t_{i'_2 j'_2} \ \cdots \ t_{i'_l j'_l}, \tag{4.24}$$

where $t_{i'_1 j'_1}, t_{i'_2 j'_2}, \cdots, t_{i'_l j'_l} \in T^D$ and $t_{i'_p j'_p}$ fixes r for $1 \leq p \leq l-1$. Since $f_r \neq r$, it follows from (4.19) that $\sigma^{(r)}$ maps f_r to r. Hence we deduce that $t_{i'_l j'_l} = t_{f_r r}$. By (4.18) and (4.24), we get

$$t_{i'_1 j'_1} t_{i'_2 j'_2} \cdots t_{i'_{l-1} j'_{l-1}} = \sigma^{(r-1)}.$$

So we arrive at

$$\tilde{\mathrm{l}}'(\sigma^{(r-1)}) \leq l - 1.$$

By (4.18), we see that

$$l \leq \tilde{\mathrm{l}}'(\sigma^{(r-1)}) + 1.$$

Thus we conclude that

$$l = \tilde{\mathrm{l}}'(\sigma^{(r-1)}) + 1. \tag{4.25}$$

This completes the proof of (4.17). ∎

Using the E-code and the F-code, we can define a bijection $\rho\colon D_n \longrightarrow D_n$ as given by

$$\rho = \text{F-code}^{-1} \circ \text{E-code}.$$

Combining Proposition 4.4 and Proposition 4.5, we obtain the following property.

Theorem 4.6 *The bijection ρ transforms $(\mathrm{inv_D}, \mathrm{nmin_D})$ to $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})$, that is, for any $\sigma \in D_n$, we have*

$$(\mathrm{inv_D}, \mathrm{nmin_D})\ \sigma = \ (\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})\ \rho(\sigma). \tag{4.26}$$

Proof. For $\sigma \in D_n$, let $g = (g_1, g_2, \ldots, g_n)$ be the E-code of σ. It is clear that g is also the F-code of $\rho(\sigma)$. It follows from Proposition 4.4 and Proposition 4.5 that

$$(\mathrm{inv_D}, \mathrm{nmin_D})\ \sigma = (\ \sum_{r=1}^{n}(r - g_r - 2\chi(g_r < 0)), n - \sum_{r=1}^{n}\chi(g_r = r)\ ),$$

$$(\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})\ \rho(\sigma) = (\ \sum_{r=1}^{n}(r - g_r - 2\chi(g_r < 0)), n - \sum_{r=1}^{n}\chi(g_r = r)\ ).$$

Thus we obtain $(\mathrm{inv_D}, \mathrm{nmin_D})\ \sigma = (\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})\ \rho(\sigma)$. This completes the proof. ∎

We now present a proof of Theorem 4.2 based on two factorizations of the diagonal sum $\sum_{\sigma \in D_n} \sigma$ in the group algebra $\mathbb{Z}[D_n]$. It turns out that the bivariate generating functions of $(\mathrm{inv_D}, \mathrm{nmin_D})$ and $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})$ are both equal to

$$D_n(q,t) = \prod_{r=1}^{n-1}(1 + q^r t + qt \cdot [2r]_q).$$

To derive the bivariate generating function of $(\mathrm{inv_D}, \mathrm{nmin_D})$, we recall Petersen's factorization of the diagonal sum $\sum_{\sigma \in D_n} \sigma$. The elements $\Psi_1, \Psi_2, \ldots, \Psi_{n-1}$ of the group algebra of D_n are recursively defined as follows. Recall that $s_i = (i, i+1)$ for $1 \le i < n$ and $s_{\bar{1}} = (\bar{1}, 2)$. For $i = 1$, let

$$\Psi_1 = 1 + s_1 + s_{\bar{1}} + s_1 s_{\bar{1}}.$$

For $i \ge 2$, let

$$\Psi_i = 1 + s_i \Psi_{i-1} + s_i \cdots s_2 s_1 s_{\bar{1}} s_2 \cdots s_i.$$

Petersen found the following factorization.

Proposition 4.7 *For $n \geq 2$, we have*

$$\sum_{\sigma \in D_n} \sigma = \Psi_1 \Psi_2 \cdots \Psi_{n-1}.$$

For an element $\sigma \in D_n$, we define the weight of σ to be

$$\mu(\sigma) = q^{\mathrm{inv_D}(\sigma)} t^{\mathrm{nmin_D}(\sigma)}.$$

As usual, the weight function is considered as a linear map on $\mathbb{Z}[D_n]$. It can be easily checked that

$$\mu(\Psi_i) = 1 + tq^i + tq\,(1 + q + \cdots + q^{2i-1}) = 1 + tq^i + tq\,[2i]_q. \tag{4.27}$$

We are now ready to finish the proof of relation (4.4) concerning the bivariate generating function of $(\mathrm{inv_D}, \mathrm{nmin_D})$.

Proof of (4.4) in Theorem 4.2. By Proposition 4.7 and relation (4.27), we see that (4.4) can be rewritten as

$$\mu(\Psi_1 \cdots \Psi_{n-1}) = \mu(\Psi_1) \cdots \mu(\Psi_{n-1}).$$

Notice that for $i \geq 1$ and $i + 2 \leq k \leq n$, each term of Ψ_i fixes k. Here we say that an element $\sigma \in D_n$ fixes k if σ maps k to k. Thus Ψ_i can be considered as an element of $\mathbb{Z}[D_j]$ for $i < j < n$. It is evident the weight function μ is well-defined in this sense. Therefore we only need to show that

$$\mu(\Psi_1 \cdots \Psi_{n-2}\, \Psi_{n-1}) = \mu(\Psi_1 \cdots \Psi_{n-2})\, \mu(\Psi_{n-1}).$$

It suffices to prove that

$$\mu(\sigma \cdot \Psi_{n-1}) = \mu(\sigma) \cdot \mu(\Psi_{n-1}) \tag{4.28}$$

for any $\sigma = \sigma_1 \cdots \sigma_{n-1} \in D_{n-1}$. Note that σ is considered as an element of D_n which fixes n. It is easy to see that

$$\begin{aligned} \sigma \cdot \Psi_{n-1} \quad &= \quad \sigma_1 \cdots \sigma_{n-1} n + \sigma_1 \cdots \sigma_{n-2} n \sigma_{n-1} + \cdots + \sigma_1 n \cdots \sigma_{n-1} + n \sigma_1 \cdots \sigma_{n-1} \\ &\quad\quad + \bar{n} \bar{\sigma}_1 \cdots \sigma_{n-1} + \bar{\sigma}_1 \bar{n} \cdots \sigma_{n-1} + \cdots + \bar{\sigma}_1 \cdots \sigma_{n-1} \bar{n}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\mu(\sigma \cdot \Psi_{n-1}) \\ &\quad = \quad \mu(\sigma_1 \cdots \sigma_{n-1} n) + \mu(\sigma_1 \cdots \sigma_{n-2} n \sigma_{n-1}) + \cdots + \mu(\sigma_1 n \cdots \sigma_{n-1}) + \mu(n \sigma_1 \cdots \sigma_{n-1}) \\ &\quad\quad\quad + \mu(\bar{n} \bar{\sigma}_1 \cdots \sigma_{n-1}) + \mu(\bar{\sigma}_1 \bar{n} \cdots \sigma_{n-1}) + \cdots + \mu(\bar{\sigma}_1 \cdots \sigma_{n-1} \bar{n}) \\ &\quad = \quad \mu(\sigma) + qt\,\mu(\sigma) + \cdots + q^{n-2} t\,\mu(\sigma) + q^{n-1} t\,\mu(\sigma) \\ &\quad\quad\quad + q^{n-1} t\,\mu(\sigma) + q^n t\,\mu(\sigma) + \cdots + q^{2n-2} t\,\mu(\sigma) \end{aligned}$$

$$= \quad (1+tq^{n-1}+tq(1+q+\cdots+q^{2n-3}))\,\mu(\sigma).$$

Therefore, (4.28) can be deduced from (4.27). This completes the proof. ∎

To prove formula (4.5) for the bivariate generating function of $(\mathrm{sor_D}, \tilde{\mathrm{l}}'_{\mathrm{D}})$, we shall use another factorization of the diagonal sum $\sum_{\sigma\in D_n}\sigma$ due to Petersen. For $2\le j\le n$, let

$$\Phi_j = 1 + \sum_{\substack{i\neq 0\\ \bar{j}\le i<j}} t_{ij}.$$

Proposition 4.8 *For $n\ge 2$, we have*

$$\sum_{\sigma\in D_n}\sigma = \Phi_2\,\Phi_3\cdots\Phi_n.$$

For an element $\sigma\in D_n$, we define another weight function

$$\nu(\sigma) = q^{\mathrm{sor_D}(\sigma)}t^{\tilde{\mathrm{l}}'_{\mathrm{D}}(\sigma)}.$$

Again, the weight function ν is considered as a linear map. It can be checked that

$$\nu(\Phi_i) = 1+tq^{i-1}+tq(1+q+\cdots+q^{2i-3}) = 1+tq^{i-1}+tq[2i-2]_q. \tag{4.29}$$

Proof of (4.5) in Theorem 4.2. By Proposition 4.8 and relation (4.29), we find that (4.5) can be expressed in the following form

$$\nu(\Phi_2\cdots\Phi_n) = \nu(\Phi_2)\cdots\nu(\Phi_n).$$

As in the proof of (4.4), we only need to show that

$$\nu(\Phi_2\cdots\Phi_n) = \nu(\Phi_2\cdots\Phi_{n-1})\,\nu(\Phi_n).$$

It suffices to prove that

$$\nu(\sigma\cdot\Phi_n) = \nu(\sigma)\cdot\nu(\Phi_n), \tag{4.30}$$

for any $\sigma=\sigma_1\cdots\sigma_{n-1}\in D_{n-1}$. Again, σ is considered as an element of D_n which fixes n. Since

$$\begin{aligned}\sigma\cdot\Phi_n \quad = \quad & \sigma_1\cdots\sigma_{n-1}n+\sigma_1\cdots\sigma_{n-2}n\sigma_{n-1}+\cdots+\sigma_1 n\cdots\sigma_{n-1}\sigma_2+n\sigma_2\cdots\sigma_{n-1}\sigma_1\\ & +\bar{n}\sigma_2\cdots\sigma_{n-1}\bar{\sigma}_1+\sigma_1\bar{n}\cdots\sigma_{n-1}\bar{\sigma}_2+\cdots+\bar{\sigma}_1\cdots\sigma_{n-1}\bar{n},\end{aligned}$$

we get

$$\begin{aligned}&\nu(\sigma\cdot\Phi_n)\\ &\quad = \quad \nu(\sigma_1\cdots\sigma_{n-1}n)+\nu(\sigma_1\cdots\sigma_{n-2}n\sigma_{n-1})+\cdots+\nu(\sigma_1 n\cdots\sigma_{n-1}\sigma_2)+\nu(n\sigma_2\cdots\sigma_{n-1}\sigma_1)\end{aligned}$$

$$
\begin{aligned}
& +\nu(\bar{n}\sigma_2\cdots\sigma_{n-1}\bar{\sigma_1}) + \nu(\sigma_1\bar{n}\cdots\sigma_{n-1}\bar{\sigma_2}) + \cdots + \nu(\bar{\sigma_1}\sigma_2\cdots\sigma_{n-1}\bar{n}) \\
= \; & \nu(\sigma) + qt\,\nu(\sigma) + \cdots + q^{n-2}t\,\nu(\sigma) + q^{n-1}t\,\nu(\sigma) \\
& +q^{n-1}t\,\nu(\sigma) + q^{n}t\,\nu(\sigma) + \cdots + q^{2n-2}t\,\nu(\sigma) \\
= \; & (1 + tq^{n-1} + tq(1+q+\cdots+q^{2n-3}))\,\nu(\sigma).
\end{aligned}
$$

Hence (4.30) follows from (4.29). This completes the proof. ∎

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