GLOBAL WELL-POSEDNESS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN $H^{\frac{1}{2}}(\mathbb{R})$

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Abstract. We prove that the derivative nonlinear Schrödinger equation is globally well-posed in $H^{\frac{1}{2}}(\mathbb{R})$ when the mass of initial data is strictly less than 4π.

1. Introduction

In this note, we study the Cauchy problem to the derivative nonlinear Schrödinger equation (DNLS):

$$\begin{aligned} i\partial_t u + \partial_x^2 u =& i\partial_x(|u|^2 u), \qquad t\in\mathbb{R},\ x\in\mathbb{R}, \\ u(0,x) =& u_0(x). \end{aligned} \tag{1.1}$$

This equation was derived by [12, 13] for studying the propagation of the circular polarised nonlinear Alfvén waves in plasma, and has been extensively studied since then. It is well-known that (1.1) is completely integrable (see [9, 8, 17]), and thus has infinite number of conservation laws. In particular, in this paper we will use the following three conservation laws: if u is a H^1-solution of (1.1) then

$$\begin{aligned} M_D(u) &:= \int_{\mathbb{R}} |u|^2 dx = M_D(u_0), \\ E_D(u) &:= \int_{\mathbb{R}} |u_x|^2 + \frac{3}{2}|u|^2 \operatorname{Im}(u\bar{u}_x) + \frac{1}{2}|u|^6 dx = E_D(u_0), \\ P_D(u) &:= \int_{\mathbb{R}} \operatorname{Im}(\bar{u}u_x) - \frac{1}{2}|u|^4 dx = P_D(u_0). \end{aligned}$$

Equation (1.1) has been extensively studied. On the well-posedness, Hayashi and Ozawa [5, 6, 7, 14] proved local well-posedness in $H^1(\mathbb{R})$, and moreover global well-posedness for initial data in H^1 satisfying

$$\int_{\mathbb{R}} |u_0(x)|^2 dx < 2\pi. \tag{1.2}$$

The condition above appears naturally in the sharp Galiardo-Nirenberg inequality to ensure an apriori estimate of H^1-norm by mass and energy conservation. Later, Local well-posedness in H^s for $s \geq 1/2$ was obtained by Takaoka [15], and this result is sharp in the sense that the solution map fails to be uniformly continuous in a ball of H^s if $s < 1/2$. Low regularity global well-posedness was also studied, for example, global well-posedness in $H^s(\mathbb{R})$ under (1.2) was obtained in [16, 2, 3] for $s > 1/2$, and finally in [11] for $s = 1/2$. On the long-time behavior and modified scattering theory, see [4] and references therein.

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A natural question is whether blowup occurs for (1.1). To the authors' knowledge, this problem is still open. See [10] for a numerical blowup analysis on a class of DNLS. Recently, the second author [19] showed the global well-posedness in H^1 under a weaker condition

$$\int_{\mathbb{R}} |u_0(x)|^2 dx < 4\pi, \tag{1.3}$$

improving his previous result [18]. This result shows a striking difference between DNLS and other mass critical equations like focusing generalized KdV and quintic focusing nonlinear Schrödinger equation. The key ingredient is the use of the momemtum conservation.

The purpose of this paper is to prove the low-regularity global well-posedness under (1.3). The main result is

Theorem 1.1. *The Cauchy problem* (1.1) *is global well-posed in* $H^{\frac{1}{2}}(\mathbb{R})$ *under* (1.3)*.*

We explain the ideas of the proof of the theorem. Inspired by [19], we derive directly an apriori estimate using the conservation laws of mass, momentum and energy as well as the sharp Gagliardo-Nirenberg inequality, and thus provide a simplified proof of the result of [19]. We do not prove by contradiction and can get a clear bound of H^1-norm. Then we combine it with the I-method to prove the theorem.

2. Apriori estimate

To prove the theorem, it suffices to control the $H^{1/2}$-norm of the solution. For convenience, we use the following gauge transformation. If u is a solution to (1.1) with $u_0 \in H^{1/2}$, let

$$v(t,x) := e^{-\frac{3}{4}i\int_{-\infty}^{x} |u(t,y)|^2\, dy} u(t,x). \tag{2.1}$$

Then v solves

$$i\partial_t v + \partial_x^2 v = \frac{i}{2}|v|^2 v_x - \frac{i}{2} v^2 \bar{v}_x - \frac{3}{16}|v|^4 v \tag{2.2}$$

with initial data $v(0,x) = v_0(x) := e^{-\frac{3}{4}i\int_{-\infty}^{x} |u_0|^2\, dy} u_0$. It's easy to see the map $u \to v$ is a bijection in $H^{1/2}$. Indeed, by fractional Leibniz rule we get

$$\begin{aligned}\|D^{1/2}v\|_2 \lesssim& \|D^{1/2}u\|_2 + \|uD^{1/2}[e^{-\frac{3}{4}i\int_{-\infty}^{x} |u(t,y)|^2\, dy}]\|_2 \\ \lesssim& \|D^{1/2}u\|_2 + \|u\|_4 \|\partial_x[e^{-\frac{3}{4}i\int_{-\infty}^{x} |u(t,y)|^2\, dy}]\|_{4/3} \lesssim C(\|u\|_{H^{1/2}}).\end{aligned}$$

From now on, we only consider the equation (2.2) and we need to control the $H^{1/2}$-norm of v.

Under the gauge transformation, the conservation laws reduce to: for solution v of (2.2) then

$$M(v(t)) := \|v(t)\|_{L_x^2}^2 = M(v_0), \qquad \text{(mass)} \tag{2.3}$$

$$P(v(t)) := \mathrm{Im}\int_{\mathbb{R}} \bar{v}(t) v_x(t)\, dx + \frac{1}{4}\int_{\mathbb{R}} |v(t)|^4\, dx = P(v_0), \qquad \text{(momentum)} \tag{2.4}$$

$$E(v(t)) := \|v_x(t)\|_{L_x^2}^2 - \frac{1}{16}\|v(t)\|_{L_x^6}^6 = E(v_0). \qquad \text{(energy)} \tag{2.5}$$

We denote $\|\cdot\|_p = \|\cdot\|_{L^p_x}$ for $1 \le p \le \infty$. By the sharp Galiardo-Nirenberg inequality

$$\|f\|_6^6 \le \frac{4}{\pi^2}\|f\|_2^4\|f_x\|_2^2, \tag{2.6}$$

then we get

$$E(v) \ge \|v_x\|_2^2(1 - \frac{1}{4\pi^2}\|v\|_2^4).$$

Thus under the condition (1.2) we can get the apriori bound on $\|v\|_{H^1}$.

However, as observed in [19] the momentum conservation for (2.2) played a significant role. Inspired by [19] we derive directly a-priori estimate using the momentum and the following sharp GN inequality (see [1]):

$$\|f\|_6 \le C_{GN}\|f\|_4^{8/9}\|f_x\|_2^{1/9}, \tag{2.7}$$

where $C_{GN} = 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$.

Lemma 2.1. *If $v \in H^1(\mathbb{R})$ and $v \neq 0$, then*

$$P(v) \ge \frac{1}{4}\|v\|_4^4(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2) - \frac{4\sqrt{\pi}E(v)\|v\|_2}{\|v\|_4^4}. \tag{2.8}$$

Proof. Let $u = e^{i\alpha x}v(t,x)$ with $\alpha > 0$ being determined later. Then

$$|u_x|^2 = |v_x|^2 + \alpha^2|v|^2 + 2\alpha \operatorname{Im} v_x\bar{v},$$

and thus

$$\int \operatorname{Im} v_x\bar{v}dx = -\frac{E(v)}{2\alpha} - \frac{\alpha M(v)}{2} + \frac{E(u)}{2\alpha}.$$

Now by the sharp GN inequality we have

$$\begin{aligned} E(u) =& \|u_x\|_2^2 - \frac{1}{16}\|u\|_6^6 \\ \ge& C_{GN}^{-18}\|u\|_6^{18}\|u\|_4^{-16} - \frac{1}{16}\|u\|_6^6 \\ =& (C_{GN}^{-18}\|v\|_6^{12}\|v\|_4^{-16} - \frac{1}{16})\|v\|_6^6. \end{aligned}$$

Thus,

$$\begin{aligned} P(v) \ge& -\left[\frac{1}{16} - C_{GN}^{-18}\|v\|_6^{12}\|v\|_4^{-16}\right]\frac{\|v\|_6^6}{2\alpha} + \frac{\|v\|_4^4}{4} - \frac{\alpha\|v\|_2^2}{2} - \frac{E(v)}{2\alpha} \\ \ge& -f(\|v\|_6^6\|v\|_4^{-8})\frac{\|v\|_4^8}{2\alpha} + \frac{\|v\|_4^4}{4} - \frac{\alpha\|v\|_2^2}{2} - \frac{E(v)}{2\alpha} \end{aligned}$$

where $f(x) = (\frac{1}{16} - C_{GN}^{-18}x^2)x$. By calculus we know

$$\max_x f(x) = f(\frac{C_{GN}^9}{4\sqrt{3}}) = \frac{C_{GN}^9}{96\sqrt{3}} = \frac{1}{64\pi}.$$

Therefore

$$P(v) \ge -\frac{\|v\|_4^8}{128\pi\alpha} + \frac{\|v\|_4^4}{4} - \frac{\alpha\|v\|_2^2}{2} - \frac{E(v)}{2\alpha}$$

Take $\alpha = \frac{1}{8\sqrt{\pi}}\|v\|_4^4\|v\|_2^{-1}$, then $P(v) \ge \frac{1}{4}\|v\|_4^4(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2) - \frac{E(v)}{2\alpha}$. □

Lemma 2.2. *If $v \in H^1(\mathbb{R})$, $v \neq 0$ and $\|v\|_2^2 < 4\pi$, then*

$$\|v_x\|_{L^2}^2 \leq 2E(v) + \frac{P(v)^2 + 2\sqrt{\pi}|E(v)|\|v\|_2}{(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2)^2}. \tag{2.9}$$

Proof. Let $x = \|v\|_4^4$. Then (2.8) gives a estimate of the form

$$c \geq ax - \frac{b}{x}. \tag{2.10}$$

with $a = \frac{1}{4}(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2)$, $b = 4\sqrt{\pi}|E(v)|\|v\|_2$, $c = |P(v)|$. (2.10) implies

$$(ax^2 - cx - b) \leq 0.$$

Since $a > 0$, thus we get

$$x^2 \leq \Big(\frac{c + \sqrt{c^2 + 4ab}}{2a}\Big)^2 \leq \frac{c^2 + 2ab}{a^2}.$$

Thus we obtain

$$\|v\|_4^8 \leq 16(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2)^{-2}\Big(P(v)^2 + 2(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2)\sqrt{\pi}|E(v)|\|v\|_2\Big). \tag{2.11}$$

Then by (2.7) and mean value inequality we have

$$\|v_x\|_{L^2}^2 \leq 2E(v) + 2^{-4}\|v\|_{L^4}^8. \tag{2.12}$$

Therefore by (2.11) we prove the lemma. □

With this lemma, we can get that if v is a H^1-solution of (2.2) satisfying (1.3), then $\|v_x\|_2 \leq C$. Therefore, global well-posedness of (2.2) in H^1 under (1.3) follows immediately.

3. Proof of the main theorem

In this section we prove Theorem 1.1 using the I-method as the previous works [3, 11]. The main difference is that we need to use the momentum conservation.

First we recall the definition of I-operator. Let $N \gg 1$ be fixed, and the Fourier multiplier operator I_N be defined as

$$\widehat{I_N f}(\xi) = m_N(\xi)\hat{f}(\xi). \tag{3.1}$$

Here $m_N(\xi)$ is a smooth, radially decreasing function satisfying $0 < m_N(\xi) \leq 1$ and

$$m_N(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ N^{\frac{1}{2}}|\xi|^{-\frac{1}{2}}, & |\xi| > 2N. \end{cases} \tag{3.2}$$

For simplicity we denote I_N by I and m_N by m if there is no confusion. I_N maps $H^{\frac{1}{2}}$ to H^1, moreover, we have the following estimates,

$$\|f\|_{H^{\frac{1}{2}}} \lesssim \|I_N f\|_{H^1} \lesssim N^{\frac{1}{2}}\|f\|_{H^{\frac{1}{2}}}, \tag{3.3}$$

where the implicit constants are indenpendent on N.

Next we use the rescaling. For $v_0 \in H^{1/2}$, let v be the solution to (2.2). For $\lambda > 0$, let

$$v_\lambda = \lambda^{-\frac{1}{2}}v(\frac{x}{\lambda}, \frac{t}{\lambda^2}) \quad \text{and} \quad v_{0,\lambda} = \lambda^{-\frac{1}{2}}v_0(\frac{x}{\lambda}).$$

Then v_λ is a solution of (2.2) with the initial data $v_\lambda(0) = v_{0,\lambda}(x)$. Meanwhile, v_λ exists on $[0, T]$ if and only if v exists on $[0, \lambda^{-2}T]$. We have

$$\|Iv_{0,\lambda}\|_2 \leq \|v_{0,\lambda}\|_2 = \|v_0\|_2 \tag{3.4}$$

and

$$\|\partial_x I v_{0,\lambda}\|_2 \lesssim N^{1/2}\lambda^{-1/2}\|v_0\|_{\dot{H}^{1/2}}. \tag{3.5}$$

Thus choosing

$$\lambda \sim N,$$

we can make

$$\|\partial_x I v_{0,\lambda}\|_2 \leq \varepsilon_0 \ll 1 \tag{3.6}$$

where ε_0 will be determined later.

We recall a variant local well-posedness obtained in [11].

Lemma 3.1. *The Cauchy problem* (2.2) *is locally well-posed for the initial data v_0 satisfying $Iv_0 \in H^1(\mathbb{R})$. Moreover, the solution exists on the interval $[0, \delta]$ with the lifetime*

$$\delta \sim \|Iv_0\|_{H^1}^{-\mu} \tag{3.7}$$

for some $\mu > 0$, where the implicit constant is independent of N. Furthermore, the solution satisfies the estimate

$$\|Iv\|_{L^\infty((0,\delta);H^1)} \leq 2\|Iv_0\|_{H^1}. \tag{3.8}$$

By the above lemma, we need to control the growth of $\|Iv_\lambda(t)\|_{H^1}$. By mass conservation we have $\|Iv_\lambda(t)\|_{L^2_x} \leq \|v_\lambda\|_{L^2_x} \leq C$. It suffices to control $\|\partial_x I v_\lambda\|_2$. We will use (2.9) since $\|Iv_\lambda\|_2^2 \leq \|v_\lambda\|_2^2 = \|v_0\|_2^2 < 4\pi$. We define the modified momentum and energy as follows

$$P_I(v_\lambda) := P(Iv_\lambda), \quad E_I(v_\lambda) := E(Iv_\lambda). \tag{3.9}$$

Then by (3.6), Hölder's and Sobolev's inequalities, we have

$$P_I(v_{0,\lambda}) \lesssim 1; \quad E_I(v_{0,\lambda}) \lesssim 1$$

Moreover,

$$P(v_{0,\lambda}) = \frac{1}{\lambda}P(v_0) \sim N^{-1}P(v_0).$$

If $N \to \infty$, I_N tends to the identity operator. Thus $P_I(v_\lambda)$ and $E_I(v_\lambda)$ increases slowly in t if N is large enough. Indeed, in the previous works the growth of $E_I(v_\lambda)$ was already studied. Collecting the results obtained in [11] (see Section 7), we have

Lemma 3.2. *Suppose that for $T > 0$*

$$\sup_{t\in[0,T]} \|Iv_\lambda\|_{H^1} \lesssim 1, \tag{3.10}$$

then the modified energy $E_I(v_\lambda)$ obeys the following estimate: there exists $C, \alpha > 0$ such that for any $t \in [0, T]$ and any $\varepsilon > 0$

$$\begin{aligned} |E_I(v_\lambda(t))| \leq& \|\partial_x I v_{0,\lambda}\|_{L^2}^2 + CN^{-\alpha} \sup_{\tau\in[0,t]} \big(\|Iv_\lambda(\tau)\|_{H^1}^4 + \|Iv_\lambda(\tau)\|_{H^1}^6\big) \\ &+ CtN^{-\frac{5}{2}+\epsilon} \sup_{\tau\in[0,t]} \big(\|Iv_\lambda(\tau)\|_{H^1}^6 + \|Iv_\lambda(\tau)\|_{H^1}^{10}\big). \end{aligned} \tag{3.11}$$

On the modified momentum we have the following estimate. Indeed, since the momentum lies in the regularity of $H^{1/2}$, we can estimate it in a simple way.

Lemma 3.3. *We have*

$$\big|P_I(v_\lambda)-P(v_\lambda)\big| \lesssim N^{-1}\big(\|Iv_\lambda\|_{H^1}^2+\|Iv_\lambda\|_{H^1}^4\big)$$

Proof. By the definition of momentum, we need to bound

$$\Big|\,\mathrm{Im}\int_{\mathbb{R}}(I\bar v_\lambda\partial_x Iv_\lambda-\bar v_\lambda\partial_x v_\lambda)\,dx\Big|+\Big|\int|Iv_\lambda|^4\,dx-\int|v_\lambda|^4\,dx\Big|:=I+II.$$

For the first term I, since

$$\mathrm{Im}\int_{\mathbb{R}}\Big(I\bar v_\lambda\partial_x Iv_\lambda-\bar v_\lambda\partial_x v_\lambda\Big)\,dx=\mathrm{Im}\int_{\mathbb{R}}\Big(I\partial_x v_\lambda-\partial_x v_\lambda\Big)\big(I\bar v_\lambda+\bar v_\lambda\big)\,dx,$$

and $P_{\le N}\big(I\partial_x v_\lambda-\partial_x v_\lambda\big)=0$, then we get

$$I\lesssim\big\|I\partial_x v_\lambda-\partial_x v_\lambda\big\|_{\dot H^{-\frac12}}\Big(\big\|P_{\ge N}I\bar v_\lambda\big\|_{\dot H^{\frac12}}+\big\|P_{\ge N}\bar v_\lambda\big\|_{\dot H^{\frac12}}\Big).$$

By the definition of I-operator, we have

$$\big\|I\partial_x v_\lambda-\partial_x v_\lambda\big\|_{\dot H^{-\frac12}}+\big\|P_{\ge N}I\bar v_\lambda\big\|_{\dot H^{\frac12}}+\big\|P_{\ge N}\bar v_\lambda\big\|_{\dot H^{\frac12}}\lesssim N^{-\frac12}\|Iv_\lambda\|_{H^1},$$

and thus

$$I\lesssim N^{-1}\|Iv_\lambda\|_{H^1}^2.$$

For the second term II, we have

$$\int|Iv_\lambda|^4\,dx-\int|v_\lambda|^4\,dx=\int\big(Iv_\lambda-v_\lambda\big)\,P_{\ge N}(v_\lambda^3)\,dx+\text{similar terms.}\tag{3.12}$$

Using the Hölder inequality, the Sobolev's embedding, and the fractional Leibniz inequalities, we get

$$\begin{aligned}\Big|\int\big(Iv_\lambda-v_\lambda\big)\,P_{\ge N}(v_\lambda^3)\,dx\Big|&\lesssim\|Iv_\lambda-v_\lambda\|_6\big\|P_{\ge N}(v_\lambda^3)\big\|_{\frac65}\\&\lesssim\|Iv_\lambda-v_\lambda\|_{H^{\frac12}}\,N^{-\frac12}\big\|\langle\nabla\rangle^{\frac12}(v_\lambda^3)\big\|_{\frac65}\\&\lesssim N^{-1}\|Iv_\lambda\|_{H^1}\,\big\|\langle\nabla\rangle^{\frac12}v_\lambda\big\|_2\|v_\lambda\|_6^2\\&\lesssim N^{-1}\|Iv_\lambda\|_{H^1}\|v_\lambda\|_{H^{\frac12}}^3\\&\lesssim N^{-1}\|Iv_\lambda\|_{H^1}^4.\end{aligned}$$

The similar terms in (3.12) can be handled in the same way. Thus we prove the lemma. □

By Lemma 3.2 and the mass conservation law $\|v_\lambda\|_2\le C$ we have under the assumption (3.10)

$$\begin{aligned}E_I(v_\lambda(t))\le&\|\partial_x Iv_{0,\lambda}\|_{L^2}^2+CN^{-\alpha}\sup_{\tau\in[0,t]}\big(\|\partial_x Iv_\lambda(\tau)\|_2^4+\|\partial_x Iv_\lambda(\tau)\|_2^6+1\big)\\&+CtN^{-\frac52+\epsilon}\sup_{\tau\in[0,t]}\big(\|\partial_x Iv_\lambda(\tau)\|_2^6+\|\partial_x Iv_\lambda(\tau)\|_2^{10}+1\big).\end{aligned}\tag{3.13}$$

Note that (3.6). We will prove by continuity argument that for $T\le T_0:=N^{\frac52-2\epsilon}$,

$$\sup_{t\in[0,T]}\|\partial_x Iv_\lambda(t)\|_2\le4\gamma_0\varepsilon_0,\tag{3.14}$$

where $\gamma_0=\sqrt{1+\frac{\sqrt{\pi}\|v_0\|_2}{(1-\frac{1}{2\sqrt{\pi}}\|v_0\|_2)^2}}$. We choose $\varepsilon_0\ll1$ such that $100\gamma_0\varepsilon_0<1$.

Assuming (3.14), we get that the solution v_λ exists on $[0, T_0]$. Hence, v exists on $[0, \lambda^{-2}T_0]$. Note that

$$\lambda^{-2}T_0 \sim N^{-2}N^{\frac{5}{2}-2\epsilon} = N^{\frac{1}{2}-2\epsilon}.$$

Therefore,we get that v exists till arbitrarily large T by choosing sufficient large N, and thus completes the proof of Theorem 1.1.

It remains to prove (3.14). We may assume $\sup\limits_{t\in[0,T]} \|\partial_x I v_\lambda(t)\|_2 \leq 3\varepsilon_0 \ll 1$. Then the estimate (3.13) gives

$$|E_I(v_\lambda(t))| \leq \varepsilon_0^2 + CN^{-\epsilon}, \quad t \leq T. \tag{3.15}$$

On the other hand, by Lemma 3.3 we have

$$|P_I(v_\lambda(t))|^2 \leq 2|P_I(v_\lambda(t)) - P(v_\lambda(t))|^2 + 2|P(v_\lambda(t))|^2 \leq CN^{-2}. \tag{3.16}$$

By (2.9), we have

$$\begin{aligned}\left\|\partial_x I v_\lambda(t)\right\|_{L^2}^2 \leq & 2E(Iv_\lambda(t)) + \frac{P(Iv_\lambda(t))^2 + 2\sqrt{\pi}|E(Iv_\lambda)|\|v_0\|_2}{(1 - \frac{1}{2\sqrt{\pi}}\|v_0\|_2)^2} \\ \leq & 2\gamma_0^2(\varepsilon_0^2 + CN^{-\epsilon}) + CN^{-2}(1 - \frac{1}{2\sqrt{\pi}}\|v_0\|_2)^{-2}.\end{aligned}$$

Choosing N sufficiently large, we obtain (3.14) as desired.

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