POSITIVE SCALAR CURVATURE ON FOLIATIONS

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ABSTRACT. We generalize classical theorems due to Lichnerowicz and Hitchin on the existence of Riemannian metrics of positive scalar curvature on spin manifolds to the case of foliated spin manifolds. As a consequence, we show that there is no foliation of positive leafwise scalar curvature on any torus, which generalizes the famous theorem of Schoen-Yau and Gromov-Lawson on the non-existence of metrics of positive scalar curvature on torus to the case of foliations. Moreover, our method, which is partly inspired by the analytic localization techniques of Bismut-Lebeau, also applies to give a new proof of the celebrated Connes vanishing theorem without using noncommutative geometry.

CONTENTS

0. INTRODUCTION

It has been an important subject in differential geometry to study when a smooth manifold carries a Riemannian metric of positive scalar curvature (cf. [18, Chap. IV] and [12]). In this paper, we study related problems on foliations.

Let F be an integrable subbundle of the tangent vector bundle TM of a smooth manifold M. For any Euclidean metric g^F on F, let $k^F \in C^\infty(M)$, which will be called the leafwise scalar curvature associated to g^F, be defined as follows: for any $x \in M$, the integrable subbundle F determines a leaf $\mathcal{F}_x$ passing through x such that $F|_{\mathcal{F}_x} = T\mathcal{F}_x$. Then, g^F determines a Riemannian metric on $\mathcal{F}_x$. Let $k^{\mathcal{F}_x}$ denote the scalar curvature

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of this Riemannian metric. We define

(0.1) $$k^F(x) = k^{\mathcal{F}_x}(x).$$

For a closed spin manifold M, let $\widehat{\mathcal{A}}(M)$ be the canonical KO-characteristic number of M defined by that if $\dim M = 8k + 4i$ with $i = 0$ or 1, then $\widehat{\mathcal{A}}(M) = \frac{3 + (-1)^i}{4} \widehat{A}(M)$;[1] if $\dim M = 8k + i$ with $i = 1$ or 2, then $\widehat{\mathcal{A}}(M) \in \mathbf{Z}_2$ is the Atiyah-Milnor-Singer α invariant;[2] while in other dimensions one takes $\widehat{\mathcal{A}}(M) = 0$.

The main result of this paper can be stated as follows.

Theorem 0.1. *Let F be an integrable subbundle of the tangent bundle of a closed spin manifold M. If F carries a metric of positive leafwise scalar curvature, then $\widehat{\mathcal{A}}(M) = 0$.*

When $F = TM$, one recovers the classical theorems due to Lichnerowicz [19] (for the case of $\dim M = 4k$) and Hitchin [17] (for the cases of $\dim M = 8k + 1$ and $8k + 2$).

Example 0.2. Take any $8k+1$ dimensional closed spin manifold M such that $\widehat{\mathcal{A}}(M) \neq 0$. By a result of Thurston [27], there always exists a codimension one foliation on M. However, by our result, there is no metric of positive leafwise scalar curvature on the associated integrable subbundle of TM.

Remark 0.3. It is a longstanding open question in foliation theory (cf. [33, Remark C14]) that whether the existence of g^F with $k^F > 0$ implies the existence of g^{TM} with $k^{TM} > 0$. This question admits an easy positive answer in the case where (M, F) carries a transverse Riemannian structure (when such a transverse Riemannian structure exists, (M, F) is called a Riemannian foliation). An approach to this question for codimension one foliations is outlined in the long paper of Gromov [12, page 193].

Combining Theorem 0.1 with the well-known results of Gromov-Lawson [14] and Stolz [26], one gets the following consequence which provides a positive answer to the above question for simply connected manifolds of dimension greater than or equal to five.

Corollary 0.4. *Let F be an integrable subbundle of the tangent bundle of a closed simply connected manifold M with $\dim M \geq 5$. If F carries a metric of positive leafwise scalar curvature, then M admits a Riemannian metric of positive scalar curvature.*

For non-simply connected manifolds, recall that a famous result due to Schoen-Yau [25] and Gromov-Lawson [13] states that there is no metrics of positive scalar curvature on any torus. By combining Theorem 0.1 with the techniques of Lusztig [23] and Gromov-Lawson [13], one obtains the following generalization to the case of foliations.

Corollary 0.5. *There exists no foliation (T^n, F) on any torus T^n such that the integrable subbundle F of $T(T^n)$ carries a metric of positive leafwise scalar curvature.*

If F is further assumed to be spin, then Corollaries 0.4 and 0.5 can also be deduced from the following celebrated vanishing theorem of Connes, which provides another kind of generalization of the Lichnerowicz theorem [19] to the case of foliations.

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[1] Cf. [31, pp. 13] for a definition of the Hirzebruch $\widehat{A}$-genus $\widehat{A}(M)$.

[2] Cf. [18, §2.7] for a definition.

Theorem 0.6. (Connes [10, Theorem 0.2]**)** *Let F be a spin integrable subbundle of the tangent bundle of a compact oriented manifold M. If F carries a metric of positive leafwise scalar curvature, then $\widehat{A}(M) = 0$.*

Recall that the proof of Theorem 0.6 outlined in [10] makes use of noncommutative geometry in an essential way. It is based on the Connes-Skandalis longitudinal index theorem for foliations [11] as well as the techniques of cyclic cohomology. Thus it relies on the spin structure on F, and we do not see how to adapt it to prove Theorem 0.1, where one assumes TM being spin instead.

On the other hand, while Theorem 0.1 is different from Connes' result and also covers the cases of dim $M = 8k + 1$ and $8k + 2$ where the Hirzebruch $\widehat{A}$-genus vanishes tautologically, a common difficulty for both Theorems 0.1 and 0.6 is that there might be no transverse Riemannian structure on the underlying foliated manifold.

To overcome this difficulty, Connes [10] introduces an important geometric idea, which reduces the original problem to that on a fibration[3] over the foliation under consideration. The key advantage of this fibration is that the lifted (from the original) foliation is almost isometric, i.e., very close to Riemannian foliations. On the other hand, however, this fibration is noncompact. This makes the proof of Theorem 0.6 in [10], which relies essentially on the noncommutative techniques, highly nontrivial.

Our proof of Theorem 0.1 is differential geometric and does not use noncommutative geometry. It makes use of the sub-Dirac operators constructed in [22, §2b)] on the Connes fibration, as well as the adiabatic limit computations on foliations also considered in [22]. The key point is that while Connes' noncommutative proof of Theorem 0.6 relies heavily on the analysis near the (fiberwise) infinity of the associated Connes fibration, our main concern is on a compact subset of the Connes fibration. To be more precise, inspired by [5], [6] and [10], we introduce a specific deformation of the sub-Dirac operator on the Connes fibration and show that the deformed operator is "invertible" on certain compact subsets of the Connes fibration (cf. (2.21) in Section 2.2 for more details).

Moreover, by modifying the sub-Dirac operators mentioned above (see Section 1.4 for more details), our method applies to give a purely geometric proof of Theorem 0.6. This new proof provides a positive answer to a longstanding question in index theory (cf. [16, Page 5 of Lecture 9]).

We would like to mention that the idea of constructing sub-Dirac operators has also been used in [20] to prove a generalization of the Atiyah-Hirzebruch vanishing theorem for circle actions [1] to the case of foliations.

This paper is organized as follows. In Section 1, we discuss the case of almost isometric foliations and carry out the local computations. We also introduce the sub-Dirac operator in this case and prove Theorem 0.6 in the case where the underlying foliation is compact. In Section 2, we work on noncompact Connes fibrations and carry out the proofs of Theorems 0.1 and 0.6. We also include some new results in the end of the paper.

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[3]Which will be called a Connes fibration in what follows.

1. Adiabatic limit and almost isometric foliations

In this section, we discuss the geometry of almost isometric foliations in the sense of Connes [10]. We introduce for this kind of foliations a rescaled metric and show that the leafwise scalar curvature shows up from the limit behavior of the rescaled scalar curvature. We also introduce in this setting the sub-Dirac operators inspired by the original construction given in [22]. Finally, by combining the above two procedures, we prove a vanishing result when the almost isometric foliation under discussion is compact.

This section is organized as follows. In Section 1.1, we recall the definition of the almost isometric foliation in the sense of Connes. In Section 1.2 we introduce a rescaling of the given metric on the almost isometric foliation and study the corresponding limit behavior of the scalar curvature. In Section 1.3, we study Bott type connections on certain bundles transverse to the integrable subbundle. In Section 1.4, we construct the required sub-Dirac operator and compute the corresponding Lichnerowicz type formula. In Section 1.5 we prove a vanishing result when the almost isometric foliation is compact and verifies the conditions in Theorem 0.6.

1.1. **Almost isometric foliations.** Let (M, F) be a foliated manifold, where F is an integrable subbundle of the tangent vector bundle TM of a smooth manifold M, i.e., for any smooth sections X, $Y \in \Gamma(F)$, one has

$$(1.1) \qquad\qquad\qquad [X, Y] \in \Gamma(F).$$

Take a splitting $TM = F \oplus TM/F$. Let $p^{TM/F} : TM = F \oplus TM/F \to TM/F$ be the canonical projection. Following [7], we define the Bott connection to be any connection $\nabla^{TM/F}$ on TM/F so that for any $X \in \Gamma(F)$ and $U \in \Gamma(TM/F)$, one has

$$(1.2) \qquad\qquad\qquad \nabla_X^{TM/F} U = p^{TM/F}[X, U].$$

The key property of the Bott connection is that it is leafwise flat, that is, for any X, $Y \in \Gamma(F)$, one has (cf. [31, Lemma 1.14])

$$(1.3) \qquad\qquad\qquad \left(\nabla^{TM/F}\right)^2 (X, Y) = 0.$$

However, it may happen that $\nabla^{TM/F}$ does not preserve any metric on TM/F.

Let G be the holonomy groupoid of (M, F) (cf. [28]).

We make the assumption that there is a proper subbundle E of TM/F and choose a splitting

$$(1.4) \qquad\qquad\qquad TM/F = E \oplus (TM/F)/E.$$

Let q_1, q_2 denote the ranks of E and $(TM/F)/E$ respectively.

Definition 1.1. (Connes [10, Section 4]**)** If there exists a metric $g^{TM/F}$ on TM/F with its restrictions to E and $(TM/F)/E$ such that the action of G on TM/F takes the form

$$(1.5) \qquad\qquad\qquad \begin{pmatrix} O(q_1) & 0 \\ A & O(q_2) \end{pmatrix},$$

where $O(q_1)$, $O(q_2)$ are orthogonal matrices of ranks q_1, q_2 respectively, and A is a $q_2 \times q_1$ matrix, then we say that (M, F) carries an almost isometric structure.

Clearly, the existence of the almost isometric structure does not depend on the splitting (1.4). We assume from now on that (M, F) carries an almost isometric structure as above.

For simplicity, we denote E, $(TM/F)/E$ by $F_1^\perp$, $F_2^\perp$ respectively.

Let g^F be a metric on F. Let $g^{F_1^\perp}$, $g^{F_2^\perp}$ be the restrictions of $g^{TM/F}$ to $F_1^\perp$, $F_2^\perp$. Let g^{TM} be a metric on TM so that we have the orthogonal splitting

(1.6) $$TM = F \oplus F_1^\perp \oplus F_2^\perp, \qquad g^{TM} = g^F \oplus g^{F_1^\perp} \oplus g^{F_2^\perp}.$$

Let ∇^{TM} be the Levi-Civita connection associated to g^{TM}.

From the almost isometric condition (1.5), one deduces that for any $X \in \Gamma(F)$, U_i, $V_i \in \Gamma(F_i^\perp)$, $i = 1,\ 2$, the following identities, which may be thought of as infinitesimal versions of (1.5), hold (cf. [22, (A.5)]):

(1.7)
$$\langle [X, U_i], V_i \rangle + \langle U_i, [X, V_i] \rangle = X \langle U_i, V_i \rangle,$$
$$\langle [X, U_2], U_1 \rangle = 0.$$

Equivalently,

(1.8)
$$\langle X, \nabla_{U_i}^{TM} V_i + \nabla_{V_i}^{TM} U_i \rangle = 0,$$
$$\langle \nabla_X^{TM} U_2, U_1 \rangle + \langle X, \nabla_{U_2}^{TM} U_1 \rangle = 0.$$

In this paper, for simplicity, we also make the following assumption. This assumption holds by the Connes fibration to be dealt with in the next section.

Definition 1.2. We call an almost isometric foliation as above verifies Condition (C) if $F_2^\perp$ is also integrable. That is, for any U_2, $V_2 \in \Gamma(F_2^\perp)$, one has

(1.9) $$[U_2, V_2] \in \Gamma\left(F_2^\perp\right).$$

1.2. **Adiabatic limit and the scalar curvature.** In this subsection, we study the relationship between the leafwise scalar curvature and the scalar curvature on the total manifold of an almost isometric foliation. For convenience, we recall the formula for the Levi-Civita connection (cf. [4, (1.18)]) that for any X, Y, $Z \in \Gamma(TM)$,

(1.10) $\ 2 \langle \nabla_X^{TM} Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle$
$$+ \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle.$$

Recall that by [22, Proposition A.2], if one rescales the metric $g^{F_1^\perp}$ to $\frac{1}{\varepsilon^2} g^{F_1^\perp}$ and takes $\varepsilon \to 0$, then the almost isometric foliation in the sense of Definition 1.1 becomes an almost Riemannian foliation in the sense of [22, Definition 2.1]. In order to get information on the leafwise scalar curvature, one further rescales the metric $\frac{1}{\varepsilon^2} g^{F_1^\perp} \oplus g^{F_2^\perp}$ (standardly) to $\frac{1}{\beta^2}(\frac{1}{\varepsilon^2} g^{F_1^\perp} \oplus g^{F_2^\perp})$ (compare with [22, (1.4)] and [21]), which is equivalent to rescaling g^F to $\beta^2 g^F$. Putting these two rescaling procedures together, it is natural to introduce the following defomation of g^{TM}.

For any β, $\varepsilon > 0$, let $g_{\beta,\varepsilon}^{TM}$ be the rescaled Riemannian metric on TM defined by

(1.11) $$g_{\beta,\varepsilon}^{TM} = \beta^2 g^F \oplus \frac{1}{\varepsilon^2} g^{F_1^\perp} \oplus g^{F_2^\perp}.$$

We will always assume that $0 < \beta,\ \varepsilon \leq 1$.

We will use the subscripts and/or superscripts "β, ε" to decorate the geometric data associated to $g_{\beta,\varepsilon}^{TM}$. For example, $\nabla^{TM,\beta,\varepsilon}$ will denote the Levi-Civita connection associated to $g_{\beta,\varepsilon}^{TM}$. When the corresponding notation does not involve "β, ε", we will mean that it corresponds to the case of $\beta = \varepsilon = 1$.

Let p, $p_1^\perp$, $p_2^\perp$ be the orthogonal projections from TM to F, $F_1^\perp$, $F_2^\perp$ with respect to the orthogonal splitting (1.6). Let $\nabla^{F,\beta,\varepsilon}$, $\nabla^{F_1^\perp,\beta,\varepsilon}$, $\nabla^{F_2^\perp,\beta,\varepsilon}$ be the Euclidean connections on F, $F_1^\perp$, $F_2^\perp$ defined by

$$(1.12) \qquad \nabla^{F,\beta,\varepsilon} = p\nabla^{TM,\beta,\varepsilon}p, \quad \nabla^{F_1^\perp,\beta,\varepsilon} = p_1^\perp\nabla^{TM,\beta,\varepsilon}p_1^\perp, \quad \nabla^{F_2^\perp,\beta,\varepsilon} = p_2^\perp\nabla^{TM,\beta,\varepsilon}p_2^\perp.$$

In particular, one has

$$(1.13) \qquad \nabla^F = p\nabla^{TM}p, \quad \nabla^{F_1^\perp} = p_1^\perp\nabla^{TM}p_1^\perp, \quad \nabla^{F_2^\perp} = p_2^\perp\nabla^{TM}p_2^\perp.$$

By (1.10)-(1.13) and the integrability of F, the following identities hold for $X \in \Gamma(F)$:

$$(1.14) \qquad \nabla^{F,\beta,\varepsilon} = \nabla^F, \quad p\nabla_X^{TM,\beta,\varepsilon}p_i^\perp = p\nabla_X^{TM}p_i^\perp, \quad i = 1, \, 2,$$

$$p_1^\perp\nabla_X^{TM,\beta,\varepsilon}p = \beta^2\varepsilon^2 p_1^\perp\nabla_X^{TM}p, \quad p_2^\perp\nabla_X^{TM,\beta,\varepsilon}p = \beta^2 p_2^\perp\nabla_X^{TM}p.$$

From (1.7)-(1.11), we deduce that for $X \in \Gamma(F)$, U_i, $V_i \in \Gamma(F_i^\perp)$, $i = 1, \, 2$,

$$(1.15) \qquad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon}V_1, X \right\rangle = \left\langle \nabla_{U_1}^{TM}V_1, X \right\rangle = \frac{1}{2}\left\langle [U_1, V_1], X \right\rangle,$$

while

$$(1.16) \qquad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon}V_2, X \right\rangle = \left\langle \nabla_{U_2}^{TM}V_2, X \right\rangle = \frac{1}{2}\left\langle [U_2, V_2], X \right\rangle = 0.$$

Equivalently, for any $U_i \in \Gamma(F_i^\perp)$, $i = 1, \, 2$,

$$(1.17) \qquad p_1^\perp\nabla_{U_1}^{TM,\beta,\varepsilon}p = \beta^2\varepsilon^2 p_1^\perp\nabla_{U_1}^{TM}p, \quad p_2^\perp\nabla_{U_2}^{TM,\beta,\varepsilon}p = 0.$$

Similarly, one verifies that

$$(1.18) \qquad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon}X, U_2 \right\rangle = \frac{1}{2}\left\langle [U_1, X], U_2 \right\rangle - \frac{\beta^2}{2}\left\langle [U_1, U_2], X \right\rangle,$$

$$\left\langle \nabla_{U_2}^{TM,\beta,\varepsilon}X, U_1 \right\rangle = \frac{\varepsilon^2}{2}\left\langle [U_1, X], U_2 \right\rangle + \frac{\beta^2\varepsilon^2}{2}\left\langle [U_1, U_2], X \right\rangle.$$

For convenience of the later computations, we collect the asymptotic behavior of various covariant derivatives in the following lemma. These formulas can be derived by applying (1.7)-(1.18). The inner products appear in the lemma correspond to $\beta = \varepsilon = 1$.

Lemma 1.3. *The following formulas hold for X, Y, $Z \in \Gamma(F)$, U_i, V_i, $W_i \in \Gamma(F_i^\perp)$ with $i = 1, \, 2$, when $\beta > 0$, $\varepsilon > 0$ are small,*

(1.19)
$$\left\langle \nabla_X^{TM,\beta,\varepsilon}Y, Z \right\rangle = O(1), \quad \left\langle \nabla_X^{TM,\beta,\varepsilon}Y, U_1 \right\rangle = O\left(\beta^2\varepsilon^2\right), \quad \left\langle \nabla_X^{TM,\beta,\varepsilon}Y, U_2 \right\rangle = O\left(\beta^2\right),$$

(1.20)
$$\left\langle \nabla_X^{TM,\beta,\varepsilon}U_1, Y \right\rangle = O\left(1\right), \quad \left\langle \nabla_X^{TM,\beta,\varepsilon}U_1, V_1 \right\rangle = O\left(1\right), \quad \left\langle \nabla_X^{TM,\beta,\varepsilon}U_1, U_2 \right\rangle = O\left(1\right),$$

(1.21)
$$\left\langle \nabla_X^{TM,\beta,\varepsilon} U_2, Y \right\rangle = O(1), \quad \left\langle \nabla_X^{TM,\beta,\varepsilon} U_2, U_1 \right\rangle = O(\varepsilon^2), \quad \left\langle \nabla_X^{TM,\beta,\varepsilon} U_2, V_2 \right\rangle = O(1),$$

(1.22)
$$\left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} X, Y \right\rangle = O(1), \quad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} X, V_1 \right\rangle = O(\beta^2 \varepsilon^2), \quad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} X, U_2 \right\rangle = O(1),$$

(1.23)
$$\left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} V_1, X \right\rangle = O(1), \quad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} V_1, W_1 \right\rangle = O(1), \quad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} V_1, U_2 \right\rangle = O\left(\frac{1}{\varepsilon^2}\right),$$

(1.24)
$$\left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} U_2, X \right\rangle = O\left(\frac{1}{\beta^2}\right), \quad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} U_2, V_1 \right\rangle = O(1), \quad \left\langle \nabla_{U_1}^{TM,\beta,\varepsilon} U_2, V_2 \right\rangle = O(1),$$

(1.25) $\quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} X, Y \right\rangle = O(1), \quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} X, U_1 \right\rangle = O(\varepsilon^2), \quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} X, V_2 \right\rangle = 0,$

(1.26)
$$\left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} U_1, X \right\rangle = O\left(\frac{1}{\beta^2}\right), \quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} U_1, V_1 \right\rangle = O(1), \quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} U_1, V_2 \right\rangle = O(1),$$

(1.27) $\quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} V_2, X \right\rangle = 0, \quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} V_2, U_1 \right\rangle = O(\varepsilon^2), \quad \left\langle \nabla_{U_2}^{TM,\beta,\varepsilon} V_2, W_2 \right\rangle = O(1).$

Proof. Formulas in (1.19) follows from (1.14).

The first formula in (1.20) follows from (1.11) and the second formula in (1.19). The second one is trivial and the third one follows from (1.18).

The first formula in (1.21) follows from (1.11) and the third formula in (1.19). The second one follows from the second formulas in (1.7) and (1.18). The third one is trivial.

The first formula in (1.22) follows from (1.1), (1.10) and (1.11). The second one follows from (1.17) and the third one follows from the first formula in (1.18).

The first formula in (1.23) follows from (1.11) and the second formula in (1.22). The second formula is trivial. For the third formula, the $\frac{1}{\varepsilon^2}$ factor comes from the terms involving $\langle [U_1, U_2], V_1 \rangle$, $\langle [V_1, U_2], U_1 \rangle$ and $U_2 \langle U_1, V_1 \rangle$.

The first formula in (1.24) follows from the first formula in (1.18). The second one is trivial, and the third one follows from (1.9).

The first formula in (1.25) follows from the first formula in (1.14). The second one follows from the second formula in (1.18), and third one follows from (1.16).

The first formula in (1.26) follows from (1.11) and the second formula in (1.25). The second one is trivial, and the third one follows from (1.9).

The first formula in (1.27) follows from the third formula in (1.25). The second one follows from the third formula in (1.26), and the third one is trivial.

The proof of Lemma 1.3 is completed. $\qquad\square$

In what follows, when we compute the asymptotics of various covariant derivatives, we will simply use the above asymptotic formulas freely without further notice.

Let $R^{TM,\beta,\varepsilon} = (\nabla^{TM,\beta,\varepsilon})^2$ be the curvature of $\nabla^{TM,\beta,\varepsilon}$. Then for any X, $Y \in \Gamma(TM)$, one has the following standard formula,

$$(1.28) \qquad R^{TM,\beta,\varepsilon}(X,Y) = \nabla_X^{TM,\beta,\varepsilon}\nabla_Y^{TM,\beta,\varepsilon} - \nabla_Y^{TM,\beta,\varepsilon}\nabla_X^{TM,\beta,\varepsilon} - \nabla_{[X,Y]}^{TM,\beta,\varepsilon}.$$

Let $R^F = (\nabla^F)^2$ be the curvature of ∇^F. Let $k^{TM,\beta,\varepsilon}$, k^F denote the scalar curvatures of $g^{TM,\beta,\varepsilon}$, g^F respectively. Recall that k^F is defined in (0.1). The following formula for k^F is obvious,

$$(1.29) \qquad k^F = -\sum_{i,j=1}^{\mathrm{rk}(F)} \left\langle R^F\left(f_i, f_j\right) f_i, f_j \right\rangle,$$

where f_i, $i = 1, \cdots, \mathrm{rk}(F)$, is an orthonormal basis of (F, g^F). Clearly, when $F = TM$, it reduces to the usual definition of the scalar curvature k^{TM} of g^{TM}.

Proposition 1.4. *If Condition (C) holds, then when $\beta > 0$, $\varepsilon > 0$ are small, the following formula holds uniformly on any compact subset of M,*

$$(1.30) \qquad k^{TM,\beta,\varepsilon} = \frac{k^F}{\beta^2} + O\left(1 + \frac{\varepsilon^2}{\beta^2}\right).$$

Proof. By (1.1), (1.14), (1.28) and Lemma 1.3, one deduces that when $\beta > 0$, $\varepsilon > 0$ are very small, for any X, $Y \in \Gamma(F)$, one has

$$(1.31) \quad \left\langle R^{TM,\beta,\varepsilon}(X,Y)X,Y \right\rangle = \left\langle \nabla_X^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_Y^{TM,\beta,\varepsilon}X, Y \right\rangle$$

$$- \left\langle \nabla_Y^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_X^{TM,\beta,\varepsilon}X, Y \right\rangle - \left\langle \nabla_{[X,Y]}^{TM,\beta,\varepsilon}X, Y \right\rangle$$

$$= \left\langle R^F(X,Y)X,Y \right\rangle - \beta^2\varepsilon^2\left\langle p_1^\perp\nabla_Y^{TM}X, \nabla_X^{TM}Y \right\rangle - \beta^2\left\langle p_2^\perp\nabla_Y^{TM}X, \nabla_X^{TM}Y \right\rangle$$

$$+ \beta^2\varepsilon^2\left\langle p_1^\perp\nabla_X^{TM}X, \nabla_Y^{TM}Y \right\rangle + \beta^2\left\langle p_2^\perp\nabla_X^{TM}X, \nabla_Y^{TM}Y \right\rangle$$

$$= \left\langle R^F(X,Y)X,Y \right\rangle + O\left(\beta^2\right).$$

For $X \in \Gamma(F)$, $U \in \Gamma(F_1^\perp)$, by (1.7)-(1.28), one finds that when β, $\varepsilon > 0$ are small,

$$(1.32) \quad \left\langle R^{TM,\beta,\varepsilon}(X,U)X,U \right\rangle = \left\langle \nabla_X^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_U^{TM,\beta,\varepsilon}X, U \right\rangle$$

$$- \left\langle \nabla_U^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_X^{TM,\beta,\varepsilon}X, U \right\rangle - \left\langle \nabla_{\left(p+p_1^\perp+p_2^\perp\right)[X,U]}^{TM,\beta,\varepsilon}X, U \right\rangle$$

$$= \beta^2\varepsilon^2\left\langle \nabla_X^{TM}p\nabla_U^{TM}X, U \right\rangle + \beta^2\varepsilon^2\left\langle \nabla_X^{TM,\beta,\varepsilon}p_1^\perp\nabla_U^{TM}X, U \right\rangle - \varepsilon^2\left\langle p_2^\perp\nabla_U^{TM,\beta,\varepsilon}X, \nabla_X^{TM,\beta,\varepsilon}U \right\rangle$$

$$- \beta^2\varepsilon^2\left\langle \nabla_U^{TM}p\nabla_X^{TM}X, U \right\rangle - \beta^2\varepsilon^2\left\langle \nabla_U^{TM,\beta,\varepsilon}p_1^\perp\nabla_X^{TM}X, U \right\rangle + \varepsilon^2\left\langle p_2^\perp\nabla_X^{TM,\beta,\varepsilon}X, \nabla_U^{TM,\beta,\varepsilon}U \right\rangle$$

$$- \beta^2\varepsilon^2\left\langle \nabla_{\left(p+p_1^\perp\right)[X,U]}^{TM}X, U \right\rangle - \left\langle \nabla_{p_2^\perp[X,U]}^{TM,\beta,\varepsilon}X, U \right\rangle = O\left(\beta^2 + \varepsilon^2\right).$$

Similarly, for $X \in \Gamma(F)$, $U \in \Gamma(F_2^\perp)$, one has that when $\beta > 0$, $\varepsilon > 0$ are small,

$$(1.33) \quad \left\langle R^{TM,\beta,\varepsilon}(X,U)X,U \right\rangle = \left\langle \nabla_X^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_U^{TM,\beta,\varepsilon}X,U \right\rangle$$

$$- \left\langle \nabla_U^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_X^{TM,\beta,\varepsilon}X,U \right\rangle - \left\langle \nabla_{\left(p+p_1^\perp+p_2^\perp\right)[X,U]}^{TM,\beta,\varepsilon}X,U \right\rangle$$

$$= \beta^2 \left\langle \nabla_X^{TM}p\nabla_U^{TM}X,U \right\rangle - \frac{1}{\varepsilon^2}\left\langle p_1^\perp\nabla_U^{TM,\beta,\varepsilon}X,\nabla_X^{TM,\beta,\varepsilon}U \right\rangle + \beta^2 \left\langle \nabla_X^{TM,\beta,\varepsilon}p_2^\perp\nabla_U^{TM}X,U \right\rangle$$

$$-\beta^2 \left\langle \nabla_U^{TM}p\nabla_X^{TM}X,U \right\rangle - \beta^2\varepsilon^2 \left\langle \nabla_U^{TM,\beta,\varepsilon}p_1^\perp\nabla_X^{TM}X,U \right\rangle - \beta^2 \left\langle \nabla_U^{TM,\beta,\varepsilon}p_2^\perp\nabla_X^{TM}X,U \right\rangle$$

$$-\beta^2 \left\langle \nabla_{p[X,U]}^{TM}X,U \right\rangle - \beta^2 \left\langle \nabla_{p_2^\perp[X,U]}^{TM}X,U \right\rangle = O\left(\beta^2 + \varepsilon^2\right).$$

For U, $V \in \Gamma(F_1^\perp)$, one verifies that

$$(1.34) \quad \left\langle R^{TM,\beta,\varepsilon}(U,V)U,V \right\rangle = \left\langle \nabla_U^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_V^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$- \left\langle \nabla_V^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_U^{TM,\beta,\varepsilon}U,V \right\rangle - \left\langle \nabla_{\left(p+p_1^\perp+p_2^\perp\right)[U,V]}^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$= \beta^2\varepsilon^2 \left\langle \nabla_U^{TM}p\nabla_V^{TM,\beta,\varepsilon}U,V \right\rangle + \left\langle \nabla_U^{TM}p_1^\perp\nabla_V^{TM}U,V \right\rangle - \varepsilon^2 \left\langle p_2^\perp\nabla_V^{TM,\beta,\varepsilon}U,\nabla_U^{TM,\beta,\varepsilon}V \right\rangle$$

$$-\beta^2\varepsilon^2 \left\langle \nabla_V^{TM}p\nabla_U^{TM,\beta,\varepsilon}U,V \right\rangle - \left\langle \nabla_V^{TM}p_1^\perp\nabla_U^{TM}U,V \right\rangle + \varepsilon^2 \left\langle p_2^\perp\nabla_U^{TM,\beta,\varepsilon}U,\nabla_V^{TM,\beta,\varepsilon}V \right\rangle$$

$$- \left\langle \nabla_{p[U,V]}^{TM,\beta,\varepsilon}U,V \right\rangle - \left\langle \nabla_{p_1^\perp[U,V]}^{TM}U,V \right\rangle - \left\langle \nabla_{p_2^\perp[U,V]}^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$= -\varepsilon^2 \left\langle p_2^\perp\nabla_V^{TM,\beta,\varepsilon}U,\nabla_U^{TM,\beta,\varepsilon}V \right\rangle + \varepsilon^2 \left\langle p_2^\perp\nabla_U^{TM,\beta,\varepsilon}U,\nabla_V^{TM,\beta,\varepsilon}V \right\rangle + O\left(1\right) = O\left(\frac{1}{\varepsilon^2}\right),$$

from which one gets that when $\beta > 0$, $\varepsilon > 0$ are small,

$$(1.35) \qquad\qquad \varepsilon^2 \left\langle R^{TM,\beta,\varepsilon}(U,V)U,V \right\rangle = O\left(1\right).$$

For U, $V \in \Gamma(F_2^\perp)$, one verifies directly that

$$(1.36) \quad \left\langle R^{TM,\beta,\varepsilon}(U,V)U,V \right\rangle = \left\langle \nabla_U^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_V^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$- \left\langle \nabla_V^{TM,\beta,\varepsilon}\left(p + p_1^\perp + p_2^\perp\right)\nabla_U^{TM,\beta,\varepsilon}U,V \right\rangle - \left\langle \nabla_{[U,V]}^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$= \beta^2 \left\langle \nabla_U^{TM}p\nabla_V^{TM,\beta,\varepsilon}U,V \right\rangle - \frac{1}{\varepsilon^2}\left\langle p_1^\perp\nabla_V^{TM,\beta,\varepsilon}U,\nabla_U^{TM,\beta,\varepsilon}V \right\rangle + \left\langle \nabla_U^{TM}p_2^\perp\nabla_V^{TM}U,V \right\rangle$$

$$-\beta^2 \left\langle \nabla_V^{TM}p\nabla_U^{TM,\beta,\varepsilon}U,V \right\rangle + \frac{1}{\varepsilon^2}\left\langle p_1^\perp\nabla_U^{TM,\beta,\varepsilon}U,\nabla_V^{TM,\beta,\varepsilon}V \right\rangle - \left\langle \nabla_V^{TM}p_2^\perp\nabla_U^{TM}U,V \right\rangle$$

$$- \left\langle \nabla_{[U,V]}^{TM}U,V \right\rangle = O(1).$$

For $U \in \Gamma(F_1^\perp)$, $V \in \Gamma(F_2^\perp)$, one verifies directly that,

$$(1.37) \quad \left\langle R^{TM,\beta,\varepsilon}(U,V)U,V \right\rangle = \left\langle \nabla_U^{TM,\beta,\varepsilon} \left(p + p_1^\perp + p_2^\perp\right) \nabla_V^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$- \left\langle \nabla_V^{TM,\beta,\varepsilon} \left(p + p_1^\perp + p_2^\perp\right) \nabla_U^{TM,\beta,\varepsilon}U,V \right\rangle - \left\langle \nabla_{[U,V]}^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$= -\beta^2 \left\langle p\nabla_V^{TM,\beta,\varepsilon}U, \nabla_U^{TM,\beta,\varepsilon}V \right\rangle - \frac{1}{\varepsilon^2} \left\langle p_1^\perp \nabla_V^{TM,\beta,\varepsilon}U, \nabla_U^{TM,\beta,\varepsilon}V \right\rangle + \left\langle \nabla_U^{TM,\beta,\varepsilon} p_2^\perp \nabla_V^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$+ \beta^2 \left\langle p\nabla_U^{TM,\beta,\varepsilon}U, \nabla_V^{TM,\beta,\varepsilon}V \right\rangle + \frac{1}{\varepsilon^2} \left\langle p_1^\perp \nabla_U^{TM,\beta,\varepsilon}U, \nabla_V^{TM,\beta,\varepsilon}V \right\rangle - \left\langle \nabla_V^{TM} p_2^\perp \nabla_U^{TM,\beta,\varepsilon}U,V \right\rangle$$

$$+ \frac{1}{\varepsilon^2} \left\langle U, \nabla_{[U,V]}^{TM,\beta,\varepsilon}V \right\rangle = O\left(\frac{1}{\varepsilon^2} + \frac{1}{\beta^2}\right),$$

from which one gets that when $\beta > 0$, $\varepsilon > 0$ are small,

$$(1.38) \qquad \varepsilon^2 \left\langle R^{TM,\beta,\varepsilon}(U,V)U,V \right\rangle = \left\langle R^{TM,\beta,\varepsilon}(V,U)V,U \right\rangle = O\left(1 + \frac{\varepsilon^2}{\beta^2}\right).$$

From (1.29), (1.31)-(1.33), (1.35), (1.36) and (1.38), one gets (1.30). □

1.3. Bott connections on $F_1^\perp$ and $F_2^\perp$.
From (1.7) and (1.9)-(1.12), one verifies directly that for $X \in \Gamma(F)$, U_i, $V_i \in \Gamma(F_i^\perp)$, $i = 1$, 2, one has

$$(1.39) \qquad \left\langle \nabla_X^{F_1^\perp,\beta,\varepsilon}U_1,V_1 \right\rangle = \langle [X,U_1],V_1 \rangle - \frac{\beta^2\varepsilon^2}{2} \langle [U_1,V_1],X \rangle,$$

$$\left\langle \nabla_X^{F_2^\perp,\beta,\varepsilon}U_2,V_2 \right\rangle = \langle [X,U_2],V_2 \rangle.$$

By (1.39), one has that for $X \in \Gamma(F)$, $U_i \in \Gamma(F_i^\perp)$, $i = 1$, 2,

$$(1.40) \qquad \lim_{\varepsilon \to 0^+} \nabla_X^{F_i^\perp,\beta,\varepsilon}U_i = \widetilde{\nabla}_X^{F_i^\perp}U_i := p_i^\perp [X,U_i].$$

Let $\widetilde{\nabla}^{F_i^\perp}$ be the connection on $F_i^\perp$ defined by the second equality in (1.40) and by $\widetilde{\nabla}_U^{F_i^\perp}U_i = \nabla_U^{F_i^\perp}U_i$ for $U \in \Gamma(F_1^\perp \oplus F_2^\perp)$. In view of (1.2) and (1.40), we call $\widetilde{\nabla}^{F_i^\perp}$ a Bott connection on $F_i^\perp$ for $i = 1$ or 2. Let $\widetilde{R}^{F_i^\perp}$ denote the curvature of $\widetilde{\nabla}^{F_i^\perp}$ for $i = 1$, 2.

The following result holds without Condition (C).

Lemma 1.5. *For X, $Y \in \Gamma(F)$ and $i = 1$, 2, the following identity holds,*

$$(1.41) \qquad \widetilde{R}^{F_i^\perp}(X,Y) = 0.$$

Proof. We proceed as in [31, Proof of Lemma 1.14]. By (1.40) and the standard formula for the curvature (cf. [31, (1.3)]), for any $U \in \Gamma(F_i^\perp)$, $i = 1$, 2, one has,

$$(1.42) \quad \widetilde{R}^{F_i^\perp}(X,Y)U = \widetilde{\nabla}_X^{F_i^\perp}\widetilde{\nabla}_Y^{F_i^\perp}U - \widetilde{\nabla}_Y^{F_i^\perp}\widetilde{\nabla}_X^{F_i^\perp}U - \widetilde{\nabla}_{[X,Y]}^{F_i^\perp}U$$

$$= p_i^\perp \left([X,[Y,U]] + [Y,[U,X]] + [U,[X,Y]]\right) - p_i^\perp \left[X,(\mathrm{Id} - p_i^\perp)[Y,U]\right]$$

$$- p_i^\perp \left[Y,(\mathrm{Id} - p_i^\perp)[U,X]\right]$$

$$= -p_i^\perp \left[X,(p_1^\perp + p_2^\perp - p_i^\perp)[Y,U]\right] - p_i^\perp \left[Y,(p_1^\perp + p_2^\perp - p_i^\perp)[U,X]\right],$$

where the last equality follows from the Jacobi identity and the integrability of F.

Now if $i = 1$, then by (1.7), one has $U \in \Gamma(F_1^\perp)$ and

$$(1.43) \qquad p_1^\perp \left[X, p_2^\perp [Y, U]\right] = p_1^\perp \left[Y, p_2^\perp [U, X]\right] = 0.$$

While if $i = 2$, still by (1.7), one has $U \in \Gamma(F_2^\perp)$ and

$$(1.44) \qquad p_1^\perp [Y, U] = p_1^\perp [U, X] = 0.$$

From (1.42)-(1.44), one gets (1.41). The proof of Lemma 1.5 is completed. $\qquad \square$

Remark 1.6. For $i = 1, 2$, let $R^{F_i^\perp, \beta, \varepsilon}$ denote the curvature of $\nabla^{F_i^\perp, \beta, \varepsilon}$. From (1.39)-(1.41), one finds that for any $X, Y \in \Gamma(F)$, when $\beta > 0$, $\varepsilon > 0$ are small, the following identity holds:

$$(1.45) \qquad R^{F_i^\perp, \beta, \varepsilon}(X, Y) = O\left(\beta^2 \varepsilon^2\right).$$

On the other hand, for $i = 1, 2$, by using (1.7), (1.9), (1.10), (1.12) and (1.28), one verifies directly that when $\beta > 0$, $\varepsilon > 0$ are small, the following identity holds,

$$(1.46) \qquad R^{F_i^\perp, \beta, \varepsilon} = O\left(1\right).$$

1.4. Sub-Dirac operators associated to spin integrable subbundles.

We assume for simplicity that TM, F, $F_i^\perp$, $i = 1, 2$, are all oriented and of even rank, with the orientation of TM being compatible with the orientations on F, $F_1^\perp$ and $F_2^\perp$ through (1.6). We further assume that F is spin and carries a fixed spin structure.

Let $S(F) = S_+(F) \oplus S_-(F)$ be the Hermitian bundle of spinors associated to (F, g^F). For any $X \in \Gamma(F)$, the Clifford action $c(X)$ exchanges $S_\pm(F)$.

Let $i = 1$ or 2. Let $\Lambda^*(F_i^\perp)$ denote the exterior algebra bundle of $F_i^{\perp,*}$. Then $\Lambda^*(F_i^\perp)$ carries a canonically induced metric $g^{\Lambda^*(F_i^\perp)}$ from $g^{F_i^\perp}$. For any $U \in F_i^\perp$, let $U^* \in F_i^{\perp,*}$ correspond to U via $g^{F_i^\perp}$. For any $U \in \Gamma(F_i^\perp)$, set

$$(1.47) \qquad c(U) = U^* \wedge - i_U, \quad \widehat{c}(U) = U^* \wedge + i_U,$$

where $U^* \wedge$ and i_U are the exterior and interior multiplications by U^* and U on $\Lambda^*(F_i^\perp)$.

Denote $q = \mathrm{rk}(F)$, $q_i = \mathrm{rk}(F_i^\perp)$.

Let $h_1, \cdots, h_{q_i}$ be an oriented orthonormal basis of $F_i^\perp$. Set

$$(1.48) \qquad \tau\left(F_i^\perp, g^{F_i^\perp}\right) = \left(\frac{1}{\sqrt{-1}}\right)^{\frac{q_i(q_i+1)}{2}} c(h_1) \cdots c(h_{q_i}).$$

Then

$$(1.49) \qquad \tau\left(F_i^\perp, g^{F_i^\perp}\right)^2 = \mathrm{Id}_{\Lambda^*\left(F_i^\perp\right)}.$$

Set

$$(1.50) \qquad \Lambda_\pm^*\left(F_i^\perp\right) = \left\{h \in \Lambda^*\left(F_i^\perp\right) : \tau\left(F_i^\perp, g^{F_i^\perp}\right) h = \pm h\right\}.$$

Since q_i is even, for any $h \in F_i^\perp$, $c(h)$ anti-commutes with $\tau(F_i^\perp, g^{F_i^\perp})$, while $\widehat{c}(h)$ commutes with $\tau(F_i^\perp, g^{F_i^\perp})$. In particular, $c(h)$ exchanges $\Lambda_\pm^*(F_i^\perp)$.

Let $\widetilde{\tau}(F_i^\perp)$ denote the $\mathbf{Z}_2$-grading of $\Lambda^*(F_i^\perp)$ defined by

$$(1.51) \qquad \widetilde{\tau}\left(F_i^\perp\right)\big|_{\Lambda^{\frac{\mathrm{even}}{\mathrm{odd}}}\left(F_i^\perp\right)} = \pm \mathrm{Id}\big|_{\Lambda^{\frac{\mathrm{even}}{\mathrm{odd}}}\left(F_i^\perp\right)}.$$

Now we have the following $\mathbf{Z}_2$-graded vector bundles over M:

$$(1.52) \qquad S(F) = S_+(F) \oplus S_-(F),$$

$$(1.53) \qquad \Lambda^* \left(F_i^\perp \right) = \Lambda_+^* \left(F_i^\perp \right) \oplus \Lambda_-^* \left(F_i^\perp \right), \quad i = 1,\, 2,$$

and

$$(1.54) \qquad \Lambda^* \left(F_i^\perp \right) = \Lambda^{\mathrm{even}} \left(F_i^\perp \right) \oplus \Lambda^{\mathrm{odd}} \left(F_i^\perp \right), \quad i = 1,\, 2.$$

We form the following $\mathbf{Z}_2$-graded tensor product, which will play a role in Section 2:

$$(1.55) \qquad W \left(F, F_1^\perp, F_2^\perp \right) = S(F) \widehat{\otimes} \Lambda^* \left(F_1^\perp \right) \widehat{\otimes} \Lambda^* \left(F_2^\perp \right),$$

with the $\mathbf{Z}_2$-grading operator given by

$$(1.56) \qquad \tau_W = \tau_{S(F)} \cdot \tau \left(F_1^\perp, g^{F_1^\perp} \right) \cdot \widetilde{\tau} \left(F_2^\perp \right),$$

where $\tau_{S(F)}$ is the $\mathbf{Z}_2$-grading operator defining the splitting in (1.52). We denote by

$$(1.57) \qquad W \left(F, F_1^\perp, F_2^\perp \right) = W_+ \left(F, F_1^\perp, F_2^\perp \right) \oplus W_- \left(F, F_1^\perp, F_2^\perp \right)$$

the $\mathbf{Z}_2$-graded decomposition with respect to τ_W.

Recall that the connections ∇^F, $\nabla^{F_1^\perp}$ and $\nabla^{F_2^\perp}$ have been defined in (1.13). They lift canonically to Hermitian connections $\nabla^{S(F)}$, $\nabla^{\Lambda^*(F_1^\perp)}$, $\nabla^{\Lambda^*(F_2^\perp)}$ on $S(F)$, $\Lambda^*(F_1^\perp)$, $\Lambda^*(F_2^\perp)$ respectively, preserving the corresponding $\mathbf{Z}_2$-gradings. Let $\nabla^{W(F,F_1^\perp,F_2^\perp)}$ be the canonically induced connection on $W(F, F_1^\perp, F_2^\perp)$ which preserves the canonically induced Hermitian metric on $W(F, F_1^\perp, F_2^\perp)$, and also the $\mathbf{Z}_2$-grading of $W(F, F_1^\perp, F_2^\perp)$.

For any vector bundle E over M, by an integral polynomial of E we will mean a bundle $\phi(E)$ which is a polynomial in the exterior and symmetric powers of E with integral coefficients.

For $i = 1,\, 2$, let $\phi_i(F_i^\perp)$ be an integral polynomial of $F_i^\perp$. We denote the complexification of $\phi_i(F_i^\perp)$ by the same notation. Then $\phi_i(F_i^\perp)$ carries a naturally induced Hermitian metric from $g^{F_i^\perp}$ and also a naturally induced Hermitian connection $\nabla^{\phi_i(F_i^\perp)}$ from $\nabla^{F_i^\perp}$.

Let $W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$ be the $\mathbf{Z}_2$-graded vector bundle over M,

$$(1.58) \quad W \left(F, F_1^\perp, F_2^\perp \right) \otimes \phi_1 \left(F_1^\perp \right) \otimes \phi_2 \left(F_2^\perp \right) = W_+ \left(F, F_1^\perp, F_2^\perp \right) \otimes \phi_1 \left(F_1^\perp \right) \otimes \phi_2 \left(F_2^\perp \right)$$
$$\oplus\, W_- \left(F, F_1^\perp, F_2^\perp \right) \otimes \phi_1 \left(F_1^\perp \right) \otimes \phi_2 \left(F_2^\perp \right).$$

Let $\nabla^{W \otimes \phi_1 \otimes \phi_2}$ denote the naturally induced Hermitian connection on $W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$ with respect to the naturally induced Hermitian metric on it. Clearly, $\nabla^{W \otimes \phi_1 \otimes \phi_2}$ preserves the $\mathbf{Z}_2$-graded decomposition in (1.58).

Let S be the $\mathrm{End}(TM)$-valued one form on M defined by

$$(1.59) \qquad \nabla^{TM} = \nabla^F + \nabla^{F_1^\perp} + \nabla^{F_2^\perp} + S.$$

Let $e_1, \cdots, e_{\dim M}$ be an orthonormal basis of TM. Let $\nabla^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$ be the Hermitian connection on $W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$ defined by that for any $X \in \Gamma(TM)$,

$$(1.60) \qquad \nabla_X^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} = \nabla_X^{W \otimes \phi_1 \otimes \phi_2} + \frac{1}{4} \sum_{i,j=1}^{\dim M} \langle S(X) e_i, e_j \rangle\, c\,(e_i)\, c\,(e_j).$$

Let the linear operator $D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)} : \Gamma(W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)) \to$
$\Gamma(W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp))$ be defined by

$$(1.61) \qquad D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)} = \sum_{i=1}^{\dim M} c(e_i) \nabla_{e_i}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}.$$

We call $D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}$ a sub-Dirac operator with respect to the spin vector bundle F.

One verifies that $D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}$ is a first order formally self-adjoint elliptic differential operator. Moreover, it exchanges $\Gamma(W_\pm(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp))$. We denote by $D_\pm^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}$ the restrictions of $D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}$ to $\Gamma(W_\pm(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp))$. Then one has

$$(1.62) \qquad \left(D_+^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)} \right)^* = D_-^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}.$$

Remark 1.7. In the special case of $F = \{0\}$, the above sub-Dirac operator is simply the sub-Signature operator constructed in [30] (cf. [32]). On the other hand, in the case of $F_2^\perp = \{0\}$, the above sub-Dirac operator is constructed in [22, Section 2], which is sufficient for the proof of Theorem 0.1. The sub-Dirac operator constructed above will be used in Section 2.5 to prove the Connes vanishing theorem, i.e., Theorem 0.6.

Remark 1.8. When $F_1^\perp$, $F_2^\perp$ are also spin and carry fixed spin structures, then $TM = F \oplus F_1^\perp \oplus F_2^\perp$ is spin and carries an induced spin structure from the spin structures on F, $F_1^\perp$ and $F_2^\perp$. Moreover, one has the following identifications of $\mathbf{Z}_2$-graded vector bundles (cf. [18]) for $i = 1, 2$,

$$(1.63) \qquad \Lambda_+^* \left(F_i^\perp \right) \oplus \Lambda_-^* \left(F_i^\perp \right) = S_+ \left(F_i^\perp \right) \otimes S \left(F_i^\perp \right)^* \oplus S_- \left(F_i^\perp \right) \otimes S \left(F_i^\perp \right)^*,$$

$$(1.64) \quad \Lambda^{\text{even}} \left(F_i^\perp \right) \oplus \Lambda^{\text{odd}} \left(F_i^\perp \right) = \left(S_+ \left(F_i^\perp \right) \otimes S_+ \left(F_i^\perp \right)^* \oplus S_- \left(F_i^\perp \right) \otimes S_- \left(F_i^\perp \right)^* \right)$$
$$\oplus \left(S_+ \left(F_i^\perp \right) \otimes S_- \left(F_i^\perp \right)^* \oplus S_- \left(F_i^\perp \right) \otimes S_+ \left(F_i^\perp \right)^* \right).$$

By (1.48)-(1.61), (1.63) and (1.64), $D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}$ is simply the twisted Dirac operator

$$(1.65) \quad D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)} : \Gamma \left(S(TM) \widehat{\otimes} S \left(F_2^\perp \right)^* \otimes S \left(F_1^\perp \right)^* \otimes \phi_1 \left(F_1^\perp \right) \otimes \phi_2 \left(F_2^\perp \right) \right)$$
$$\longrightarrow \Gamma \left(S(TM) \widehat{\otimes} S \left(F_2^\perp \right)^* \otimes S \left(F_1^\perp \right)^* \otimes \phi_1 \left(F_1^\perp \right) \otimes \phi_2 \left(F_2^\perp \right) \right),$$

where for $i = 1, 2$, the Hermitian (dual) bundle of spinors $S(F_i^\perp)^*$ associated to $(F_i^\perp, g^{F_i^\perp})$ carries the Hermitian connection induced from $\nabla^{F_i^\perp}$. The point of (1.61) is that it only requires F being spin. While on the other hand, (1.65) allows us to take the advantage of applying the calculations already done for usual (twisted) Dirac operators when doing local computations.

Remark 1.9. It is clear that the definition in (1.61) does not require that $F \subseteq TM$ being integrable.

Let $\Delta^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}$ denote the Bochner Laplacian defined by

$$(1.66) \qquad \Delta^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)} = \sum_{i=1}^{\dim M} \left(\nabla_{e_i}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}\right)^2 - \nabla_{\sum_{i=1}^{\dim M}\nabla_{e_i}^{TM}e_i}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}.$$

Let k^{TM} be the scalar curvature of g^{TM}, $R^{F_i^\perp}$ $(i=1, 2)$ be the curvature of $\nabla^{F_i^\perp}$. Let $R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}$ be the curvature of the tensor product connection on $\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$ induced from $\nabla^{\phi_1(F_1^\perp)}$ and $\nabla^{\phi_2(F_2^\perp)}$.

In view of Remark 1.8, the following Lichnerowicz type formula holds:

$$(1.67)$$

$$\left(D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}\right)^2 = -\Delta^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)} + \frac{k^{TM}}{4} + \frac{1}{2}\sum_{i,j=1}^{\dim M} c(e_i)\,c(e_j)\,R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}(e_i,e_j)$$

$$+ \frac{1}{8}\sum_{k=1}^{2}\sum_{i,j,s,t=1}^{\dim M}\left\langle R^{F_k^\perp}(e_i,e_j)\,e_t,e_s\right\rangle c(e_i)\,c(e_j)\,\widehat{c}(e_s)\,\widehat{c}(e_t).$$

When M is compact, by the Atiyah-Singer index theorem [2] (cf. [18]), one has

$$(1.68) \quad \mathrm{ind}\left(D_+^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp)}\right)$$

$$= 2^{\frac{q_1}{2}}\left\langle \widehat{A}(F)\widehat{L}\left(F_1^\perp\right)e\left(F_2^\perp\right)\mathrm{ch}\left(\phi_1\left(F_1^\perp\right)\right)\mathrm{ch}\left(\phi_2\left(F_2^\perp\right)\right),[M]\right\rangle,$$

where $\widehat{A}(F)$ is the Hirzebruch $\widehat{A}$-class (cf. [31, §1.6.3]) of F, $\widehat{L}(F_1^\perp)$ is the Hirzebruch $\widehat{L}$-class (cf. [18, (11.18') of Chap. III]) of $F_1^\perp$, $e(F_2^\perp)$ is the Euler class (cf. [31, §3.4]) of $F_2^\perp$, and "ch" is the notation for the Chern character (cf. [31, §1.6.4]).

1.5. A vanishing theorem for almost isometric foliations.

In this subsection, we assume M is compact and prove a vanishing theorem. Some of the computations in this subsection will be used in the next section where we will deal with the case where M is non-compact.

Let $f_1, \cdots, f_q$ be an oriented orthonormal basis of F. Let $h_1, \cdots, h_{q_1}$ (resp. $e_1, \cdots, e_{q_2}$) be an oriented orthonormal basis of $F_1^\perp$ (resp. $F_2^\perp$).

Let $\beta > 0$, $\varepsilon > 0$ and consider the construction in Section 1.4 with respect to the metric $g_{\beta,\varepsilon}^{TM}$ defined in (1.11). We still use the superscripts "β, ε" to decorate the geometric data associated to $g_{\beta,\varepsilon}^{TM}$. For example, $D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}$ now denotes the sub-Dirac operator constructed in (1.61) associated to $g_{\beta,\varepsilon}^{TM}$. Moreover, it can be written as

$$(1.69)$$

$$D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} = \beta^{-1}\sum_{i=1}^{q} c(f_i)\,\nabla_{f_i}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} + \varepsilon\sum_{j=1}^{q_1} c(h_j)\,\nabla_{h_j}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}$$

$$+ \sum_{s=1}^{q_2} c(e_s)\,\nabla_{e_s}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}.$$

By (1.69), the Lichnerowicz type formula (1.67) for $(D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon})^2$ takes the following form (compare with [22, Theorem 2.3]),

$$(1.70)\quad \left(D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}\right)^2 = -\Delta^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} + \frac{k^{TM,\beta,\varepsilon}}{4}$$

$$+\frac{1}{2\beta^2}\sum_{i,j=1}^{q} c(f_i)c(f_j)R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(f_i,f_j)+\frac{\varepsilon^2}{2}\sum_{i,j=1}^{q_1} c(h_i)c(h_j)R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(h_i,h_j)$$

$$+\frac{1}{2}\sum_{i,j=1}^{q_2} c(e_i)c(e_j)R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(e_i,e_j)+\frac{\varepsilon}{\beta}\sum_{i=1}^{q}\sum_{j=1}^{q_1} c(f_i)c(h_j)R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(f_i,h_j)$$

$$+\frac{1}{\beta}\sum_{i=1}^{q}\sum_{j=1}^{q_2} c(f_i)c(e_j)R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(f_i,e_j)+\varepsilon\sum_{i=1}^{q_1}\sum_{j=1}^{q_2} c(h_i)c(e_j)R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(h_i,e_j)$$

$$+\frac{1}{8\beta^2}\sum_{i,j=1}^{q}\sum_{s,t=1}^{q_1}\left\langle R^{F_1^\perp,\beta,\varepsilon}(f_i,f_j)h_t,h_s\right\rangle c(f_i)c(f_j)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{\varepsilon^2}{8}\sum_{i,j=1}^{q_1}\sum_{s,t=1}^{q_1}\left\langle R^{F_1^\perp,\beta,\varepsilon}(h_i,h_j)h_t,h_s\right\rangle c(h_i)c(h_j)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{1}{8}\sum_{i,j=1}^{q_2}\sum_{s,t=1}^{q_1}\left\langle R^{F_1^\perp,\beta,\varepsilon}(e_i,e_j)h_t,h_s\right\rangle c(e_i)c(e_j)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{\varepsilon}{4\beta}\sum_{i=1}^{q}\sum_{j=1}^{q_1}\sum_{s,t=1}^{q_1}\left\langle R^{F_1^\perp,\beta,\varepsilon}(f_i,h_j)h_t,h_s\right\rangle c(f_i)c(h_j)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{1}{4\beta}\sum_{i=1}^{q}\sum_{j=1}^{q_2}\sum_{s,t=1}^{q_1}\left\langle R^{F_1^\perp,\beta,\varepsilon}(f_i,e_j)h_t,h_s\right\rangle c(f_i)c(e_j)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{\varepsilon}{4}\sum_{i=1}^{q_1}\sum_{j=1}^{q_2}\sum_{s,t=1}^{q_1}\left\langle R^{F_1^\perp,\beta,\varepsilon}(h_i,e_j)h_t,h_s\right\rangle c(h_i)c(e_j)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{1}{8\beta^2}\sum_{i,j=1}^{q}\sum_{s,t=1}^{q_2}\left\langle R^{F_2^\perp,\beta,\varepsilon}(f_i,f_j)e_t,e_s\right\rangle c(f_i)c(f_j)\widehat{c}(e_s)\widehat{c}(e_t)$$

$$+\frac{\varepsilon^2}{8}\sum_{i,j=1}^{q_1}\sum_{s,t=1}^{q_2}\left\langle R^{F_2^\perp,\beta,\varepsilon}(h_i,h_j)e_t,e_s\right\rangle c(h_i)c(h_j)\widehat{c}(e_s)\widehat{c}(e_t)$$

$$+\frac{1}{8}\sum_{i,j=1}^{q_2}\sum_{s,t=1}^{q_2}\left\langle R^{F_2^\perp,\beta,\varepsilon}(e_i,e_j)e_t,e_s\right\rangle c(e_i)c(e_j)\widehat{c}(e_s)\widehat{c}(e_t)$$

$$+\frac{\varepsilon}{4\beta}\sum_{i=1}^{q}\sum_{j=1}^{q_1}\sum_{s,t=1}^{q_2}\left\langle R^{F_2^\perp,\beta,\varepsilon}(f_i,h_j)e_t,e_s\right\rangle c(f_i)c(h_j)\widehat{c}(e_s)\widehat{c}(e_t)$$

$$+\frac{1}{4\beta}\sum_{i=1}^{q}\sum_{j=1}^{q_2}\sum_{s,t=1}^{q_2}\left\langle R^{F_2^\perp,\beta,\varepsilon}(f_i,e_j)e_t,e_s\right\rangle c(f_i)c(e_j)\widehat{c}(e_s)\widehat{c}(e_t)$$

$$+\frac{\varepsilon}{4} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp,\beta,\varepsilon}(h_i, e_j) e_t, e_s \right\rangle c(h_i) c(e_j) \widehat{c}(e_s) \widehat{c}(e_t).$$

By (1.30), (1.45), (1.46) and (1.70), we get that when $\beta > 0$, $\varepsilon > 0$ are small,

$$(1.71) \qquad \left(D^{F,\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp),\beta,\varepsilon} \right)^2 = -\Delta^{F,\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp),\beta,\varepsilon} + \frac{k^F}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2} \right).$$

Proposition 1.10. *If $k^F > 0$ over M, then for any Pontrjagin classes $p(F_1^\perp)$, $p'(F_2^\perp)$ of $F_1^\perp$, $F_2^\perp$ respectively, the following identity holds,*

$$(1.72) \qquad \left\langle \widehat{A}(F) p\left(F_1^\perp\right) e\left(F_2^\perp\right) p'\left(F_2^\perp\right), [M] \right\rangle = 0.$$

Proof. Since $k^F > 0$ over M, one can take $\beta > 0$, $\varepsilon > 0$ small enough so that the corresponding terms in the right hand side of (1.71) verifies that

$$(1.73) \qquad \frac{k^F}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2} \right) > 0$$

over M. Since $-\Delta^{F,\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp),\beta,\varepsilon}$ is nonnegative, by (1.62), (1.71) and (1.73), one gets

$$(1.74) \qquad \operatorname{ind}\left(D_+^{F,\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp),\beta,\varepsilon} \right) = 0.$$

From (1.68) and (1.74), we get

$$(1.75) \qquad \left\langle \widehat{A}(F) \widehat{L}\left(F_1^\perp\right) \operatorname{ch}\left(\phi_1\left(F_1^\perp\right)\right) e\left(F_2^\perp\right) \operatorname{ch}\left(\phi_2\left(F_2^\perp\right)\right), [M] \right\rangle = 0.$$

Now as it is standard that any rational Pontrjagin class of $F_1^\perp$ (resp. $F_2^\perp$) can be expressed as a rational linear combination of classes of the form $\widehat{L}(F_1^\perp)\operatorname{ch}(\phi_1(F_1^\perp))$ (resp. $\operatorname{ch}(\phi_2(F_2^\perp))$), one gets (1.72) from (1.75). $\qquad\square$

Remark 1.11. If one changes the $\mathbf{Z}_2$-grading in the definition of the sub-Dirac operator by replacing $\widetilde{\tau}(F_2^\perp)$ in (1.56) by $\tau(F_2^\perp, g^{F_2^\perp})$, then one can prove that under the same condition as in Proposition 1.10,

$$(1.76) \qquad \left\langle \widehat{A}(F) p\left(F_1^\perp\right) p'\left(F_2^\perp\right), [M] \right\rangle = 0$$

for any Pontrjagin classes $p(F_1^\perp)$, $p'(F_2^\perp)$ of $F_1^\perp$, $F_2^\perp$.

2. Connes fibration and vanishing theorems

This section is organized as follows. In Section 2.1, we recall the definition of the Connes fibration and prove some basic properties of it. In Section 2.2, we introduce a specific deformation of the sub-Dirac operator on the Connes fibration and prove a key vanishing result for the deformed sub-Dirac operator on certain compact subsets of the Connes fibration. This motivates the proof of Theorem 0.1 for the case of $\dim M = 4k$ given in Section 2.3. In Section 2.4, we present the proof of the $\dim M = 8k+i$ ($i = 1$, 2) cases of Theorem 0.1. Finally, in Section 2.5 we present the proof of Theorem 0.6, and state some new vanishing results.

2.1. **The Connes fibration.** Let (M, F) be a compact foliation, i.e., F is an integrable subbundle of the tangent vector bundle TM of a closed manifold M. For any vector space E of rank n, let $\mathcal{E}$ be the set of all Euclidean metrics on E. It is well known that $\mathcal{E}$ is the noncompact homogeneous space $GL(n, \mathbf{R})^+/SO(n)$ (with $\dim \mathcal{E} = \frac{n(n+1)}{2}$), which carries a natural Riemannian metric of nonpositive sectional curvature (cf. [15]). In particular, any two points of $\mathcal{E}$ can be joined by a unique geodesic.

Following [10, §5], let $\pi : \mathcal{M} \to M$ be the fibration over M such that for any $x \in M$, $\mathcal{M}_x = \pi^{-1}(x)$ is the space of Euclidean metrics on the linear space $T_x M/F_x$.

Let $T^V \mathcal{M}$ denote the vertical tangent bundle of the fibration $\pi : \mathcal{M} \to M$. Then it carries a natural metric $g^{T^V \mathcal{M}}$ such that any two points p, $q \in \mathcal{M}_x$, with $x \in M$, can be joined by a unique geodesic in $\mathcal{M}_x$. Let $d^{\mathcal{M}_x}(p, q)$ denote the length of this geodesic.

By using the Bott connection on TM/F (cf. (1.2)), which is leafwise flat, one lifts F to an integrable subbundle $\mathcal{F}$ of $T\mathcal{M}$.[4] Let g^F be a Euclidean metric on F, which lifts to a Euclidean metric $g^{\mathcal{F}} = \pi^* g^F$ on $\mathcal{F}$.

For any $v \in \mathcal{M}$, $T_v \mathcal{M}/(\mathcal{F}_v \oplus T_v^V \mathcal{M})$ is identified with $T_{\pi(v)} M/F_{\pi(v)}$ under the projection $\pi : \mathcal{M} \to M$. By definition, v determines a metric on $T_{\pi(v)} M/F_{\pi(v)}$, which in turn determines a metric on $T_v \mathcal{M}/(\mathcal{F}_v \oplus T_v^V \mathcal{M})$. In this way, $T\mathcal{M}/(\mathcal{F} \oplus T^V \mathcal{M})$ carries a canonically induced metric.

Let $\mathcal{F}_1^\perp \subseteq T\mathcal{M}$ be a subbundle, which is transversal to $\mathcal{F} \oplus T^V \mathcal{M}$, such that we have a splitting $T\mathcal{M} = (\mathcal{F} \oplus T^V \mathcal{M}) \oplus \mathcal{F}_1^\perp$. Then $\mathcal{F}_1^\perp$ can be identified with $T\mathcal{M}/(\mathcal{F} \oplus T^V \mathcal{M})$ and carries a canonically induced metric $g^{\mathcal{F}_1^\perp}$. We denote from now on that $\mathcal{F}_2^\perp = T^V \mathcal{M}$.

Let $g^{T\mathcal{M}}$ be the Riemannian metric on $\mathcal{M}$ defined by the following orthogonal splitting,

$$(2.1) \qquad T\mathcal{M} = \mathcal{F} \oplus \mathcal{F}_1^\perp \oplus \mathcal{F}_2^\perp, \qquad g^{T\mathcal{M}} = g^{\mathcal{F}} \oplus g^{\mathcal{F}_1^\perp} \oplus g^{\mathcal{F}_2^\perp}.$$

Let $p_2^\perp$ be the orthogonal projection from $T\mathcal{M}$ to $\mathcal{F}_2^\perp$. Let $\nabla^{T\mathcal{M}}$ be the Levi-Civita connection of $g^{T\mathcal{M}}$. Then $\nabla^{\mathcal{F}_2^\perp} = p_2^\perp \nabla^{T\mathcal{M}} p_2^\perp$ is a Euclidean connection on $\mathcal{F}_2^\perp$ not depending on $g^{\mathcal{F}}$ and $g^{\mathcal{F}_1^\perp}$.

By [10, Lemma 5.2], $(\mathcal{M}, \mathcal{F})$ admits an almost isometric structure with respect to the metrics given by (2.1). In particular, for any $X \in \Gamma(\mathcal{F})$, U_i, $V_i \in \Gamma(\mathcal{F}_i^\perp)$ with $i = 1, 2$, one has by (1.7) that

$$(2.2) \qquad \begin{aligned} \langle [X, U_i], V_i \rangle + \langle U_i, [X, V_i] \rangle &= X \langle U_i, V_i \rangle, \\ \langle [X, U_2], U_1 \rangle &= 0. \end{aligned}$$

Take a metric on TM/F. This is equivalent to taking an embedded section $s : M \hookrightarrow \mathcal{M}$ of the Connes fibration $\pi : \mathcal{M} \to M$. Then we have a canonical inclusion $s(M) \subset \mathcal{M}$.

For any $p \in \mathcal{M} \setminus s(M)$, we connect p and $s(\pi(p)) \in s(M)$ by the unique geodesic in $\mathcal{M}_{\pi(p)}$. Let $\sigma(p) \in \mathcal{F}_2^\perp|_p$ denote the unit vector tangent to this geodesic. Let $\rho(p) = d^{\mathcal{M}_{\pi(p)}}(p, s(\pi(p)))$ denote the length of this geodesic.

The following simple result will play a key role in what follows.

---

[4] Indeed, the Bott connection on TM/F determines an integrable lift $\widetilde{\mathcal{F}}$ of F in $T\widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}} = GL(TM/F)^+$ is the $GL(q_1, \mathbf{R})^+$ (with $q_1 = \mathrm{rk}(TM/F)$) principal bundle of oriented frames over M. Now as $\widetilde{\mathcal{M}}$ is a principal $SO(q_1)$ bundle over $\mathcal{M}$, $\widetilde{\mathcal{F}}$ determines an integrable subbundle $\mathcal{F}$ of $T\mathcal{M}$.

Lemma 2.1. *There exists $A_1 > 0$, depending only on the embedding $s : M \hookrightarrow \mathcal{M}$, such that for any $X \in \Gamma(\mathcal{F})$ with $|X| \leq 1$, the following pointwise inequalities hold on $\mathcal{M} \setminus s(M)$,*

$$(2.3) \qquad\qquad\qquad\qquad |X(\rho)| \leq A_1,$$

$$(2.4) \qquad\qquad\qquad\qquad \left|\nabla_X^{\mathcal{F}_2^\perp} \sigma\right| \leq \frac{A_1}{\rho}.$$

In particular, the following inequality holds on $\mathcal{M}$,

$$(2.5) \qquad\qquad\qquad\qquad \left|\nabla_X^{\mathcal{F}_2^\perp} (\rho\sigma)\right| \leq 2A_1.$$

Proof. Since the estimates to be proved are local, we may well assume that there is $Y \in \Gamma(F)$ over M, with $|Y| \leq 1$, such that $X = \pi^*Y$. Let ϕ_t (resp. $\widetilde{\phi}_t$), $t \in \mathbf{R}$, be the one-parameter group of diffeomorphisms on M (resp. $\mathcal{M}$) generated by Y (resp. $X = \pi^*Y$). Then $\widetilde{\phi}_t$ is the lift of ϕ_t.

Take any $p \in \mathcal{M} \setminus s(M)$. By [10, Lemma 5.2] and (2.2), one sees that each $\widetilde{\phi}_t$ maps the fiber $\mathcal{M}_{\pi(p)}$ isometrically to the fiber $\mathcal{M}_{\phi_t(\pi(p))}$. Thus, it maps the geodesic connecting p and $s(\pi(p))$ in $\mathcal{M}_{\pi(p)}$ to the geodesic connecting $\widetilde{\phi}_t(p)$ and $\widetilde{\phi}_t(s(\pi(p)))$ in $\mathcal{M}_{\phi_t(\pi(p))}$, such that $\rho(p) = d^{\mathcal{M}_{\phi_t(\pi(p))}}(\widetilde{\phi}_t(p), \widetilde{\phi}_t(s(\pi(p))))$. Thus, one has

$$(2.6)$$
$$\left|\rho\left(\widetilde{\phi}_t(p)\right) - \rho(p)\right| = \left|d^{\mathcal{M}_{\phi_t(\pi(p))}}\left(\widetilde{\phi}_t(p), s(\phi_t(\pi(p)))\right) - d^{\mathcal{M}_{\phi_t(\pi(p))}}\left(\widetilde{\phi}_t(p), \widetilde{\phi}_t(s(\pi(p)))\right)\right|$$
$$\leq d^{\mathcal{M}_{\phi_t(\pi(p))}}\left(s(\phi_t(\pi(p))), \widetilde{\phi}_t(s(\pi(p)))\right) = \rho\left(\widetilde{\phi}_t(s(\pi(p)))\right).$$

Since at p one has $X(\rho) = \lim_{t \to 0^+} \frac{\rho(\widetilde{\phi}_t(p)) - \rho(p)}{t}$, (2.3) follows from (2.6) and the following lemma.

Lemma 2.2. *There exist $c_0, A_0 > 0$, depending only on the embedding $s : M \hookrightarrow \mathcal{M}$, such that for any $x \in s(M)$ and $0 \leq t \leq c_0$, one has*

$$(2.7) \qquad\qquad\qquad\qquad \rho\left(\widetilde{\phi}_t(x)\right) \leq A_0 t.$$

Proof. Take any $x \in s(M)$. If $t = 0$, then (2.7) clearly holds. Recall that $\widetilde{\phi}_t$ maps $\mathcal{M}_{\pi(p)}$ isometrically to $\mathcal{M}_{\phi_t(\pi(p))}$. Thus one has

$$(2.8) \qquad\qquad\qquad \rho\left(\widetilde{\phi}_t(x)\right) = \rho\left(\widetilde{\phi}_t^{-1}(s(\phi_t(\pi(x))))\right).$$

Since $\widetilde{\phi}_t^{-1}(s(\phi_t(\pi(x))))$ depends smoothly on t, one sees from (2.8) that (2.7) holds at $x \in s(M)$. By the compactness of $s(M)$, it holds for all $x \in s(M)$. $\qquad\square$

To prove (2.4), we first observe that by (2.2) one has that for any $U \in \Gamma(\mathcal{F}_2^\perp)$, the following identity holds (cf. (1.25)),

$$(2.9) \qquad\qquad\qquad\qquad p_2^\perp \nabla_U^{TM} X = 0.$$

From (2.9) and the fact that $[X, \sigma] = [\pi^* Y, \sigma] \in \Gamma(\mathcal{F}_2^\perp)$ (cf. [4, Lemma 10.7]), one sees that in order to prove (2.4), one need only to prove that

$$(2.10) \qquad |[X, \sigma]| \leq \frac{A_1}{\rho}.$$

To prove (2.10), recall that (cf. [9, Theorem 2.3 of Chapter 6])

$$(2.11) \qquad [X, \sigma] = \lim_{t \to 0^+} \frac{\sigma - \left(\widetilde{\phi}_t\right)_* \sigma}{t}.$$

Since $\widetilde{\phi}_t$ maps geodesics in $\mathcal{M}_{\phi_{-t}(\pi(p))}$ to geodesics in $\mathcal{M}_{\pi(p)}$, one sees as in [10, §5] that at $p \in \mathcal{M} \setminus s(M)$, $(\widetilde{\phi}_t)_* \sigma$ is the unit vector tangent to the geodesic connecting p and $\widetilde{\phi}_t(s(\phi_{-t}(\pi(p))))$.

Consider the geodesic triangle in $\mathcal{M}_{\pi(p)}$ with vertices p, $s(\pi(p))$ and $\widetilde{\phi}_t(s(\phi_{-t}(\pi(p))))$. Let α_p be the angle at p. Then one has

$$(2.12) \qquad \left| \sigma - \left(\widetilde{\phi}_t\right)_* \sigma \right|^2 = 2 \left(1 - \cos\left(\alpha_p\right)\right).$$

Since $\mathcal{M}_{\pi(p)}$ is of nonpositive curvature, one has (cf. [15, Corollary I.13.2]),

$$(2.13) \quad \left(\rho\left(\widetilde{\phi}_t\left(s\left(\phi_{-t}(\pi(p))\right)\right)\right)\right)^2 \geq 2\left(1 - \cos\left(\alpha_p\right)\right) \rho(p)\, d^{\mathcal{M}_{\pi(p)}}\left(p, \widetilde{\phi}_t\left(s\left(\phi_{-t}(\pi(p))\right)\right)\right).$$

From (2.12) and (2.13), one gets

$$(2.14) \qquad \left| \sigma - \left(\widetilde{\phi}_t\right)_* \sigma \right| \leq \frac{\rho\left(\widetilde{\phi}_t\left(s\left(\phi_{-t}(\pi(p))\right)\right)\right)}{\sqrt{\rho(p)\, d^{\mathcal{M}_{\pi(p)}}\left(p, \widetilde{\phi}_t\left(s\left(\phi_{-t}(\pi(p))\right)\right)\right)}}.$$

From (2.11), (2.14) and proceed as in Lemma 2.2, one gets (2.10). $\qquad \square$

2.2. Sub-Dirac operators and the vanishing on compact subsets.

From now on we assume that there is $\delta > 0$ such that $k^F \geq \delta$ over M. We also assume that M is spin and carries a fixed spin structure, then $\mathcal{F} \oplus \mathcal{F}_1^\perp = \pi^*(TM)$ is spin and carries an induced spin structure. For simplicity, we also assume first that $\mathcal{F}_2^\perp$ is oriented and both TM and $\mathcal{F}_2^\perp$ are of even rank.

For any β, $\varepsilon > 0$, following (1.11), let $g_{\beta,\varepsilon}^{T\mathcal{M}}$ be the deformed metric of (2.1) on $\mathcal{M}$ defined by the orthogonal splitting,

$$(2.15) \qquad T\mathcal{M} = \mathcal{F} \oplus \mathcal{F}_1^\perp \oplus \mathcal{F}_2^\perp, \qquad g_{\beta,\varepsilon}^{T\mathcal{M}} = \beta^2 g^{\mathcal{F}} \oplus \frac{g^{\mathcal{F}_1^\perp}}{\varepsilon^2} \oplus g^{\mathcal{F}_2^\perp}.$$

In what follows, we will use the subscripts (or superscripts) β, ε to decorate the geometric objects with respect to the deformed metric $g_{\beta,\varepsilon}^{T\mathcal{M}}$. It is clear that for any $X \in \mathcal{F} \oplus \mathcal{F}_1^\perp$ and $U \in \mathcal{F}_2^\perp$, $c_{\beta,\varepsilon}(X)$, $c(U)$ and $\widehat{c}(U)$ act on $S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp) \widehat{\otimes} \Lambda^*(\mathcal{F}_2^\perp)$ and exchange $(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp) \widehat{\otimes} \Lambda^*(\mathcal{F}_2^\perp))_\pm$.

Let $f_1, \cdots, f_q$ (resp. $h_1, \cdots, h_{q_1}$; resp. $e_1, \cdots, e_{q_2}$) be an orthonormal basis of $(\mathcal{F}, g^{\mathcal{F}})$ (resp. $(\mathcal{F}_1^\perp, g^{\mathcal{F}_1^\perp})$; resp. $(\mathcal{F}_2^\perp, g^{\mathcal{F}_2^\perp})$). By proceeding as in [22, Section 2] and

Sections 1.4, 1.5, we construct the sub-Dirac operator (cf. (1.61) and (1.69), where we take F in (1.61) to be $\mathcal{F} \oplus \mathcal{F}_1^\perp$, $F_1^\perp$ in (1.61) to be zero and $F_2^\perp$ in (1.61) to be $\mathcal{F}_2^\perp$)

$$(2.16) \qquad D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} : \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*\left(\mathcal{F}_2^\perp\right)\right) \longrightarrow \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*\left(\mathcal{F}_2^\perp\right)\right)$$

given by

$$(2.17) \qquad D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} = \sum_{i=1}^{q} \beta^{-1} c_{\beta,\varepsilon}\left(\beta^{-1} f_i\right) \nabla_{f_i}^{\beta,\varepsilon} + \sum_{s=1}^{q_1} \varepsilon\, c_{\beta,\varepsilon}\left(\varepsilon h_s\right) \nabla_{h_s}^{\beta,\varepsilon} + \sum_{j=1}^{q_2} c\left(e_j\right) \nabla_{e_j}^{\beta,\varepsilon},$$

where as in (1.69), $\nabla^{\beta,\varepsilon}$ is the canonical connection on $S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*(\mathcal{F}_2^\perp)$ determined by (1.60) with respect to $g_{\beta,\varepsilon}^{T\mathcal{M}}$. In particular, in view of Remark 1.8, one has

$$(2.18) \qquad \left[\nabla^{\beta,\varepsilon}, \widehat{c}(\sigma)\right] = \widehat{c}\left(\nabla^{\mathcal{F}_2^\perp} \sigma\right).$$

Let $D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon, \pm}$ be the restrictions of $D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}$ on $(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*(\mathcal{F}_2^\perp))_\pm$, then

$$(2.19) \qquad \left(D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon, +}\right)^* = D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon, -}.$$

For any $R > 0$, denote

$$(2.20) \qquad \mathcal{M}_R = \{p \in \mathcal{M} : \ \rho(p) \leq R\}.$$

Then $\mathcal{M}_R$ is a smooth manifold with boundary $\partial\mathcal{M}_R$.

Let $f : [0,1] \to [0,1]$ be a smooth function such that $f(t) = 0$ for $0 \leq t \leq \frac{1}{4}$, while $f(t) = 1$ for $\frac{1}{2} \leq t \leq 1$. Let $h : [0,1] \to [0,1]$ be a smooth function such that $h(t) = 1$ for $0 \leq t \leq \frac{3}{4}$, while $h(t) = 0$ for $\frac{7}{8} \leq t \leq 1$.

Inspired by [5] and [10], we make the following deformation of $D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}$ on $\mathcal{M}_R$, which will play a key role in what follows,

$$(2.21) \qquad D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} + \frac{f\left(\frac{\rho}{R}\right)\widehat{c}(\sigma)}{\beta}.$$

Remark 2.3. The usual deformation from the analytic localization point of view (such as in [5]) deforms $D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}$ by $T\widehat{c}(\rho\sigma)$, with $T > 0$ being independent of β and ε. On the other hand, $fc(\sigma)$ has occured in [10], where it is viewed as the symbol of a fiberwise Dirac operator. Here we use $f\widehat{c}(\sigma)/\beta$ to deform $D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}$, while Lemma 2.1 allows us to get the needed estimates given in the following Lemma.

Lemma 2.4. *There exists $R_0 > 0$ such that for any (fixed) $R \geq R_0$, when β, $\varepsilon > 0$ (which may depend on R) are small enough,*
(i) for any $s \in \Gamma(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*(\mathcal{F}_2^\perp))$ supported in $\mathcal{M}_R$, one has[5]

$$(2.22) \qquad \left\|\left(D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} + \frac{f\left(\frac{\rho}{R}\right)\widehat{c}(\sigma)}{\beta}\right) s\right\| \geq \frac{\sqrt{\delta}}{4\beta}\|s\|;$$

(ii) for any $s \in \Gamma(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*(\mathcal{F}_2^\perp))$ supported in $\mathcal{M}_R \setminus \mathcal{M}_{\frac{R}{2}}$, one has

$$(2.23) \qquad \left\|\left(h\left(\frac{\rho}{R}\right) D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} h\left(\frac{\rho}{R}\right) + \frac{\widehat{c}(\sigma)}{\beta}\right) s\right\| \geq \frac{1}{2\beta}\|s\|.$$

---

[5]The norms below denpend on β and ε. In case of no confusion, we omit the subscripts for simplicity.

Proof. In view of Remark 1.8 and (2.17), one has

$$(2.24) \quad \left(D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} + \frac{f\left(\frac{\rho}{R}\right) \widehat{c}(\sigma)}{\beta} \right)^2 = D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}^2 + \frac{f'\left(\frac{\rho}{R}\right)}{\beta R} c_{\beta, \varepsilon}(d\rho) \widehat{c}(\sigma)$$

$$+ \frac{f\left(\frac{\rho}{R}\right)}{\beta} \left[D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}, \widehat{c}(\sigma) \right] + \frac{f\left(\frac{\rho}{R}\right)^2}{\beta^2},$$

where we identify $d\rho$ with the gradient of ρ.

By definition, one has on $\mathcal{M} \setminus s(M)$ that

$$(2.25) \quad c_{\beta, \varepsilon}(d\rho) = \sum_{i=1}^{q} \beta^{-1} c_{\beta, \varepsilon} \left(\beta^{-1} f_i \right) f_i(\rho) + \sum_{s=1}^{q_1} \varepsilon \, c_{\beta, \varepsilon} \left(\varepsilon h_s \right) h_s(\rho) + \sum_{j=1}^{q_2} c\left(e_j \right) e_j(\rho).$$

By (2.17) and (2.18), one has on $\mathcal{M} \setminus s(M)$ that

$$(2.26) \quad \left[D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}, \widehat{c}(\sigma) \right] = \sum_{i=1}^{q} \beta^{-1} c_{\beta, \varepsilon} \left(\beta^{-1} f_i \right) \widehat{c} \left(\nabla_{f_i}^{\mathcal{F}_2^\perp} \sigma \right) + \sum_{s=1}^{q_1} \varepsilon \, c_{\beta, \varepsilon} \left(\varepsilon h_s \right) \widehat{c} \left(\nabla_{h_s}^{\mathcal{F}_2^\perp} \sigma \right)$$

$$+ \sum_{j=1}^{q_2} c\left(e_j \right) \widehat{c} \left(\nabla_{e_j}^{\mathcal{F}_2^\perp} \sigma \right).$$

By Lemma 2.1, (2.25) and (2.26), we find that there exists a constant $C > 0$, not depending on R, β, $\varepsilon > 0$, such that the following inequality holds on $\mathcal{M}_R \setminus s(M)$,

$$(2.27) \quad \frac{|c_{\beta, \varepsilon}(d\rho)|}{R} + f\left(\frac{\rho}{R}\right) \left| \left[D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}, \widehat{c}(\sigma) \right] \right| \leq \frac{C}{\beta R} + O_R(1),$$

where by $O_R(\cdot)$ we mean that the estimating constant might depend on $R > 0$.

On the other hand, by (1.71), the following formula holds on $\mathcal{M}_R$,

$$(2.28) \quad D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}^2 = -\Delta^{\beta, \varepsilon} + \frac{k^{\mathcal{F}}}{4\beta^2} + O_R\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2} \right),$$

where $-\Delta^{\beta, \varepsilon} \geq 0$ is the corresponding Bochner Laplacian, and $k^{\mathcal{F}} = \pi^* k^F \geq \delta$.

From (2.24), (2.27) and (2.28), one sees that if one first fixes a sufficiently large $R > 0$ and then makes $\beta > 0$, $\varepsilon > 0$ sufficiently small, one deduces (2.22) easily.

Now by (2.17) one has on $\mathcal{M}_R \setminus s(M)$ that

$$(2.29) \quad \left(h\left(\frac{\rho}{R}\right) D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} h\left(\frac{\rho}{R}\right) + \frac{\widehat{c}(\sigma)}{\beta} \right)^2 = \left(h\left(\frac{\rho}{R}\right) D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} h\left(\frac{\rho}{R}\right) \right)^2$$

$$+ \frac{h\left(\frac{\rho}{R}\right)^2}{\beta} \left[D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon}, \widehat{c}(\sigma) \right] + \frac{1}{\beta^2}.$$

From (2.27) and (2.29), one gets (2.23), where $\mathrm{Supp}(s) \subseteq \mathcal{M}_R \setminus \mathcal{M}_{\frac{R}{2}}$, similarly. $\qquad \square$

Lemma 2.4 motivates the proof of Theorem 0.1 (for the case of $\dim M = 4k$) given in Section 2.3, where we make use of a trick of Braverman [8, §14] (See also [24, §3]). This approach reflects the topological nature of the $\widehat{A}$-genus and the involved indices.

2.3. **Proof of Theorem 0.1 for the case of** $\dim M = 4k$. Let $\partial \mathcal{M}_R$ bound another
oriented manifold $\mathcal{N}_R$ so that $\widetilde{\mathcal{N}}_R = \mathcal{M}_R \cup \mathcal{N}_R$ is a closed manifold (for example, one
can take the double of $\mathcal{M}_R$). Let E be a Hermitian vector bundle over $\mathcal{M}_R$ such that
$(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^{\perp})\widehat{\otimes}\Lambda^*(\mathcal{F}_2^{\perp}))_- \oplus E$ is a trivial vector bundle over $\mathcal{M}_R$. Then $(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^{\perp})\widehat{\otimes}\Lambda^*(\mathcal{F}_2^{\perp}))_+ \oplus E$ is a trivial vector bundle near $\partial \mathcal{M}_R$, under the identification $\widehat{c}(\sigma) + \mathrm{Id}_E$.

By extending obviously the above trivial vector bundles to $\mathcal{N}_R$, we get a $\mathbf{Z}_2$-graded
Hermitian vector bundle $\xi = \xi_+ \oplus \xi_-$ over $\widetilde{\mathcal{N}}_R$ and an odd self-adjoint endomorphism
$V = v + v^* \in \Gamma(\mathrm{End}(\xi))$ (with $v : \Gamma(\xi_+) \to \Gamma(\xi_-)$, v^* being the adjoint of v) such that

$$(2.30) \qquad \xi_{\pm} = \left(S_{\beta,\varepsilon}\left(\mathcal{F} \oplus \mathcal{F}_1^{\perp}\right)\widehat{\otimes}\Lambda^*\left(\mathcal{F}_2^{\perp}\right)\right)_{\pm} \oplus E$$

over $\mathcal{M}_R$, V is invertible on $\mathcal{N}_R$ and

$$(2.31) \qquad V = f\left(\frac{\rho}{R}\right)\widehat{c}(\sigma) + \mathrm{Id}_E$$

on $\mathcal{M}_R$, which is invertible on $\mathcal{M}_R \setminus \mathcal{M}_{\frac{R}{2}}$.

Recall that $h(\frac{\rho}{R})$ vanishes near $\partial \mathcal{M}_R$. We extend it to a function on $\widetilde{\mathcal{N}}_R$ which equals
to zero on $\mathcal{N}_R$, and denote the resulting function on $\widetilde{\mathcal{N}}_R$ by $\widetilde{h}_R$. Let $\pi_{\widetilde{\mathcal{N}}_R} : T\widetilde{\mathcal{N}}_R \to \widetilde{\mathcal{N}}_R$
be the projection of the tangent bundle of $\widetilde{\mathcal{N}}_R$. Let $\gamma^{\widetilde{\mathcal{N}}_R} \in \mathrm{Hom}(\pi_{\widetilde{\mathcal{N}}_R}^* \xi_+, \pi_{\widetilde{\mathcal{N}}_R}^* \xi_-)$ be the
symbol defined by

$$(2.32) \qquad \gamma^{\widetilde{\mathcal{N}}_R}(p, w) = \pi_{\widetilde{\mathcal{N}}_R}^* \left(\sqrt{-1}\,\widetilde{h}_R^2\, c_{\beta,\varepsilon}(w) + v(p)\right), \quad \text{for } p \in \widetilde{\mathcal{N}}_R, \quad w \in T_p\widetilde{\mathcal{N}}_R.$$

By (2.31) and (2.32), $\gamma^{\widetilde{\mathcal{N}}_R}$ is singular only if $w = 0$ and $p \in \mathcal{M}_{\frac{R}{2}}$. Thus $\gamma^{\widetilde{\mathcal{N}}_R}$ is an elliptic
symbol.

On the other hand, it is clear that $\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^{\perp}, \beta, \varepsilon} \widetilde{h}_R$ is well-defined on $\widetilde{\mathcal{N}}_R$ if we define
it to equal to zero on $\widetilde{\mathcal{N}}_R \setminus \mathcal{M}_R$.

Let $A : L^2(\xi) \to L^2(\xi)$ be a second order positive elliptic differential operator on
$\widetilde{\mathcal{N}}_R$ preserving the $\mathbf{Z}_2$-grading of $\xi = \xi_+ \oplus \xi_-$, such that its symbol equals to $|\eta|^2$ at
$\eta \in T\widetilde{\mathcal{N}}_R$ (to be more precise, here A also depends on the defining metric. We omit the
corresponding subscript/superscript only for convenience). Let $P_{R,\beta,\varepsilon} : L^2(\xi) \to L^2(\xi)$
be the zeroth order pseudodifferential operator on $\widetilde{\mathcal{N}}_R$ defined by

$$(2.33) \qquad P_{R,\beta,\varepsilon} = A^{-\frac{1}{4}}\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^{\perp}, \beta, \varepsilon} \widetilde{h}_R A^{-\frac{1}{4}} + \frac{V}{\beta}.$$

Let $P_{R,\beta,\varepsilon,+} : L^2(\xi_+) \to L^2(\xi_-)$ be the obvious restriction. Then the principal symbol of
$P_{R,\beta,\varepsilon,+}$, which we denote by $\gamma(P_{R,\beta,\varepsilon,+})$, is homotopic through elliptic symbols to $\gamma^{\widetilde{\mathcal{N}}_R}$.
Thus $P_{R,\beta,\varepsilon,+}$ is a Fredholm operator. Moreover, by the Atiyah-Singer index theorem [2]
(cf. [18, Theorem 13.8 of Chap. III]), one finds

$$(2.34) \qquad \mathrm{ind}\,(P_{R,\beta,\varepsilon,+}) = \widehat{A}(M).$$

Inspired by [8, §14] (See also [24, §3]), for any $0 \le t \le 1$, set

$$(2.35) \qquad P_{R,\beta,\varepsilon,+}(t) = A^{-\frac{1}{4}}\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^{\perp}, \beta, \varepsilon} \widetilde{h}_R A^{-\frac{1}{4}} + \frac{tv}{\beta} + A^{-\frac{1}{4}}\frac{(1-t)v}{\beta}A^{-\frac{1}{4}}.$$

Then $P_{R,\beta,\varepsilon,+}(t)$ is a smooth family of zeroth order pseudodifferential operators such that the corresponding symbol $\gamma(P_{R,\beta,\varepsilon,+}(t))$ is elliptic for $0 < t \leq 1$. Thus $P_{R,\beta,\varepsilon,+}(t)$ is a continuous family of Fredholm operators for $0 < t \leq 1$ with $P_{R,\beta,\varepsilon,+}(1) = P_{R,\beta,\varepsilon,+}$.

Now since $P_{R,\beta,\varepsilon,+}(t)$ is continuous on the whole $[0, 1]$, in view of (2.34), if $P_{R,\beta,\varepsilon,+}(0)$ is Fredholm and has vanishing index, then Theorem 0.1 follows from (2.34).

Thus we need only to prove the following result.

Proposition 2.5. *There exist* R, β, $\varepsilon > 0$ *such that the following identity holds,*

$$(2.36) \qquad \dim\left(\ker\left(P_{R,\beta,\varepsilon,+}(0)\right)\right) = \dim\left(\ker\left(P_{R,\beta,\varepsilon,+}(0)^*\right)\right) = 0.$$

Proof. By definition, $P_{R,\beta,\varepsilon}(0) : L^2(\xi) \to L^2(\xi)$ is given by

$$(2.37) \qquad P_{R,\beta,\varepsilon}(0) = A^{-\frac{1}{4}}\widetilde{h}_R D_{\mathcal{F}\oplus\mathcal{F}_1^\perp,\beta,\varepsilon}\widetilde{h}_R A^{-\frac{1}{4}} + A^{-\frac{1}{4}}\frac{V}{\beta}A^{-\frac{1}{4}}.$$

By (2.19), $P_{R,\beta,\varepsilon}(0)$ is formally self-adjoint. Thus we need to show that

$$(2.38) \qquad \dim\left(\ker\left(P_{R,\beta,\varepsilon}(0)\right)\right) = 0$$

for certain R, β, $\varepsilon > 0$. Let $s \in \ker(P_{R,\beta,\varepsilon}(0))$. By (2.37) one has

$$(2.39) \qquad \left(\widetilde{h}_R D_{\mathcal{F}\oplus\mathcal{F}_1^\perp,\beta,\varepsilon}\widetilde{h}_R + \frac{V}{\beta}\right)A^{-\frac{1}{4}}s = 0.$$

Since $\widetilde{h}_R = 0$ on $\widetilde{\mathcal{N}}_R \setminus \mathcal{M}_R$, while V is invertible on $\widetilde{\mathcal{N}}_R \setminus \mathcal{M}_R$, one has by (2.39)

$$(2.40) \qquad A^{-\frac{1}{4}}s = 0 \qquad \text{on} \qquad \widetilde{\mathcal{N}}_R \setminus \mathcal{M}_R.$$

Write on $\mathcal{M}_R$ that

$$(2.41) \qquad A^{-\frac{1}{4}}s = s_1 + s_2,$$

with $s_1 \in L^2(S_{\beta,\varepsilon}(\mathcal{F} \oplus \mathcal{F}_1^\perp)\widehat{\otimes}\Lambda^*(\mathcal{F}_2^\perp))$ and $s_2 \in L^2(E)$.

By (2.31), (2.39) and (2.41), one has

$$(2.42) \qquad s_2 = 0,$$

while

$$(2.43) \qquad \left(\widetilde{h}_R D_{\mathcal{F}\oplus\mathcal{F}_1^\perp,\beta,\varepsilon}\widetilde{h}_R + \frac{f\left(\frac{\rho}{R}\right)\widehat{c}(\sigma)}{\beta}\right)s_1 = 0.$$

We need to show that (2.43) implies $s_1 = 0$.

Let $\alpha : [0, 1] \to [0, 1]$ be a smooth function such that $\alpha(t) = 0$ for $0 \leq t \leq \frac{1}{2}$, while $\alpha(t) = 1$ for $\frac{2}{3} \leq t \leq 1$.

Following [5, pp. 115], let α_1, α_2 be the smooth functions on $\mathcal{M}_R$ defined by

$$(2.44) \qquad \alpha_1 = \frac{1 - \alpha\left(\frac{\rho}{R}\right)}{\left(\alpha\left(\frac{\rho}{R}\right)^2 + \left(1 - \alpha\left(\frac{\rho}{R}\right)\right)^2\right)^{\frac{1}{2}}}, \qquad \alpha_2 = \frac{\alpha\left(\frac{\rho}{R}\right)}{\left(\alpha\left(\frac{\rho}{R}\right)^2 + \left(1 - \alpha\left(\frac{\rho}{R}\right)\right)^2\right)^{\frac{1}{2}}}.$$

Then $\alpha_1^2 + \alpha_2^2 = 1$ on $\mathcal{M}_R$. Clearly, $\alpha_1 \widetilde{h}_R = \alpha_1$, $\alpha_2 f(\frac{\rho}{R}) = \alpha_2$. Thus, one has

$$(2.45) \quad \left\| \left(\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} \widetilde{h}_R + \frac{f\left(\frac{\rho}{R}\right) \widehat{c}(\sigma)}{\beta} \right) s_1 \right\|^2 = \left\| \alpha_1 \left(D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} + \frac{f\left(\frac{\rho}{R}\right) \widehat{c}(\sigma)}{\beta} \right) s_1 \right\|^2$$

$$+ \left\| \alpha_2 \left(\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} \widetilde{h}_R + \frac{\widehat{c}(\sigma)}{\beta} \right) s_1 \right\|^2,$$

from which one gets

$$(2.46) \quad \sqrt{2} \left\| \left(\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} \widetilde{h}_R + \frac{f\left(\frac{\rho}{R}\right) \widehat{c}(\sigma)}{\beta} \right) s_1 \right\| \geq \left\| \alpha_1 \left(D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} + \frac{f\left(\frac{\rho}{R}\right) \widehat{c}(\sigma)}{\beta} \right) s_1 \right\|$$

$$+ \left\| \alpha_2 \left(\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} \widetilde{h}_R + \frac{\widehat{c}(\sigma)}{\beta} \right) s_1 \right\| \geq \left\| \left(D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} + \frac{f\left(\frac{\rho}{R}\right) \widehat{c}(\sigma)}{\beta} \right) (\alpha_1 s_1) \right\|$$

$$+ \left\| \left(\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} \widetilde{h}_R + \frac{\widehat{c}(\sigma)}{\beta} \right) (\alpha_2 s_1) \right\| - \left\| c_{\beta, \varepsilon} (d\alpha_1) s_1 \right\| - \left\| c_{\beta, \varepsilon} (d\alpha_2) s_1 \right\|,$$

where for each $i \in \{1, 2\}$, we identify $d\alpha_i$ with the gradient of α_i.

By Lemma 2.1, (2.25) and (2.44), there is $C_1 > 0$, not depending on R, β, $\varepsilon > 0$, such that

$$(2.47) \qquad |c_{\beta, \varepsilon} (d\alpha_1)| + |c_{\beta, \varepsilon} (d\alpha_2)| \leq \frac{C_1}{\beta R} + O_R(1).$$

From Lemma 2.4, (2.46) and (2.47), one finds that there exist R, β, $\varepsilon > 0$ such that

$$(2.48) \qquad \left\| \left(\widetilde{h}_R D_{\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta, \varepsilon} \widetilde{h}_R + \frac{f\left(\frac{\rho}{R}\right) \widehat{c}(\sigma)}{\beta} \right) s_1 \right\| \geq \frac{\|s_1\|}{\sqrt{\beta}}.$$

From (2.39)-(2.43), (2.48) and the invertibility of $A^{-\frac{1}{4}}$, one sees that for suitable R, β, $\varepsilon > 0$, (2.38) holds. This completes the proof of Proposition 2.5, which implies Theorem 0.1 for the case of $\dim M = 4k$, when $\mathcal{F}_2^\perp$ is orientable and of even rank. $\qquad \square$

If $\mathrm{rk}(\mathcal{F}_2^\perp)$ is not even, we can consider $M \times M \times M \times M$ to make it even. If $\mathcal{F}_2^\perp$ is not orientable, then we can consider the double covering of M with respect to $w_1(\mathcal{F}_2^\perp)$, the first Stiefel-Whitney class of $\mathcal{F}_2^\perp$, and consider the pull-back of $\mathcal{F}_2^\perp$ on the double covering. The proof of Theorem 0.1 for the case of $\dim M = 4k$ is thus completed.

Remark 2.6. One may also use $\frac{\rho}{R}$ instead of $f(\frac{\rho}{R})$ in the above proof.

2.4. **The case of the mod 2 index.** In this subsection, we consider the cases of $\dim M = 8k + i$, $i = 1, 2$. Here we deal with the case of $\dim M = 8k + 1$, where one considers real operators as in [3], in detail. By multiplying M by a Bott manifold of dimension 8, which is a compact spin manifold B^8 such that $\widehat{A}(B^8) = 1$, we may well assume that $q_1 > 1$. Then $\partial \mathcal{M}_R$ is connected.

Let $f_1, \cdots, f_{q+q_1}$ be an oriented orthonormal basis of $(\mathcal{F} \oplus \mathcal{F}_1^\perp, \beta^2 g^\mathcal{F} \oplus \frac{g^{\mathcal{F}_1^\perp}}{\varepsilon^2})$. Set

$$(2.49) \qquad \tau_{\beta, \varepsilon} = c_{\beta, \varepsilon} (f_1) \cdots c_{\beta, \varepsilon} (f_{q+q_1}).$$

Let $\widehat{\tau}$ be the $\mathbf{Z}_2$-grading operator for $\Lambda^*(\mathcal{F}_2^\perp) = \Lambda^{\mathrm{even}}(\mathcal{F}_2^\perp) \oplus \Lambda^{\mathrm{odd}}(\mathcal{F}_2^\perp)$.

Inspired by [3, §3] and [6, (3.1)] (compare with [29] which deals with the case of $\dim M = 8k + 2$), we modify the sub-Dirac operator in (2.16) by

(2.50)
$$\widehat{\tau}\,\tau_{\beta,\varepsilon} D_{\mathcal{F}\oplus\mathcal{F}_1^{\perp},\beta,\varepsilon} : \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^{\perp})\otimes\Lambda^*\left(\mathcal{F}_2^{\perp}\right)\right) \longrightarrow \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^{\perp})\otimes\Lambda^*\left(\mathcal{F}_2^{\perp}\right)\right),$$

which is formally skew-adjoint (here by dimension reason there is no $\mathbf{Z}_2$-grading of the real spinor bundle $S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^{\perp})$). We also modify $V = v + v^*$ in (2.31) by

(2.51)
$$\widehat{V} = \widehat{v} - \widehat{v}^*$$

such that one has, on $\mathcal{M}_R$, the following formula for $\widehat{v}$ acting between real vector bundles,

(2.52) $\quad \widehat{v} = f\left(\dfrac{\rho}{R}\right)\widehat{\tau c}(\sigma) + \mathrm{Id}_E : \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^{\perp})\otimes\Lambda^{\mathrm{even}}\left(\mathcal{F}_2^{\perp}\right)\oplus E\right)$

$$\longrightarrow \Gamma\left(S_{\beta,\varepsilon}(\mathcal{F}\oplus\mathcal{F}_1^{\perp})\otimes\Lambda^{\mathrm{odd}}\left(\mathcal{F}_2^{\perp}\right)\oplus E\right).$$

We then modify the operator $P_{R,\beta,\varepsilon}$ in (2.33) by

(2.53)
$$\widehat{P}_{R,\beta,\varepsilon} = A^{-\frac{1}{4}}\widetilde{h}_R\tau_{\beta,\varepsilon}\widehat{\tau}\, D_{\mathcal{F}\oplus\mathcal{F}_1^{\perp},\beta,\varepsilon}\widetilde{h}_R A^{-\frac{1}{4}} + \frac{\widehat{V}}{\beta},$$

which is clearly formally skew-adjoint. By direct computation, one has

(2.54)
$$(\widehat{\tau c}(\sigma))^* = \widehat{c}(\sigma)\widehat{\tau} = -\widehat{\tau c}(\sigma)$$

and that for any $X \in T\mathcal{M}$,

(2.55) $\qquad \widehat{\tau}\,\tau c(X)\widehat{\tau c}(\sigma) + \widehat{\tau c}(\sigma)\widehat{\tau}\,\tau c(X) = \tau c(X)\widehat{c}(\sigma) - \widehat{c}(\sigma)\tau c(X) = 0.$

From (2.53)-(2.55), one sees that $(\widehat{P}_{R,\beta,\varepsilon})^2$ has an elliptic symbol. Thus $\widehat{P}_{R,\beta,\varepsilon}$ is a zeroth order real skew-adjoint elliptic pseudodifferential operator, and thus admits a mod 2 index in the sense of [3]. Moreover, by the mod 2 index theorem in [3] (cf. [18]), one has

(2.56)
$$\alpha(M) = \dim\left(\ker\left(\widehat{P}_{R,\beta,\varepsilon}\right)\right) \mod 2.$$

Now by proceeding as in Section 2.3, one sees that there are R, β, $\varepsilon > 0$ such that

(2.57)
$$\dim\left(\ker\left(\widehat{P}_{R,\beta,\varepsilon}\right)\right) = 0.$$

From (2.56) and (2.57), one gets $\alpha(M) = 0$.

2.5. **Proof of the Connes vanishing theorem and more.** Without loss of generality, we may and we will assume that all $\mathcal{F} = \pi^* F$, $\mathcal{F}_1^{\perp}$ and $\mathcal{F}_2^{\perp}$ are oriented and of even rank. The main concern here is that we only assume F is spin, not TM. Thus, here $\mathcal{F} = \pi^* F$ is spin and carries a fixed spin structure.

Instead of the sub-Dirac operator considered in (2.16), we now consider the sub-Dirac operator constructed as in (1.61),

(2.58) $\quad D^{\mathcal{F},\phi(\mathcal{F}_1^{\perp})}_{\beta,\varepsilon} : \Gamma\left(S(\mathcal{F})\widehat{\otimes}\Lambda^*\left(\mathcal{F}_1^{\perp}\right)\widehat{\otimes}\Lambda^*\left(\mathcal{F}_2^{\perp}\right)\otimes\phi\left(\mathcal{F}_1^{\perp}\right)\right)$

$$\longrightarrow \Gamma\left(S(\mathcal{F})\widehat{\otimes}\Lambda^*\left(\mathcal{F}_1^{\perp}\right)\widehat{\otimes}\Lambda^*\left(\mathcal{F}_2^{\perp}\right)\otimes\phi\left(\mathcal{F}_1^{\perp}\right)\right).$$

Now we can proceed as in Sections 2.2 and 2.3, by replacing the sub-Dirac operator in (2.16) by $D_{\beta,\varepsilon}^{\mathcal{F},\phi(\mathcal{F}_1^\perp)}$ above.

In particular, by the Atiyah-Singer index theorem [2], the right hand side of the formula corresponding to (2.34) is now

$$(2.59) \qquad 2^{\frac{q_1}{2}} \left\langle \widehat{A}(F)\widehat{L}(TM/F)\mathrm{ch}(\phi(TM/F)), [M] \right\rangle.$$

In summary, if k^F is positive over M, then we get

$$(2.60) \qquad \left\langle \widehat{A}(F)\widehat{L}(TM/F)\mathrm{ch}(\phi(TM/F)), [M] \right\rangle = 0.$$

Now as any rational Pontrjagin class of TM/F can be expressed as a rational linear combination of classes of form $\widehat{L}(TM/F)\mathrm{ch}(\phi(TM/F))$, one gets from (2.60) that for any Pontrjagin class $p(TM/F)$ of TM/F, one has

$$(2.61) \qquad \left\langle \widehat{A}(F)p(TM/F), [M] \right\rangle = 0,$$

which has been proved in [10, Corollary 8.3]. In particular, one has

$$(2.62) \qquad \widehat{A}(M) = \left\langle \widehat{A}(TM), [M] \right\rangle = \left\langle \widehat{A}(F)\widehat{A}(TM/F), [M] \right\rangle = 0,$$

which completes the proof of Theorem 0.6.

Remark 2.7. If one modifies the sub-Dirac operator in (2.16) by twisting an integral power of $\mathcal{F}_1^\perp$, then one sees that (2.61) also holds under the condition of Theorem 0.1. This generalizes [22, Theorem 3.1].

By further modifying the sub-Dirac operators involved above, one gets the following generalization of Theorems 0.1 and 0.6 (compare with [22, Theorem 3.2]).

Theorem 2.8. *Under the assumptions of either Theorem 0.1 or 0.6, if TM/F is also oriented, then for any Pontrjagin class $p(TM/F)$ of TM/F, one has for any integer $k \geq 0$ that*

$$(2.63) \qquad \left\langle \widehat{A}(F)p(TM/F)e(TM/F)^k, [M] \right\rangle = 0.$$

In particular,

$$(2.64) \qquad \left\langle \widehat{A}(F)e(TM/F), [M] \right\rangle = 0.$$

Under the assumption of Theorem 2.8, if one assumes that $\dim M = 6$ and $\mathrm{rk}(F) = 4$, then by (2.63) one gets

$$(2.65) \qquad \left\langle e(TM/F)^3, [M] \right\rangle = 0.$$

From (2.65), one obtains the following partial complement to a classical result of Bott [7, Corollary 1.7] which states that there is no smooth codimension two foliation on the complex projective space $\mathbf{C}P^{2n+1}$ with $n \geq 2$.

Corollary 2.9. *There is no smooth codimension two foliation of positive leafwise scalar curvature on $\mathbf{C}P^3$.*

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