A path Turán problem for infinite graphs

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Abstract

Let G be an infinite graph whose vertex set is the set of positive integers, and let G_n be the subgraph of G induced by the vertices $\{1,2,\dots,n\}$. An increasing path of length k in G, denoted I_k, is a sequence of $k+1$ vertices $1 \leq i_1 < i_2 < \cdots < i_{k+1}$ such that $i_1, i_2, \dots, i_{k+1}$ is a path in G. For $k \geq 2$, let $p(k)$ be the supremum of $\liminf_{n \to \infty} \frac{e(G_n)}{n^2}$ over all I_k-free graphs G. In 1962, Czipszer, Erdős, and Hajnal proved that $p(k) = \frac{1}{4}(1 - \frac{1}{k})$ for $k \in \{2,3\}$. Erdős conjectured that this holds for all $k \geq 4$. This was disproved for certain values of k by Dudek and Rödl who showed that $p(16) > \frac{1}{4}(1 - \frac{1}{16})$ and $p(k) > \frac{1}{4} + \frac{1}{200}$ for all $k \geq 162$. Given that the conjecture of Erdős is true for $k \in \{2,3\}$ but false for large k, it is natural to ask for the smallest value of k for which $p(k) > \frac{1}{4}(1 - \frac{1}{k})$. In particular, the question of whether or not $p(4) = \frac{1}{4}(1 - \frac{1}{4})$ was mentioned by Dudek and Rödl as an open problem. We solve this problem by proving that $p(4) \geq \frac{1}{4}(1 - \frac{1}{4}) + \frac{1}{584064}$ and $p(k) > \frac{1}{4}(1 - \frac{1}{k})$ for $4 \leq k \leq 15$. We also show that $p(4) \leq \frac{1}{4}$ which improves upon the previously best known upper bound on $p(4)$. Therefore, $p(4)$ must lie somewhere between $\frac{3}{16} + \frac{1}{584064}$ and $\frac{1}{4}$.

Keywords: path Turán, infinite graphs

1. Introduction

Turán problems form a cornerstone of extremal graph theory. In general, the Turán problem asks for the maximum number of edges in a graph which does not contain another graph as a subgraph. Turán's theorem determines this maximum when the forbidden graph is a clique on a fixed number of vertices. Because of its significance, different Turán type problems have been considered in a variety of different settings. One such setting is infinite graphs. Perhaps not surprisingly, Paul Erdős was one of the pioneers of infinite graph theory and we recommend [1] and [6] for excellent discussions of his work in this area as well as many open problems. In this paper, we study a Turán problem on countably infinite graphs that was first considered by Czipszer, Erdős, and Hajnal [2].

Let G be an infinite graph with $V(G) = \{1,2,3,\dots\}$. An *increasing path of length* k, denoted I_k, is a sequence of $k+1$ vertices $i_1, \dots, i_{k+1}$ such that $i_1 < i_2 < \cdots < i_{k+1}$ and i_j is adjacent to i_{j+1} for $1 \leq j \leq k$. An infinite graph G is I_k*-free* if it does not contain an increasing path of length k. For an infinite graph G, let G_n be the subgraph of G induced by the vertices $\{1,2,\dots,n\}$ and $p(G) = \liminf\limits_{n \to \infty} \dfrac{e(G_n)}{n^2}$. Define the *path Turán number of* I_k, denoted $p(k)$, to be the value

$$p(k) = \sup\{p(G) : G \text{ is } I_k\text{-free}\}.$$

Czipszer, Erdős, and Hajnal [2] introduced these path Turán numbers and proved the following.

Theorem 1.1 (Czipszer, Erdős, Hajnal [2]). *The path Turán numbers $p(2)$ and $p(3)$ satisfy*

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$$p(2) = \tfrac{1}{8} \text{ and } p(3) = \tfrac{1}{6}.$$

They also gave a simple construction that shows

$$p(k) \geq \tfrac{1}{4}\left(1 - \tfrac{1}{k}\right) \text{ for all } k \geq 2$$

and asked if $p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right)$ holds for $k \geq 4$. Erdős conjectured in [4] and [5] that $p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right)$ holds for all $k \geq 2$. In 2008, Dudek and Rödl [3] disproved the conjecture for certain values of k by proving the following result.

Theorem 1.2 (Dudek, Rödl [3]). *The path Turán number $p(16)$ satisfies*

$$p(16) > \frac{1}{4}\left(1 - \frac{1}{16}\right).$$

Furthermore,

$$p(k) > \frac{1}{4} + \frac{1}{200}$$

for all $k \geq 162$.

The results of [3] and the conjecture $p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right)$ is mentioned in a survey paper of Komjáth [6] which discusses some of the work of Erdős in infinite graph theory.

Theorem 1.1 and Theorem 1.2 suggest the following question: for which values of k does one have

$$p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right) \tag{1}$$

and in particular, what is the smallest value of k for which (1) holds? Our first result is a construction that shows (1) does not hold for several small values of k and disproves the conjecture of Erdős in the most difficult case; the case when $k = 4$.

Theorem 1.3. *If $4 \leq k \leq 15$, then*

$$p(k) > \frac{1}{4}\left(1 - \frac{1}{k}\right).$$

By combining the results of [3] with the results and techniques of this paper, one can show that (1) fails for all $k \geq 4$. For more on this, see Section 5.

Using the argument of [2], we obtained the following upper bound on $p(4)$.

Theorem 1.4. *The path Turán number $p(4)$ satisfies*

$$p(4) \leq \frac{1}{4}.$$

In proving Theorem 1.3, we will find a positive constant c_k for which $p(k) \geq \frac{1}{4}(1 - \frac{1}{k}) + c_k$ provided $k \in \{4, 5, \ldots, 15\}$. In particular, we obtain $c_4 = \frac{1}{584064}$ (see Section 3.3) so that by Theorem 1.4,

$$\frac{1}{4}\left(1 - \frac{1}{4}\right) + \frac{1}{584064} \leq p(4) \leq \frac{1}{4}. \tag{2}$$

Determining the exact value of $p(4)$ is a challenging open problem. Probably the lower bound in (2) is closer to the true value of $p(4)$.

The next section introduces a sequence reformulation of the path Turán problem. This reformulation was a key ingredient in the constructions of [3] and we use it in our constructions as well. In Section 3.1 we give our construction method and state our main lemma. Section 3.2 contains the proof of our main lemma. In Section 3.3 we prove Theorem 1.3 and in Section 4 we prove Theorem 1.4.

2. Sequences

It will be convenient to work with the sequence formulation of the problem introduced by Dudek and Rödl. Given an I_k-free graph G with $V(G) = \mathbb{N}$, partition $\mathbb{N}$ into k sets $N_1, \ldots, N_k$ where

$$N_1(G) = \{n \in \mathbb{N} : \forall m \in \mathbb{N} \text{ with } \{n,m\} \in E(G) \text{ we have } n < m\}$$

and for $2 \leq i \leq k$,

$$N_i(G) = \{n \in \mathbb{N} \setminus \bigcup_{j=1}^{i-1} N_j(G) : \forall m \in \mathbb{N} \text{ with } \{n,m\} \in E(G)$$
$$\text{we have } n < m \text{ or } m \in \bigcup_{j=1}^{i-1} N_j(G)\}.$$

Define $C = C(G)$ to be the sequence $\{c_n\}_{n=1}^{\infty}$ where $c_n = i$ if and only if $n \in N_i(G)$. Let

$$\mathcal{S}_C(n) = |\{(i,j) : 1 \leq i < j \leq n \text{ and } c_i < c_j\}|.$$

It is shown in [3], that

$$\liminf_{n \to \infty} \frac{e(G_n)}{n^2} = \liminf_{n \to \infty} \frac{\mathcal{S}_C(n)}{n^2}.$$

Conversely, given a sequence whose terms are elements of $[k]$, the corresponding infinite graph G with vertex set $\mathbb{N}$ has edge set

$$\{(i,j) : 1 \leq i < j \text{ and } c_i < c_j\}.$$

3. Proof of Theorem 1.3

3.1. Constructing Sequences

Let $k \geq 2$ and $l \geq 1$ be integers. Let D be a $k \times l$ matrix whose entries are non-negative integers. Let $d_{i,j}$ be the (i,j)-entry of D. We will use D to construct an infinite sequence C with entries in $[k]$. Let D_j be the sequence

$$D_j = \underbrace{11 \cdots 11}_{d_{1,j}\ 1's} \underbrace{22 \cdots 22}_{d_{2,j}\ 2's} \underbrace{33 \cdots 33}_{d_{3,j}\ 3's} \cdots \underbrace{kk \cdots kk}_{d_{k,j}\ k's}.$$

We call D_j an *atom*. We remark that D_j has length $\sum_{i=1}^{k} d_{i,j}$ and since the $d_{i,j}$'s can be zero, it is possible that D_j does not contain every symbol from $[k]$. Given a finite sequence R, let us write $L(R)$ for the length of R so that, in this notation,

$$L(D_j) = \sum_{i=1}^{k} d_{i,j}.$$

Given any two finite sequences $S = s_1 s_2 \ldots s_x$ and $T = t_1 t_2 \ldots t_y$, we write

$$ST = s_1 s_2 \ldots s_x t_1 t_2 \ldots t_y$$

for the concatenation of S and T. For an integer $m \geq 1$, define B_m to be the sequence

$$B_m = \underbrace{D_1 D_1 \cdots D_1 D_1}_{2^{m-1}} \underbrace{D_2 D_2 \cdots D_2 D_2}_{2^{m-1}} \cdots \underbrace{D_l D_l \cdots D_l D_l}_{2^{m-1}}.$$

We call the sequence B_m a *block*. Since B_m contains 2^{m-1} copies of D_j for $1 \leq j \leq l$, the length of B_m is

$$L(B_m) = 2^{m-1} \sum_{j=1}^{l} L(D_j) = 2^{m-1} \sum_{j=1}^{l} \sum_{i=1}^{k} d_{i,j}.$$

Define $C = C(D)$ to be the infinite sequence

$$C = B_1 B_2 B_3 B_4 B_5 \cdots .$$

Write $C = \{c_r\}_{r=1}^{\infty}$ for this sequence. For example, if $D = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 1 \end{pmatrix}$, then $D_1 = 133$, $D_2 = 111223$, and

$$C = \underbrace{133111223}_{B_1}\underbrace{133133111223111223}_{B_2}\cdots .$$

Motivated by the infinite I_k-free graph corresponding to the sequence C, we call a pair $\{i, j\}$ for which $1 \le i < j$ and $c_i < c_j$ an *edge*. Let M be the $l \times l$ matrix whose (i, j)-entry, denoted by $m_{i,j}$, is given by

$$m_{i,j} = \sum_{x=1}^{k-1} d_{x,i} \left(\sum_{y=x+1}^{k} d_{y,j} \right).$$

The value $m_{i,j}$ is the number of edges with one endpoint in an atom D_i, the other endpoint in an atom D_j, and where D_i precedes D_j in the sequence C. In general, the matrix M is not a symmetric matrix. Define

$$w_1(M) = \sum_{i=1}^{l}\sum_{j=1}^{l} m_{i,j}, \quad w_2(M) = \sum_{i=1}^{l-1}\sum_{j=i+1}^{l} m_{i,j}, \quad w_3(M) = \sum_{i=1}^{l} m_{i,i},$$

and $w(M) = \frac{w_1(M)}{3} + \frac{w_2(M)}{3} + \frac{w_3(M)}{6}$.

We want to choose D so that for the corresponding sequence $C = C(D)$,

$$\liminf_{n\to\infty} \frac{\mathcal{S}_C(n)}{n^2} \tag{3}$$

is as large as possible. Our main tool for estimating (3) is the following lemma.

Lemma 3.1. *Given D, C, and M as above, the value of $\liminf_{n\to\infty} \frac{\mathcal{S}_C(n)}{n^2}$ is at least the minimum value of*

$$\frac{w(M) + \sum_{j=1}^{t-1}\left(\sum_{i=1}^{l} m_{i,j} + \frac{m_{j,j}}{2} + \sum_{i=1}^{j-1} m_{i,j} \right) + \epsilon \sum_{i=1}^{l} m_{i,t} + \epsilon^2 \frac{m_{t,t}}{2} + \epsilon \sum_{i=1}^{t-1} m_{i,t}}{\left(\sum_{i=1}^{k}\sum_{j=1}^{l} d_{i,j} + \sum_{j=1}^{t-1}\sum_{i=1}^{k} d_{i,j} + \epsilon \sum_{i=1}^{k} d_{i,t} \right)^2}$$

where t ranges over all integers in $\{1, 2, \ldots, l\}$, and ϵ ranges over all real numbers in the interval $[0, 1]$.

Remark: In Lemma 3.1 and in the rest of the paper, any sum of the form $\sum_{i=1}^{t-1} \alpha_i$ with $t = 1$ is taken to be 0.

3.2. Proof of Lemma 3.1

Let n be a positive integer. We choose m to be the largest integer such that

$$\sum_{x=1}^{m} \left(2^{x-1} \sum_{j=1}^{l}\sum_{i=1}^{k} d_{i,j} \right) \le n < \sum_{x=1}^{m+1} \left(2^{x-1} \sum_{j=1}^{l}\sum_{i=1}^{k} d_{i,j} \right). \tag{4}$$

The left hand side of (4) is $L(B_1 B_2 \cdots B_m)$. The right hand side is $L(B_1 B_2 \cdots B_m B_{m+1})$. We can write n in the form

$$n = L(B_1 B_2 \cdots B_m) + L(\underbrace{D_1 D_1 \cdots D_1}_{2^m}) + \cdots + L(\underbrace{D_{t-1} D_{t-1} \cdots D_{t-1}}_{2^m}) + \epsilon L(\underbrace{D_t D_t \cdots D_t}_{2^m})$$

for some $0 \leq \epsilon \leq 1$ and $t \in \{1, 2, \ldots, l\}$. Therefore,

$$\begin{aligned} n &= \sum_{x=1}^{m}\left(2^{x-1}\sum_{i=1}^{k}\sum_{j=1}^{l} d_{i,j}\right) + 2^m\left(\sum_{i=1}^{k} d_{i,1} + \cdots + \sum_{i=1}^{k} d_{i,t-1} + \epsilon\sum_{i=1}^{k} d_{i,t}\right) \\ &= 2^m\left((1-\frac{1}{2^m})\left(\sum_{i=1}^{k}\sum_{j=1}^{l} d_{i,j}\right) + \left(\sum_{x=1}^{t-1}\sum_{i=1}^{k} d_{i,x} + \epsilon\sum_{i=1}^{k} d_{i,t}\right)\right). \end{aligned}$$

This is the formula that we will use for n in the expression $\frac{\mathcal{S}_C(n)}{n^2}$. It is helpful to think of n as the location of a cut in the sequence C. When we look at the first n terms of the sequence, we see all of the terms in blocks B_1 through B_m, and only some of the terms in the block B_{m+1}. For this reason, we call B_{m+1} a *partial block*. Clearly the number of elements that we see from B_{m+1} depends on n.

Next we look for a lower bound on $\mathcal{S}_C(n)$.

Lemma 3.2. *For $1 \leq k_1 < k_2$, the number of edges between the block B_{k_1} and the block B_{k_2} is*

$$\frac{2^{k_1+k_2}}{4}\sum_{i=1}^{l}\sum_{j=1}^{l} m_{i,j}.$$

Proof. The sequence B_{k_1} contains 2^{k_1-1} atoms of type D_j for each $j \in \{1, 2, \ldots, l\}$. A similar assertion holds for B_{k_2}. There are $2^{k_1-1}2^{k_2-1}m_{i,j}$ edges from the D_i atoms in block B_{k_1} to the D_j atoms in block in B_{k_2}. The proof of the lemma is completed by summing over all i, j with $1 \leq i, j \leq l$. ∎

Lemma 3.3. *For any $k \geq 1$, the number of edges in B_k is at least*

$$\frac{4^k}{4}\sum_{i=1}^{l-1}\sum_{j=i+1}^{l} m_{i,j} + \frac{4^k}{8}\sum_{i=1}^{l} m_{i,i} - c_D 2^k$$

where c_D is a constant that depends only on D.

Proof. For any $1 \leq i < j \leq l$, the block B_k contains 2^{k-1} consecutive atoms of type D_i that precede 2^{k-1} consecutive atoms of type D_j. Therefore, B_k contains at least

$$2^{k-1}2^{k-1}\sum_{i=1}^{l-1}\sum_{j=i+1}^{l} m_{i,j}$$

edges that have end points in atoms of different types. Next we count edges that have both endpoints in an atom of type D_i. There are $\binom{2^{k-1}}{2}$ pairs of distinct atoms of type D_i in the block B_k and a total of $m_{i,i}$ edges between any two such atoms. Summing over $1 \leq i \leq l$ gives a total of

$$\binom{2^{k-1}}{2}\sum_{i=1}^{l} m_{i,i} = \frac{4^k}{8}\sum_{i=1}^{l} m_{i,i} - c_D 2^k$$

edges. ∎

A consequence of Lemmas 3.2 and 3.3 is that the number of edges contained in $B_1 B_2 \cdots B_m$ is at least

$$\frac{1}{4}w_1(M)\sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m} 2^{k_1+k_2} + \left(\frac{1}{4}w_2(M) + \frac{1}{8}w_3(M)\right)\left(\sum_{k=1}^{m} 4^k\right) - c_D 2^{m+1}.$$

A short calculation shows that this expression can be simplified to

$$4^m\left(\frac{w_1(M)}{3} + \frac{w_2(M)}{3} + \frac{w_3(M)}{6}\right) - O(2^m)$$

where the constant in the O notation only depends on D. Since $w(M) = \frac{w_1(M)}{3} + \frac{w_2(M)}{3} + \frac{w_3(M)}{6}$, we have the lower bound

$$\mathcal{S}_C(n) \geq 4^m \left(\frac{w_1(M)}{3} + \frac{w_2(M)}{3} + \frac{w_3(M)}{6} \right) - O(2^m) = 4^m w(M) - O(2^m)$$

however this is not good enough, especially in the case when t is close to l which, in terms of n, means that n is closer to $L(B_1 B_2 \cdots B_{m+1})$ than it is to $L(B_1 B_2 \cdots B_m)$. We are losing too much by not counting edges between $B_1 B_2 \cdots B_m$ and the D_i's coming from the partial block B_{m+1}, as well as the edges in the partial block B_{m+1}. To count these edges we need a few more lemmas.

Lemma 3.4. *The number of edges with one endpoint in $B_1 B_2 \cdots B_m$ and the other endpoint in an atom of type D_j in the partial block B_{m+1} where $j \in \{1, 2, \ldots, t\}$ is*

$$4^m \left(1 - \frac{1}{2^m}\right) \left(\sum_{i=1}^{l} \sum_{j=1}^{t-1} m_{i,j} + \epsilon \sum_{i=1}^{l} m_{i,t} \right).$$

Proof. For any $i \in \{1, 2, \ldots, l\}$, there are $2^m - 1$ atoms of type D_i in the sequence $B_1 B_2 \cdots B_m$. Each such atom sends $m_{i,j}$ edges to a D_j in the partial block B_{m+1}. For $j \in \{1, 2 \ldots, t-1\}$, the partial block B_{m+1} contains 2^m atoms of type D_j. The partial block B_{m+1} contains $\epsilon 2^m$ atoms of type D_t. Summing over $1 \leq j \leq t-1$ gives a total of

$$(2^m - 1) 2^m \sum_{i=1}^{l} \sum_{j=1}^{t-1} m_{i,j} + (2^m - 1)\epsilon 2^m \sum_{i=1}^{l} m_{i,t}$$

edges. ■

Lemma 3.5. *The number of edges both of whose endpoints are contained in the partial block B_{m+1} is at least*

$$\frac{4^m}{2} \left(1 - \frac{1}{2^m}\right) \sum_{j=1}^{t-1} m_{j,j} + 4^m \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} m_{i,j} + \epsilon^2 \frac{4^m}{2} \left(1 - \frac{1}{2^m \epsilon}\right) m_{t,t} + \epsilon 4^m \sum_{i=1}^{t-1} m_{i,t}. \tag{5}$$

Proof. The proof of the lemma is similar to the proofs of the previous lemmas. Instead of going through the details, we simply state what types of edges each of the four terms in (5) is counting. The sum

$$\frac{4^m}{2} \left(1 - \frac{1}{2^m}\right) \sum_{j=1}^{t-1} m_{j,j}$$

counts edges both of whose endpoints are in an atom of type D_j in the partial block B_{m+1} for $j \in \{1, 2, \ldots, t-1\}$. There are $\binom{2^m}{2}$ pairs of such D_j.

The second term counts edges in the partial block B_{m+1} where one endpoint is an atom of type D_i and the other endpoint is an atom of type D_j where $1 \leq i < j \leq t-1$.

The third term

$$\epsilon^2 \frac{4^m}{2} \left(1 - \frac{1}{2^m \epsilon}\right) m_{t,t}$$

counts edges whose endpoints are in an atom of type D_t. There are $\binom{\epsilon 2^m}{2}$ pairs of distinct atoms of type D_t in the partial block B_{m+1}.

The final term counts edges with one endpoint in an atom of type D_i where $1 \leq i < t$, and the other endpoint is in an atom of type D_t. There are $2^m \epsilon 2^m$ such pairs and we then sum this over $1 \leq i \leq t-1$. ■

From Lemmas 3.4 and 3.5, we now have

$$\begin{aligned}\mathcal{S}_C(n) &\geq 4^m\left(w(M)+\sum_{j=1}^{t-1}\left((1-1/2^m)\sum_{i=1}^{l}m_{i,j}+(1-1/2^m)\frac{m_{j,j}}{2}+\sum_{i=1}^{j-1}m_{i,j}\right)\right)\\ &+ 4^m\left((1-1/2^m)\epsilon\sum_{i=1}^{l}m_{i,t}+\epsilon^2\frac{1}{2}\left(1-\frac{1}{2^m\epsilon}\right)m_{t,t}+\epsilon\sum_{i=1}^{t-1}m_{i,t}\right)-O(2^m).\end{aligned}$$

Now as n goes to infinity, m must also tend to infinity. Combining this lower bound on $\mathcal{S}_C(n)$ together with

$$n=2^m\left((1-\frac{1}{2^m})\left(\sum_{i=1}^{k}\sum_{j=1}^{l}d_{i,j}\right)+\left(\sum_{j=1}^{t-1}\sum_{i=1}^{k}d_{i,j}+\epsilon\sum_{i=1}^{k}d_{i,t}\right)\right)$$

completes the proof of Lemma 3.1.

3.3. Choosing Matrices

In this section we give several matrices which, when combined with Lemma 3.1, improve the lower bound

$$p(k)\geq\frac{1}{4}\left(1-\frac{1}{k}\right)$$

for different values of k. We first list the matrices and then give the corresponding lower bounds obtained from Lemma 3.1. Computations were done using Mathematica [7] and the code used for the computations is given in the Appendix.

The matrices used to improve $p(k)\geq\frac{1}{4}\left(1-\frac{1}{k}\right)$.

$$D(5)=\begin{pmatrix}6&2&1\\1&7&3\\2&2&8\\6&2&2\\4&5&2\end{pmatrix}\qquad D(6)=\begin{pmatrix}6&0&2\\3&6&0\\2&3&5\\2&5&3\\3&2&4\\6&2&0\end{pmatrix}\qquad D(7)=\begin{pmatrix}3&0&2&3\\3&4&2&0\\1&2&5&1\\0&2&4&3\\2&0&4&3\\3&1&3&4\\3&3&2&0\end{pmatrix}$$

$$D(8)=\begin{pmatrix}5&2&0&0\\0&5&2&2\\0&2&7&1\\0&1&2&7\\1&1&4&3\\2&0&1&6\\4&2&2&0\\1&5&2&0\end{pmatrix}\qquad D(9)=\begin{pmatrix}7&0&0&1&2\\0&6&0&1&3\\1&1&6&2&1\\1&2&2&6&2\\1&1&1&0&8\\2&2&6&2&1\\0&1&2&6&0\\3&2&0&1&7\\5&3&1&0&0\end{pmatrix}$$

$$D(10)=\begin{pmatrix}10&7&2&5&8\\6&10&11&4&3\\4&7&11&9&6\\4&3&9&14&9\\9&3&2&9&10\\10&9&6&3&7\\9&11&6&4&4\\6&9&12&8&1\\7&5&7&9&8\\7&7&6&6&9\end{pmatrix}\qquad D(11)=\begin{pmatrix}7&1&0&1&2&1\\0&7&1&0&0&0\\0&1&6&0&0&2\\1&0&0&6&0&2\\1&1&0&1&6&1\\0&0&2&1&1&6\\1&1&0&6&0&2\\1&1&0&2&6&2\\0&1&2&2&1&5\\7&0&1&0&0&0\\2&6&2&1&0&0\end{pmatrix}$$

$$D(12) = \begin{pmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 & 1 \\ 0 & 0 & 1 & 6 & 1 & 0 \\ 0 & 1 & 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 1 & 0 & 6 \\ 1 & 0 & 0 & 6 & 1 & 1 \\ 1 & 0 & 1 & 1 & 6 & 0 \\ 1 & 0 & 1 & 0 & 1 & 6 \\ 6 & 0 & 1 & 1 & 1 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 \\ 1 & 1 & 6 & 0 & 0 & 0 \end{pmatrix} \qquad D(13) = \begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 7 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 8 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 7 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 8 \\ 1 & 1 & 0 & 0 & 8 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 7 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 8 \\ 8 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 7 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$D(14) = \begin{pmatrix} 7 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 7 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 7 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 7 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 8 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 8 \\ 1 & 0 & 1 & 8 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 7 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 7 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 8 \\ 8 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 8 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 8 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad D(15) = \begin{pmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 9 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 9 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 8 \\ 0 & 0 & 1 & 0 & 9 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 9 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 9 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ 9 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 9 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Corresponding lower bounds

$p(5) \geq \frac{1688}{8427} = \frac{1}{4}\left(1 - \frac{1}{5}\right) + \frac{13}{42135}$ $\qquad$ $p(6) \geq \frac{3683}{17672} = \frac{1}{4}\left(1 - \frac{1}{6}\right) + \frac{1}{13254}$

$p(7) \geq \frac{365}{1701} = \frac{1}{4}\left(1 - \frac{1}{7}\right) + \frac{1}{3402}$ $\qquad$ $p(8) \geq \frac{19325}{87846} = \frac{1}{4}\left(1 - \frac{1}{8}\right) + \frac{1739}{1405536}$

$p(9) \geq \frac{9448}{42483} = \frac{1}{4}\left(1 - \frac{1}{9}\right) + \frac{22}{127449}$ $\qquad$ $p(10) \geq \frac{83234}{369603} = \frac{1}{4}\left(1 - \frac{1}{10}\right) + \frac{2933}{14784120}$

$p(11) \geq \frac{18033}{79202} = \frac{1}{4}\left(1 - \frac{1}{11}\right) + \frac{179}{435611}$ $\qquad$ $p(12) \geq \frac{13511}{58482} = \frac{1}{4}\left(1 - \frac{1}{12}\right) + \frac{871}{467856}$

$p(13) \geq \frac{57931}{249696} = \frac{1}{4}\left(1 - \frac{1}{13}\right) + \frac{4015}{3246048}$ $\qquad$ $p(14) \geq \frac{16743}{71824} = \frac{1}{4}\left(1 - \frac{1}{14}\right) + \frac{487}{502768}$

$p(15) \geq \frac{36251}{154568} = \frac{1}{4}\left(1 - \frac{1}{15}\right) + \frac{2777}{2318520}$

We take a moment to briefly describe the method in which these matrices were obtained. The matrix $D(5)$ was obtained by starting with the matrix

$$R(5) := \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

The idea to use this matrix as a starting point comes from the fact that the constructions in [3] are good for large values of k and so, while they do not improve the lower bound $p(k) \geq \frac{1}{4}(1 - \frac{1}{k})$ for small k, they still give a reasonable bound for small k. The matrix $R(5)$ is a natural modification of the construction in [3]. We then added a random 0-1 matrix to $R(5)$. This was repeated many times until we found a new matrix that provided a better lower bound on $p(5)$ than the lower bound given by $R(5)$. This process was repeated until we arrived at the matrix $D(5)$ given above. Looking closely at each of the matrices above, one may be able to find the "dominant diagonal" entries. For instance in $D(8)$, the positions

$$(1,1),(2,2),(3,3),(4,4),(5,3),(6,4),(7,1),(8,2)$$

contain entries that are larger than the other entries. The initial matrix used to construct $D(8)$ is

$$R(8) := \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

which is again a natural modification of the construction from [3].

In the case that $k = 4$, the matrix

$$D(4) = \begin{pmatrix} 6 & 2 & 5 & 8 & 4 & 6 & 6 & 9 \\ 7 & 5 & 7 & 6 & 5 & 5 & 4 & 4 \\ 4 & 5 & 7 & 6 & 5 & 8 & 4 & 7 \\ 7 & 2 & 5 & 8 & 4 & 5 & 6 & 8 \end{pmatrix}$$

was obtained by a more ad hoc method. It leads to the lower bound

$$p(4) \geq \frac{109513}{584064} = \frac{1}{4}\left(1 - \frac{1}{4}\right) + \frac{1}{584064}.$$

The corresponding sequence on 4 symbols has 8 atoms.

4. Proof of Theorem 1.4

We follow the method of [2]. Let G be an infinite graph that is I_4-free. Let $C = C(G) = \{c_n\}$ be its associated sequence on the symbols $\{1,2,3,4\}$. Define three sequences $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ as follows.

1. For $i \in \mathbb{N}$, $u_i = k$ if and only if $c_k \in \{2,3,4\}$ and $|\{r : c_r \in \{2,3,4\}, r \leq k\}| = i$.
2. For $i \in \mathbb{N}$, $v_i = k$ if and only if $c_{u_k} \in \{3,4\}$ and $|\{r : c_r \in \{3,4\}, r \leq u_k\}| = i$.
3. For $i \in \mathbb{N}$, $w_i = k$ if and only if $c_{u_{v_k}} \in \{4\}$ and $|\{r : c_r \in \{4\}, r \leq u_{v_k}\}| = i$.

We give the following example for convenience. In several of our counting arguments it may be quite useful for the reader to refer back to this example.

n	1	2	3	4	5	6	7	8	9	10
c_n	1	2	1	3	2	4	1	4	2	3
u_i		$u_1 = 2$		$u_2 = 4$	$u_3 = 5$	$u_4 = 6$		$u_5 = 8$	$u_6 = 9$	$u_7 = 10$
v_i				$v_1 = 2$		$v_2 = 4$		$v_3 = 5$		$v_4 = 7$
w_i						$w_1 = 2$		$w_2 = 3$		

Example 1

Given $n \in \mathbb{N}$, we call a 4-tuple of positive integers (n, j, k, l) a *cut* if

$$u_j \leq n < u_{j+1}, \;\; v_k \leq j < v_{k+1}, \;\text{ and }\; w_l \leq k < w_{l+1}.$$

For instance, $(9,6,3,2)$ and $(10,7,4,2)$ are cuts for the sequence in Example 1.

If $n \in \mathbb{N}$ and (n,j,k,l) is a cut, then

$$e(G_n) \leq \sum_{i=1}^{j}(u_i - i) + \sum_{i=1}^{k}(v_i - i) + \sum_{i=1}^{l}(w_i - i). \tag{6}$$

Indeed, the sum $\sum_{i=1}^{j}(u_i - i)$ counts all pairs of the form (c_s, c_t) with $1 \leq s < t \leq n$, $c_s = 1$, and $c_t \in \{2,3,4\}$. Similarly, the sum $\sum_{i=1}^{k}(v_i - i)$ counts all pairs of the form (c_s, c_t) with $1 \leq s < t \leq n$, $c_s = 2$, and $c_t \in \{3,4\}$. The sum $\sum_{i=1}^{l}(w_i - i)$ counts all pairs of the form (c_s, c_t) with $1 \leq s < t \leq n$, $c_s = 3$, and $c_t = 4$.

Given $S \subseteq \{1,2,3,4\}$, define

$$\alpha_S(i) = |\{r : c_r \in S \text{ and } r < i\}|.$$

We claim that if (n,j,k,l) is a cut, then

$$jk = \sum_{i=1}^{k} v_i + \sum_{i=1}^{j} \alpha_{\{3,4\}}(u_i) \tag{7}$$

and

$$kl = \sum_{i=1}^{l} w_i + \sum_{i=1}^{k} \alpha_{\{4\}}(u_{v_i}). \tag{8}$$

To prove these equalities, we will count pairs of the form (u_s, v_t) with $1 \leq s \leq j$ and $1 \leq t \leq k$, as well as pairs of the form (v_s, w_t) with $1 \leq s \leq k$ and $1 \leq t \leq l$.

Clearly there are jk pairs (u_s, v_t) with $1 \leq s \leq j$ and $1 \leq t \leq k$. The sum $\sum_{i=1}^{k} v_i$ counts all pairs (u_s, v_t) for which $s \leq v_t$ while the sum $\sum_{i=1}^{j} \alpha_{\{3,4\}}(u_i)$ counts all pairs (u_s, v_t) for which $s > v_t$. This double counting is best illustrated by referring to Example 1. In terms of Example 1, v_t is precisely the number of u_s's for $s \leq v_t$ so that the sum $\sum_{i=1}^{k} v_i$ counts pairs where the u_s is directly above or to the left of v_t. The sum $\sum_{i=1}^{j} \alpha_{\{3,4\}}(u_i)$ then counts all pairs where the u_s is to the right of v_t. This shows that (7) holds and a similar argument gives (8).

Combining (6), (7), and (8) we have

$$e(G_n) \leq \sum_{i=1}^{j}(u_i - \alpha_{\{3,4\}}(u_i)) - \sum_{i=1}^{k} \alpha_{\{4\}}(u_{v_i}) + jk + kl - \frac{1}{2}((j+1)^2 + (k+1)^2 + (l+1)^2) \tag{9}$$

for any cut (n,j,k,l).

To estimate the first sum, we use the following lemma.

Lemma 4.1 (Czipszer, Erdős, Hajnal [2]). *Let s_n, t_n be nondecreasing sequences of natural numbers such that $s_n - t_n > 0$ for all $n \in \mathbb{N}$. There exists a sequence $n_1 < n_2 < \dots$ such that for all r,*

$$\sum_{i=1}^{n_r}(s_i - t_i) \leq \frac{1}{2} n_r (s_{n_r} - t_{n_r}) + o(s_{n_r}^2).$$

We apply Lemma 4.1 to the sequence

$$\{u_j - \alpha_{\{3,4\}}(u_j)\}_{j=1}^{\infty}$$

to obtain a sequence $j_1 < j_2 < \dots$ such that

$$\sum_{i=1}^{j_r}(u_i - \alpha_{\{3,4\}}(u_i)) \leq \frac{1}{2} j_r (u_{j_r} - \alpha_{\{3,4\}}(u_{j_r})) + o(u_{j_r}^2) \tag{10}$$

for all $r \in \mathbb{N}$. For $r \in \mathbb{N}$, define sequences n_r, k_r, and l_r, in terms of j_r, by

1. $n_r = u_{j_r}$,
2. k_r is the largest index for which $v_{k_r} \leq j_r$, and
3. l_r is the largest index for which $w_{l_r} \leq k_r$.

We then consider the sequence $\{(n_r, j_r, k_r, l_r)\}_{r=1}^{\infty}$ of cuts. By considering these sequence of cuts, we are now looking at the subgraphs G_{n_1}, $G_{n_2}, \ldots$ and for these subgraphs, we know that (10) holds.

For any r, we have $u_{j_r} \geq j_r \geq k_r \geq l_r$ so that

$$j_r = o(u_{j_r}^2) \text{ and } k_r = o(u_{j_r}^2) \text{ and } l_r = o(u_{j_r}^2) \tag{11}$$

as $u_{j_r} \to \infty$. Furthermore,

$$\sum_{i=1}^{k_r} \alpha_{\{4\}}(u_{v_i}) \geq (l_r - 1) + (l_r - 2) + \cdots + 2 + 1. \tag{12}$$

In terms of Example 1, the sum on the left hand side of (12) moves across the entries in the v_i row, and counts 4's that are above and to the left of the current entry. A simple calculation shows that one would have $\sum_{i=1}^{4} \alpha_{\{4\}}(u_{v_i}) = 0 + 0 + 1 + 2$ for Example 1. The sum $\sum_{i=1}^{k_r} \alpha_{\{4\}}(u_{v_i})$ is minimized when all of the l_r 4's that appear in the cut (n_r, j_r, k_r, l_r) come after all of the 3's in the cut. Lastly, for any r,

$$\alpha_{\{3,4\}}(u_{j_r}) \geq k_r - 1. \tag{13}$$

To see this inequality, one notes that since k_r is the largest index for which $v_{k_r} \leq j_r$, there must be $k_r - 1$ terms of the sequence c_n that are 3 or 4 and come before u_{j_r}.

Combining (9), (10), (12), and (13) gives

$$e(G_{n_r}) \leq \frac{1}{2} j_r(u_{j_r} - k_r) - \frac{1}{2}(l_r - 1)^2 + j_r k_r + k_r l_r - \frac{1}{2}((j_r + 1)^2 + (k_r + 1)^2 + (l_r + 1)^2) + o(u_{j_r}^2) \tag{14}$$

for any $r \in \mathbb{N}$. We divide through by $n_r^2 = u_{j_r}^2$ to get

$$\frac{e(G_{n_r})}{n_r^2} \leq \frac{1}{u_{j_r}^2} \left(\frac{j_r u_{j_r}}{2} + \frac{k_r j_r}{2} - \frac{1}{2}(l_r - 1)^2 + k_r l_r - \frac{1}{2}(j_r^2 + l_r^2 + k_r^2) \right) + o(1). \tag{15}$$

Since $l_r \leq k_r$, we can write $l_r = \epsilon_r k_r$ for some $0 \leq \epsilon_r \leq 1$. The remaining analysis does not depend on r and so, for ease of notation, we

omit all occurrences of r as a subscript on all terms.

Using $l = \epsilon k$, the inequality (15) can be rewritten as

$$\frac{e(G_n)}{n^2} \leq \frac{1}{u_j^2} \left(\frac{j u_j}{2} + \frac{kj}{2} - \frac{j^2}{2} - k^2(\epsilon^2 - \epsilon + \frac{1}{2}) + l \right) + o(1). \tag{16}$$

The minimum value of $f(\epsilon) = \epsilon^2 - \epsilon + \frac{1}{2}$ with $0 \leq \epsilon \leq 1$ is $\frac{1}{4}$ and occurs when $\epsilon = \frac{1}{2}$. Thus, (16) together with (11) implies

$$\frac{e(G_n)}{n^2} \leq \frac{1}{u_j^2} \left(\frac{j u_j}{2} + \frac{kj}{2} - \frac{j^2}{2} - \frac{k^2}{4} \right) + o(1).$$

We can rewrite this as

$$\frac{e(G_n)}{n^2} \leq \frac{1}{u_j^2} \left(\frac{j u_j}{2} - \frac{j^2}{2} + \frac{j}{2} k(1 - \frac{k}{2j}) \right) + o(1).$$

The maximum value of $g(k) = k(1 - \frac{k}{2j})$ with $0 \leq k \leq j$ is $\frac{j}{2}$ and occurs when $k = j$. Therefore,

$$\frac{e(G_n)}{n^2} \leq \frac{1}{u_j^2} \left(\frac{j u_j}{2} - \frac{j^2}{2} + \frac{j^2}{4} \right) + o(1) = \frac{1}{u_j^2} \left(\frac{j u_j}{2} - \frac{j^2}{4} \right) + o(1).$$

We now have

$$\frac{e(G_n)}{n^2} \le \frac{1}{u_j^2}\left(\frac{ju_j}{2} - \frac{j^2}{4}\right) + o(1) = \frac{1}{4} - \frac{1}{4}\left(\frac{j}{u_j} - 1\right)^2 + o(1) \le \frac{1}{4} + o(1).$$

This shows that for any $r \in \mathbb{N}$, $e(G_{n_r}) \le n_r^2 \left(\frac{1}{4} + o(1)\right)$ where $o(1) \to 0$ as $r \to \infty$. We conclude that

$$\liminf_{n\to\infty} \frac{e(G_n)}{n^2} \le \frac{1}{4}.$$

This completes the proof of Theorem 1.4.

5. Concluding Remarks

Theorem 1.3 shows that

$$p(k) > \frac{1}{4}\left(1 - \frac{1}{k}\right) \tag{17}$$

for $4 \le k \le 15$ and Theorem 1.3 of [3] shows (17) holds for $k = 16$. Remark 4.3 of [3] asserts that by using the methods of that paper, one can show that (17) is true for all $k \ge 24$. Using the construction of [3] together with our main lemma, we have verified that (17) holds for $17 \le k \le 23$. We conclude that the only k for which $p(k) = \frac{1}{4}(1 - \frac{1}{k})$ is $k = 2$ or $k = 3$.

Perhaps the most interesting problem concerning $p(k)$ is to estimate the limit

$$\lim_{k\to\infty} p(k).$$

The sequence $\{p(k)\}_{k=2}^{\infty}$ is monotone increasing and bounded above by $\frac{1}{2}$ (trivially $e(G_n) \le \frac{n^2}{2}$ for all n) so this limit does exist. From [3] we know that this limit is more than $\frac{1}{4} + \frac{1}{200}$. The main idea behind our Theorem 1.3 is to slightly modify the construction of Dudek and Rödl. We were not able to use this strategy to improve their construction for large k, say $k \ge 20$. Perhaps with more computation this can be done for $k \in \{16, , 17, \dots, 20\}$, but our computations seem to indicate that our method is highly unlikely to improve the construction of Dudek and Rödl for $k \ge 50$. The best that we could achieve for large k was to feed their construction into our Lemma 3.1. For example, we have $p(200) \ge \frac{5111}{19992}$ which improves the lower bound $\frac{1}{4} + \frac{1}{200}$ coming from $p(162) > \frac{1}{4} + \frac{1}{200}$ by less than 0.0007.

[1] F. Chung, R. Graham, Erdős on Graphs: his legacy of unsolved problems, A K Peters, Ltd., Natick, MA 01760, 1999.

[2] J. Czipszer, P. Erdős, A. Hajnal, Some extremal problems on infinite graphs, *Publications of the Math. Inst. of the Hungarian Academy of Sci.* Ser. A **7** (1962), p. 441–456.

[3] A. Dudek, V. Rödl, On the Turán properties of infinite graphs, *Electron. J. of Combin*, **15** (2008), #R47.

[4] P. Erdős, Problems and results in combinatorial analysis, *Combinatorics (Proc. Sympos. Pure Math., Vol XIX, Univ. California, Los Angeles, Calif., 1968)*, pp. 77–89, Amer. Math. Soc., Providence, R. I., 1971.

[5] P. Erdős, Problems and results on finite and infinite combinatorial analysis, *Infinite and finite sets*, Vol. I; Colloq. Math. Soc. János Bolyai, Vol. 10, pp. 403–424, North-Holland, Amsterdam, 1975.

[6] P. Komjáth, Erdős's work on infinite graphs, *Erdős centennial*, 325–345, Bolyai Soc. Math. Stud., 25, *János Bolyai Math. Soc., Budapest*, 2013.

[7] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, IL (2012).

6. Appendix

In this section we give the Mathematica [7] code that is used to evaluate the optimization problem from Lemma 3.1 once we have chosen a matrix D. The function **lowerbound** depends on four inputs d, m, l, and k. The d represents the matrix D in Lemma 3.1 and the m represents the M in Lemma 3.1. The input l is the number of columns of the input d and the input k is the number of rows of d. While d determines m, we found it easier to use a command to produce m first, and then enter this as an input into **lowerbound**. The code for obtaining m from d is

```
m = Table[(Sum[d[[a,i]](Sum[d[[b, j]],{b,a+1,k}]),{a,1,k-1}]),
{i,l},{j,l}]
```

The code for **lowerbound** is

```
lowerbound[d_, m_, l_, k_] := Min[
  Table[
   Minimize[
     {
      (
        (1/3) ( Sum[ m[[i, j]], {i, l}, {j, l} ] ) +
          (1/3) ( Sum[ m[[i, j]] , {i, l - 1}, {j, i + 1, l}] ) +
          (1/6) ( Sum[ m[[i, i]] , {i, l} ] ) +
           Sum[

          Sum[ m[[i, y]], {i, 1, l}] + (1/2) m[[y, y]] +
           Sum[ m[[i, y]], {i, 1, y - 1} ] , {y, 1, t - 1}
                ] +
         x ( Sum[ m[[i, t]] , {i, l} ] ) + x^2 (1/2) m[[t, t]] +
         x ( Sum[ m[[i, t]], {i, t - 1} ] )
        )/
       (
        ( Sum[ d[[i, j]] , {i, k}, {j, l} ] +
           Sum[Sum[ d[[i, y]], {i, 1, k}], {y, 1, t - 1}] +
           x ( Sum[ d[[i, t]] , {i, k} ] )
          )^2
         )
      , 0 <= x <= 1 }, {x} ][[1]] , {t, 1, l}]]
```