Bifurcation on compact spin manifold

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Abstract

We consider the bifurcation phenomenon of solutions to a class of nonlinear Dirac equations $\mu D\psi = \psi + h(\psi)$ on compact spin manifolds. There are two main results. The first is bifurcation from zero, which asserts that the equations allow a sequence of bifurcation points (μ_k, θ), and for each $\mu \in \mathbb{R}$ near μ_k the corresponding equation has at least two nontrivial solutions near θ. The second result is bifurcation from infinity. The point (μ_k, ∞) is also proved to be a bifurcation point and we give a characterization of the bifurcation behavior.

1 Introduction and main results

Since M.F. Atiyah introduced the Dirac operator for spin manifolds in 1962, it has become one of the basic elliptic differential operators in analysis and geometry. Various forms of Dirac equations have come to play important role in modern development of both mathematics and mathematical physics. In this paper, we are concerned with a class of nonlinear Dirac equations on compact spin manifolds. We study the distributions of solutions from a bifurcation point of view.

Let M be an m-dimensional compact spin manifold. We denote $\mathbb{S}(M) = Spin(TM) \times_{\delta_m} \mathbb{S}$ as the spinor bundle on M and $D : C^\infty(M, \mathbb{S}(M)) \to C^\infty(M, \mathbb{S}(M))$ the Atiyah-Singer Dirac operator. Section 2 will introduce the spin structure and the definition of the Dirac operator on M. For a fiber preserving map $h : \mathbb{S}(M) \to \mathbb{S}(M)$, which is in general nonlinear, we consider the following equation:

$$\mu D\psi(x) = \psi(x) + h(\psi(x)) \quad \textit{on } M \tag{1.1}$$

where $\psi(x) \in C^\infty(M, \mathbb{S}(M))$ is a spinor. Eq(1.1) could be seen as a perturbed eigenvalue problem of the Dirac operator D. We intent to prove bifurcation occurs and analyze how it occurs both at zero and at infinity.

On general compact spin manifolds, Ammann studied the special form of the equation $D\psi = \lambda|\psi|^{p-2}\psi$ for $\lambda > 0$ and $2 < p \leq \frac{2m}{m-1}$ in [1]. This equation is known as the spinorial Yamabe equation. By solving an equivalent dual variational problem, Ammann obtained a nontrivial solution for each subcritical case $2 < p < \frac{2m}{m-1}$ and gave an existence criterion for the critical case $p = \frac{2m}{m-1}$. After Ammann's work, Takeshi Isobe considered equations $D\psi = h(x, \psi)$ for a general class of nonlinearities h in [9]. He treated both superlinear and sublinear cases using the classical variational methods. Almost at the same time, Takeshi Isobe also studied another form of equation $D\psi = \lambda\psi + |\psi|^{\frac{2}{m-1}}\psi$. In [8], he considered this equation as spinorial analogue of the Brezis-Nirenberg problem. The main theorem says that if $\lambda \notin Spec(D)$ and $\lambda > 0$, then there exists a nontrivial solution.

As for the authors' knowledge, the basic analytical problems for nonlinear Dirac equations on compact spin manifolds always focus on the existence or multiplicity results of solutions. They often be studied via classical variational methods such as linking and dual variational principle. As we all know, bifurcation plays an important role in characterizing the solutions of equations. On abstract Hilbert spaces, the researches about bifurcation are due to Rabinowitz([13],[12]), Chang and Wang([3]), K.Schmitt and Wang([14])and so on. However, on spin manifolds, there has not any bifurcation results been established up to now. For Eq(1.1) on M, assume E is the solution space with norm $\|\cdot\|$, θ denotes the trivial solution in E. (μ_k, θ) is a bifurcation point means any neighborhood of (μ_k, θ) in the space $\mathbb{R} \times E$ contains at least one nontrivial solution (μ, ψ) of (1.1). If for any $\varepsilon > 0$ and constant $A > 0$, there exists at least one solution (μ, ψ) of (1.1) such that $|\mu - \mu_k| < \varepsilon$ and $\|\psi\| > A$, then we say that (μ_k, ∞) is a bifurcation point from infinity.

In this paper, we concentrate on the bifurcation phenomenon that occurs both at zero and at infinity. A combined methods of variational and Morse theory will be used. In the first case, we transform the original strongly indefinite problem to an equivalent finite dimensional problem. Then with the help of the Gromoll-Meyer pair , critical group and Morse relations, we obtain a conclusion. In the other one, the problem is actually a spinoral analogue of Landesman-Lazer type problem(see [11]), because of our assumption (h_2) on h. By combining some a priori estimates, we get a bifurcation result.

Assume h is a potential operator, i.e. ,there exists a real valued continuously differentiable function H such that $\nabla_\psi H(\psi) = h(\psi)$. For the first case, we work on the following class of nonlinearities H:

(H_1) $H(\theta) = 0$ and $H(\psi) \geq 0$ for any $\psi \in C^1(M, \mathbb{S}(M))$.

(H_2) There exist positive numbers α, β satisfying $1 < \beta < \alpha \leq \frac{m}{m-1}$, and constants $C_1, C_2 > 0$ such that

$$C_1|\psi|^\beta \leq |\nabla_\psi H(\psi)| \leq C_2|\psi|^\alpha$$

Theorem 1.1. *Let D be the Dirac operator with $0 \notin Spec(D)$. Assume $h \in C^1(\mathbb{S}(M))$, $h(\psi) = o(\|\psi\|)$ as $\|\psi\| \to 0$ and H satisfies (H_1),(H_2). Then, for any $k \in \mathbb{Z}$ with $1/\mu_k \in Spec(D)$, (μ_k, θ) is a bifurcation point for* (1.1). *Precisely, the following occurs:*

(i) when $\mu_k > 0$, there exists a right-side neighborhood Λ of μ_k;

(ii) when $\mu_k < 0$, there exists a left-side neighborhood Λ of μ_k,

such that for each $\mu \in \Lambda \setminus \{\mu_k\}$, (1.1) *has at least* 2 *distinct nontrivial solutions in $W^{1/2,2}(M, \mathbb{S}(M))$.*

We next consider the case of bifurcation from infinity. In this case, the following hypothesis are raised:

(h_1) There exist $0 < \alpha < 1$, and constants $a > 0$, $b > 0$ such that

$$|h(\psi)| \leq a|\psi|^\alpha + b$$

(h_2) Assume $\mu \in \mathbb{R}$, and $\mu \neq 0$, for all convergent sequences $\{\omega_n\} \subset Ker(\mu D - I)$, $\omega_n \to \omega$, $\|\omega\| = 1$, all bounded sequence $\{\varphi_n\} \subset Ker(\mu D - I)^\perp$, all unbounded sequences of positive numbers $\{t_n\} \subset \mathbb{R}$, the following holds:

$$\liminf_{n\to\infty} \int_M h(t_n\omega_n + \varphi_n)\omega dx > 0$$

Theorem 1.2. *Let D be the Dirac operator with $0 \notin Spec(D)$. Assume $k \in \mathbb{Z}$. If $1/\mu_k \in Spec(D)$, h is completely continuous with $h(\psi) = o(\|\psi\|)$ as $\|\psi\| \to \infty$ in $W^{1/2,2}(M, \mathbb{S}(M))$. Then (μ_k, ∞) is a bifurcation point from infinity for* (1.1). *Moreover, if h satisfies (h_1) and (h_2), then there exists a right-side neighborhood Λ of μ_k such that for each $\mu \in \Lambda \setminus \{\mu_k\}$,* (1.1) *has at least one nontrivial solution ψ_μ with $\|\psi_\mu\| \to \infty$ as $\mu \to \mu_k$.*

2 Preliminaries

2.1 Spin structure and the Dirac operator

To introduce a spin structure on an m dimensional oriented Riemannian manifold (M, g), we should first have a look at the spin group. The spin group in dimension m, denoted by $Spin(m)$, is the nontrivial 2-fold covering of the special orthogonal group $SO(m)$. It is a compact Lie group. Denote $\xi : Spin(m) \to SO(m)$ as the covering map. If $m = 2$, $Spin(m)$ is connected and $\xi(z) = z^2$ for any $z \in Spin(2) \cong S^1 = \{z \in \mathbb{C} : |z| = 1\}$. And if $m \geq 3$, $Spin(m)$ is simply connected and ξ is the universal covering map. In particular, the following short exact sequence holds:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(m) \xrightarrow{\xi} SO(m) \longrightarrow 1$$

Given a trivialization $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$, we denote the $SO(m)$-principle bundle of (M, g) by $SO(TM)$. Then the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(m)$ satisfy:

$$\begin{aligned} g_{\alpha\alpha}(x) = id_{SO(m)} \quad & \text{for } x \in U_\alpha; \\ g_{\alpha\beta}(x)g_{\beta\alpha}(x) = id_{SO(m)} \quad & \text{for } x \in U_\alpha \cap U_\beta \\ g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = id_{SO(m)} \quad & \text{for } x \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

Let $Spin(TM)$ is a fiber bundle on M such that

$$\begin{tikzcd} Spin(TM) \arrow[rr, "\xi"] \arrow[dr, "Spin(m)"'] & & SO(TM) \arrow[dl, "SO(m)"] \\ & M & \end{tikzcd}$$

By taking $\{U_\alpha\}$ nice we can lift $g_{\alpha\beta}$ to the function $\tilde{g}_{\alpha\beta}$ which satisfy:

$$\tilde{g}_{\alpha\beta}(x) = id_{Spin(m)} \quad \text{for } x \in U_\alpha$$

$$\tilde{g}_{\alpha\beta}(x)\tilde{g}_{\beta\alpha}(x) = id_{Spin(m)} \quad \text{for } x \in U_\alpha \cap U_\beta$$

The cocycle condition $\tilde{g}_{\alpha\beta}(x)\tilde{g}_{\beta\gamma}(x)\tilde{g}_{\gamma\alpha}(x) = id_{Spin(m)}$, for $x \in U_\alpha \cap U_\beta \cap U_\gamma$, can also be satisfied if the second Stiefel-Whitney class $\omega_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes. Thus $Spin(TM)$ is a principle bundle. We can lift the $SO(m)$-principle bundle to a principle $Spin(m)$-bundle. Then we say (M, g) possesses a spin structure. A spin manifold is an oriented Riemannian manifold admitting a spin structure.

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$. The Clifford algebra $\mathcal{C}_m$ of the negative definite quadratic form $(\mathbb{R}^m, -x_1^2 -$

$... - x_m^2$) is multiplicatively generated by the elements $v_1, ..., v_m \in \mathbb{R}^m$ of a normal basis with the relations $v_i^2 = -1$, $v_i v_j + v_j v_i = 0$, where $1 \le i, j \le m$, $i \ne j$. In fact, $\mathcal{C}_m$ is a vector space. The elements 1 and $v_{i_1} v_{i_2} ... v_{i_s}$ form a basis of $\mathcal{C}_m$, where $1 \le i_1 < i_2 < ... < i_s \le m$, $1 \le s \le m$.

We denote $\mathcal{C}_m^c$ as the complexification of $\mathcal{C}_m$, i.e. $\mathcal{C}_m^c = \mathcal{C}_m \otimes_{\mathbb{R}} \mathbb{C}$. Actually, $\mathcal{C}_m^c$ coincides with the clifford algebra of the quadratic form $(\mathbb{C}^m, z_1^2 + z_2^2 + ... + z_m^2)$. Since $\mathcal{C}_2^c = M_2(\mathbb{C}) = End(\mathbb{C}^2)$ and $\mathcal{C}_1^c = \mathcal{C}_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, we have:

$$\mathcal{C}_m^c = M_2(\mathbb{C}) \otimes ... \otimes M_2(\mathbb{C}) = End(\mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2) = End(\mathbb{C}^{2^k})$$

when $m = 2k$ is even, and

$$\mathcal{C}_m^c = \mathcal{C}_{2k+1}^c = \mathcal{C}_{2k}^c \oplus \mathcal{C}_{2k}^c = End(\mathbb{C}^{2^k}) \oplus End(\mathbb{C}^{2^k})$$

when $m = 2k+1$ is odd. Let $\mathbb{S} = \mathbb{C}^{2^k}$, we call the complex vector space $\mathbb{S}$ as the spinor space. Now let us consider the representation κ_m of $\mathcal{C}_m^c$. If m is even, $\kappa_m : \mathcal{C}_m^c \to End(\mathbb{S})$; If m is odd, κ_m consists of the isomorphism $\mathcal{C}_m^c = End(\mathbb{S}) \oplus End(\mathbb{S})$ followed by the projection onto the first component, we also have $\kappa_m : \mathcal{C}_m^c \to End(\mathbb{S})$. Here, $\mathbb{S}$ is called the spinor module of $\mathcal{C}_m^c$.

Each element of $Spin(m)$ is multiplicativly generated by even number of unit vectors in $\mathbb{R}^m$. So $Spin(m) \subset \mathcal{C}_m \subset \mathcal{C}_m^c$. Let $\delta_m = \kappa_m|_{Spin(m)}$. Then $\delta_m : Spin(m) \to Aut(\mathbb{S})$ is a representation of the group $Spin(m)$. When m is even, the representation permits some greater detail. Define $f = i^k \kappa_m(v_1 \cdot ... \cdot v_m) : \mathbb{S} \to \mathbb{S}$ is an endomorphism on $\mathbb{S}$. The volume form $v_1 \cdot ... \cdot v_m$ of $\mathbb{R}^m$ belongs to the center of the even part of the Clifford algebra. Moreover, $(v_1 \cdot ... \cdot v_m)^2 = (-1)^{m/2}$. Then f can be proved to be an involution. Thus the spinor space $\mathbb{S}$ decomposes into two parts, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, where $\mathbb{S}^\pm = \{x \in \mathbb{S} \mid f(x) = \pm x\}$. Therefore, we have two inequivalent but equal dimensional representation $\delta_m^\pm : Spin(m) \to Aut(\mathbb{S}^\pm)$.

The spinor bundle of (M, g) is the complex vector bundle $\mathbb{S}$ associated to the principal bundle $Spin(TM)$ via the spinor representation δ_m. We have a unified expression:

$$\mathbb{S}(M) := Spin(TM) \times_{\delta_m} \mathbb{S}.$$

Here $\delta_m = \delta_m^+ \oplus \delta_m^-$, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ for m even.

The real Hermitian inner product on $\mathbb{S}$ is induced from a Hermitian inner product on the spinor space. We denote it by $(\cdot, \cdot)$. The sections of the spinor bundle $\mathbb{S}$ are usually called the spinors. The compatible covariant derivative on $\mathbb{S}_m$, denoted by ∇, can be locally expressed as:

$$\nabla \varphi_i = \frac{1}{4} \sum_{j,k=1}^{m} g(\nabla e_j, e_k) e_j \cdot e_k \cdot \varphi_i$$

where $\{e_j\}_{1\leq j\leq m}$ is a local positively-oriented orthonormal basis of TM and $\{\varphi_i\}_{1\leq i\leq 2^{[\frac{m}{2}]}}$ is a local spinorial frame. The symbol ∇ on the right side is Levi-Civita covariant connection on (M,g). With all these preliminaries, we now give a definition of the Dirac operator. It is a map $D: C^\infty(M,\mathbb{S}(M)) \to C^\infty(M,\mathbb{S}(M))$ defined by:

$$D\psi = \sum_{j=1}^{m} e_j \cdot \nabla_{e_j}\psi$$

where $\psi \in C^\infty(M,\mathbb{S}(M))$.

Remark 1. When m is even, the Dirac operator D splits into $D = D^+ \oplus D^-$, where $D^\pm : C^\infty(M,\mathbb{S}^\pm(M) \to C^\infty(M,\mathbb{S}^\mp(M))$.

Remark 2. If the reader want to know more details about spin manifold and Dirac operator, they could refer to [6],[5],[10].

2.2 The Dirac Spectrum

Let $Spec(D)$ denote the spectrum of Dirac operator D on compact spin manifold M. It is a closed subset of $\mathbb{R}$ consisting of a two-sided unbounded discrete sequence of eigenvalues. Moreover, $Spec(D)$ is symmetric about the origin if $m \neq 3(or\ 4)$. For every $j \in \mathbb{Z}$, $\lambda_j \in Spec(D)$ is an isolated eigenvalue of finite multiplicity, ranged by $-\infty < \cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots < +\infty$(counted in multiplicity). The eigenspaces of D form a complete orthonormal decomposition of $L^2(M,\mathbb{S}(M))$, i.e.,

$$L^2(M,\mathbb{S}(M)) = \overline{\bigoplus_{\lambda_j\in Spec(D)} Ker(D-\lambda_j I)}.$$

We always assume there are no harmonic spinors on M in this paper. A harmonic spinor is a spinor field which lies in the kernel of D. The existence of it depends on various factors, such as spin structure. Take an example of manifold $\mathbb{T}^n$ with flat metric. The set of harmonic spinor is empty when we choose any spin structure on $\mathbb{T}^n$ except for the trivial spin structure. In fact, there exist many manifolds which contain no harmonic spinors.

2.3 $H^{1/2}$ spinors

Since we will prove Theorem 1.1 by a variational method, it is necessary to give a suitable function space.

Recall that the Dirac operator on a compact spin manifold M is essentially self-adjoint in $L^2(M,\mathbb{S}(M))$. There exists a complete orthonormal basis $\{\psi_j\}$,

$j \in \mathbb{Z} \setminus \{0\}$, of $L^2(M, \mathbb{S}(M))$ consisting of the eigenspinors of the Dirac operator $D : D\psi_j = \lambda_j \psi_j$, where $|\lambda_j| \to \infty$ as $j \to \infty$. Suppose $\lambda_k \in Spec(D)$ is any fixed eigenvalue. Let us consider the self-adjoint operator $D - \lambda_k$. It is not difficult to see that each ψ_j is also an eigenspinor of $D - \lambda_k$ with respect to the eigenvalue $\lambda_j - \lambda_k$. Moreover, the complete orthonormal basis $\{\psi_j\}$ of $L^2(M, \mathbb{S}(M))$ can be decomposed into three parts:$\{\psi_j\} = \{\psi_j^-\}_{\lambda_j < \lambda_k} \cup \{\psi_j^0\}_{\lambda_j = \lambda_k} \cup \{\psi_j^+\}_{\lambda_j > \lambda_k}$. We choose ψ_j^-, ψ_j^0, ψ_j^+ as follows: $(D - \lambda_k)\psi_j^- = (\lambda_j - \lambda_k)\psi_j^-$ with $\lambda_j - \lambda_k < 0$; $(D - \lambda_k)\psi_j^0 = 0$; $(D - \lambda_k)\psi_j^+ = (\lambda_j - \lambda_k)\psi_j^+$ with $\lambda_j - \lambda_k > 0$.

Now we define an unbounded operator $|D - \lambda_k|^{1/2} : L^2(M, \mathbb{S}(M)) \to L^2(M, \mathbb{S}(M))$ by

$$|D - \lambda_k|^{1/2}\psi = \sum_{j \in \mathbb{Z} \setminus \{0\}} |\lambda_j - \lambda_k|^{1/2} a_j \psi_j$$

where $\psi = \sum a_j \psi_j \in L^2(M, \mathbb{S}(M))$. We denote $H^{1/2}(M, \mathbb{S}(M))$ as the domain of the operator $|D - \lambda_k|^{1/2}$. Thus $\psi = \sum a_j \psi_j \in H^{1/2}(M, \mathbb{S}(M))$ if and only if $\sum |\lambda_j - \lambda_k| \cdot |a_j|^2 < \infty$. We define inner product on $H^{1/2}(M, \mathbb{S}(M))$ as follows:

$$(\psi, \varphi)_{1/2,2} := \big(|D - \lambda_k|^{1/2}\psi,\ |D - \lambda_k|^{1/2}\varphi\big)_2 + (\psi, \varphi)_2$$

where $(\psi, \varphi)_2 = \int_M (\psi, \varphi) dx$ is the L^2-inner product on spinors. In order to unify the notations with solution space E, which has mentioned in the introduction, we use the $\| \cdot \|$ to denote the norm of elements in $H^{1/2}(M, \mathbb{S}(M))$. The norm in $L^p(M, \mathbb{S}(M))$ is denoted by $\| \cdot \|_p$ in our paper. $H^{1/2}(M, \mathbb{S}(M))$ coincides with the Sobolev space $W^{1/2,2}(M, \mathbb{S}(M))$. We have continuous embedding $H^{1/2}(M, \mathbb{S}(M)) \hookrightarrow L^p(M, \mathbb{S}(M))$ for $1 \le p \le \frac{2m}{m-1}$. It is compact when $1 \le p < \frac{2m}{m-1}$ (details refer to [6]). Remark that, for $\psi^- = \sum_{\lambda_j < \lambda_k} a_j \psi_j^-$,

$$\|\psi^-\|^2 = \sum_{\lambda_j < \lambda_k} \big(\lambda_k - \lambda_j + 1\big)|a_j|^2,$$

and for $\psi^+ = \sum_{\lambda_j > \lambda_k} a_j \psi_j^+$,

$$\|\psi^+\|^2 = \sum_{\lambda_j > \lambda_k} \big(\lambda_j - \lambda_k + 1\big)|a_j|^2,$$

hence,

$$c_k \|\varphi\|^2 \ge \|\varphi\|_2^2 \quad \text{for all } \varphi \notin \ker(D - \lambda_k) \tag{2.1}$$

where

$$c_k = \max\Big\{\frac{1}{1 + |\lambda_j - \lambda_k|}; \quad \lambda_j \neq \lambda_k\Big\}.$$

It is clear that $0 < c_k < 1$.

3 Bifurcation from zero

In this section we aim to prove Theorem1.1. Variational methods will be used to prove this bifurcation theorem. There are 2 main steps consist the whole proof. The first one is the so-called Lyapunov-Schmidt reduction. This method reduce Eq(1.1) to an equivalent finite-dimensional problem. In the second step, we treat the new problem as variational problem and use the Morse theory. Gromoll-Meyer pair will play an important role.

3.1 The Lyapunov-Schimidt reduction

Let $\lambda = 1/\mu$, then the original equation (1.1) is equivalent to the following:

$$D\psi = \lambda\psi + \lambda h(\psi) \tag{3.1}$$

Assume $X = Ker(D - \lambda_k I)$. Since $\lambda_k \in Spec(D)$ is of finite multiplicity, we could suppose $dimX = n$ and identify X with $\mathbb{R}^n$. Let $X^\perp$ denote the orthogonal complement of X in $H^{1/2}(M, \mathbb{S}(M))$, P, $P^\perp$ the orthogonal projectors of $H^{1/2}$ onto X, $X^\perp$, respectively.

Then (3.1) is equivalent to the following pair of equations:

$$D\omega_1 = \lambda\omega_1 + \lambda P h(\omega_1 + \omega_2) \tag{3.2}$$

$$D\omega_2 = \lambda\omega_2 + \lambda P^\perp h(\omega_1 + \omega_2) \tag{3.3}$$

where $\psi = \omega_1 + \omega_2$, $\omega_1 \in X$, $\omega_2 \in X^\perp$. Now, we define:

$$F(\lambda, \omega_1, \omega_2) = (D - \lambda I)\omega_2 - \lambda P^\perp h(\omega_1 + \omega_2).$$

Here F is a continuously differentiable functional in a neighborhood of the point $(\lambda_k, 0, 0) \in \mathbb{R} \times X \times X^\perp$. Moreover, $F(\lambda_k, 0, 0) = 0$ and the Fréchet derivative of F with respect to ω_2, $F_{\omega_2}(\lambda_k, 0, 0) = D - \lambda_k I$ is an isomorphism of $X^\perp$ to $X^\perp$. According to the implicit function theorem, there exists a neighborhood $\mathcal{O}$ of $(\lambda_k, 0) \in \mathbb{R} \times X$ and a functional $\varphi \in C^1(\mathcal{O}, X^\perp)$ such that the solutions of the equation $F(\lambda, \omega_1, \omega_2) = 0$ near the point $(\lambda_k, 0, 0)$ are given by $(\lambda, \omega_1, \varphi(\lambda, \omega_1))$ for $(\lambda, \omega_1) \in \mathcal{O}$. Now we replace ω_1 with ω. With the previous argument, we conclude that solving the equation (1.1) is equivalent to solving the finite-dimensional problem (3.2) with $\omega_2 = \varphi(\lambda, \omega)$.

Since $D - \lambda_k I$ is an isomorphism of $X^\perp$ onto $X^\perp$, for (λ, ω) in a small neighborhood of (λ_k, θ), we have the following expression:

$$\varphi(\lambda, \omega) = \lambda(D - \lambda I)^{-1} P^\perp h(\omega + \varphi).$$

Since $h(\omega+\varphi) = o(\|\omega+\varphi\|)$ as $\|\omega+\varphi\| \to 0$. It is not difficult to see that:

$$\varphi(\lambda,\omega) = o(\|\omega+\varphi\|) \quad as\ \|\omega+\varphi\| \to 0$$

Then we can conclude:

$$\varphi(\lambda,\omega) = o(\|\omega\|) \quad as\ \|\omega\| \to 0$$

From a variational point of view, for fixed $\lambda \in \mathbb{R}$ the solutions of Eq(1.1) can be obtained as critical points of the Euler-Lagrange functional:

$$\mathcal{L}_\lambda(\psi) = \frac{1}{2}\int_M (D\psi,\psi)dx - \frac{\lambda}{2}\int_M (\psi,\psi)dx - \lambda\int_M H(\psi)dx \tag{3.4}$$

where H denote the primitive of h with $H(\theta) = 0$. If ψ is a critical point of $\mathcal{L}_\lambda$, by applying the Lyapunov-Schmidt reduction we can write $\psi = \omega + \varphi$, where $\omega \in X$ and $\varphi \in X^\perp$. Substitute $\psi = \omega + \varphi$ for ψ in (3.4), we obtain the following functional:

$$\mathcal{J}_\lambda(\omega) = \frac{(\lambda_k-\lambda)}{2}\|\omega\|^2 + \frac{1}{2}\int_M (\varphi,(D-\lambda I)\varphi)dx - \lambda\int_M H(\omega+\varphi)dx$$

$\mathcal{J}_\lambda \in C^2(X,\mathbb{R})$. The form of $\mathcal{J}_\lambda$ implies that the critical points of $\mathcal{J}_\lambda(\cdot)$ near $\omega = \theta$ are solutions of (3.2). Moreover $\omega = \theta$ is a critical point of $\mathcal{J}_\lambda(\cdot)$ for all $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Choose $\Omega \subset X$ to be a compact neighborhood of θ. $\mathcal{J}_\lambda \in C^2(\Omega,\mathbb{R})$. In order to prove (μ_k,θ) is a bifurcation point, we need only analyze the critical points of $\mathcal{J}_\lambda(\cdot)$ in Ω for λ near $\lambda_k = 1/\mu_k \in Spec(D)$.

3.2 Gromoll-Meyer Pairs

As we mentioned above, Gromoll-Meyer pair plays an important role to solve our finite-dimensional problem. We first give a definition of the Gromoll-Meyer pair for the functional $\mathcal{J}_\lambda$ with $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.

Definition 3.1. Let K_λ be the whole critical point set of $\mathcal{J}_\lambda$ and S_λ be a subset of K_λ. A pair of topological sets (W, W_-) is called a Gromoll-Meyer pair for S_λ with respect to a pseudo-gradient vector filed V of $\mathcal{J}_\lambda$, if

(1) W is a closed neighborhood of S_λ possessing the mean value property, i.e., for any $t_1 < t_2$, $\eta(t_i) \in W$, $i = 1, 2$, implies $\eta(t) \in W$ for all $t \in [t_1, t_2]$, where $\eta(t)$ is the decreasing flow with respect to V. $W \bigcap K_\lambda = S_\lambda$ and $W \bigcap (\mathcal{J}_\lambda)_c = \emptyset$ for some c;

(2) $W_- := \{x \in W \mid \eta(t,x) \notin W, \forall t > 0\}$ is closed in W;

(3) W_- is a piecewise submanifold, and the flow η is transversal to W_-.

A subset $S_\lambda \subset K_\lambda$ is said to be a dynamically isolated critical set if there exist a closed neighborhood $\mathcal{O}_\lambda$ of S_λ and regular values a_1, a_2 of $\mathcal{J}_\lambda$ such that $\mathcal{O}_\lambda \subset \mathcal{J}_\lambda^{-1}([a_1,a_2])$ and $\overline{\cup_{t\in\mathbb{R}}\eta(\mathcal{O}_\lambda,t)} \bigcap K_\lambda \bigcap \mathcal{J}_\lambda^{-1}([a_1,a_2]) = S_\lambda$, where η is the flow with respect to a pseudo-gradient vector field V of $\mathcal{J}_\lambda$. More details about the dynamically isolated critical set can refer to [4].

Before constructing a Gromoll-Meyer pair, let us choose a suitable dynamically isolated critical set. If c is an isolated critical level, i.e., there is no critical points on the levels $[c-\varepsilon, c+\varepsilon] \setminus \{0\}$ for some $\varepsilon > 0$, then the set $S_\lambda = K_\lambda \bigcap \mathcal{J}_\lambda^{-1}(c)$ is a dynamically isolated critical set. In the following, we will consider the trivial critical point of $\mathcal{J}_{\lambda_k}$, $\omega = \theta$, where $\lambda_k \in Spec(D)$ and prove $S_{\lambda_k} = \{\theta\}$ is a dynamically isolated critical set of $\mathcal{J}_{\lambda_k}$.

When $\lambda_k > 0$, according to the space decomposition $H^{1/2} = X^- \oplus X \oplus X^+$, write $\varphi(\lambda_k,\omega) = \varphi^-(\lambda_k,\omega) + \varphi^+(\lambda_k,\omega)$, where $\varphi^-(\lambda_k,\omega) \in X^-$, $\varphi^+(\lambda_k,\omega) \in X^+$. For simplicity, let us write these notations as φ, φ^+ and φ^-. Then:

$$\begin{aligned}
\mathcal{J}_{\lambda_k}(\omega) =& \frac{1}{2}\int_M \big(\varphi,\ (D-\lambda_k I)\varphi\big)dx - \lambda_k \int_M H(\omega+\varphi)dx \\
\le& \frac{1}{2}\big(\|\varphi^+\|^2 - \|\varphi^+\|_2^2 + \|\varphi^-\|_2^2 - \|\varphi^-\|^2\big) - \lambda_k C_1 \int_M |\omega+\varphi|^{\beta+1}dx \\
\le& \frac{1}{2}\|\varphi^+\|^2 + \frac{1}{2}\|\varphi^-\|_2^2 - \lambda_k C_1 \int_M |\omega+\varphi|^{\beta+1}dx \\
\le& C\|\varphi\|^2 - \lambda_k C_1 \int_M |\omega+\varphi|^{\beta+1}dx\,.
\end{aligned}$$

Next let us give some more estimates to $\|\varphi\|^2$ and $\int_M |\omega+\varphi|^{\beta+1}dx$.

We have known that φ satisfies the equation $(D-\lambda_k I)\varphi = \lambda_k P^\perp h(\omega+\varphi)$, then on one hand,

$$\begin{aligned}
&\int_M \big((D-\lambda_k I)\varphi, \varphi^+ - \varphi^-\big)dx = \lambda_k \int_M P^\perp h(\omega+\varphi)(\varphi^+ - \varphi^-)dx \\
&\le \lambda_k \Big(\int_M \big|P^\perp h(\omega+\varphi)\big|^2 dx\Big)^{1/2} \cdot \Big(\int_M |\varphi^+ - \varphi^-|^2 dx\Big)^{1/2} \\
&\le \lambda_k \|P^\perp h(\omega+\varphi)\|_2 \cdot \|\varphi^+ - \varphi^-\|_2 \le \lambda_k \|h(\omega+\varphi)\|_2 \cdot \|\varphi^+ - \varphi^-\|_2
\end{aligned}$$

and

$$\|h(\omega+\varphi)\|_2 = \Big(\int_M |h(\omega+\varphi)|^2\Big)^{1/2} \le C_2\Big(\int_M |\omega+\varphi|^{2\alpha}dx\Big)^{\frac{1}{2\alpha}\cdot\alpha} = C_2\|\omega+\varphi\|_{2\alpha}^{\alpha}$$

We have:

$$
\begin{aligned}
&\int_M \big((D-\lambda_k I)\varphi, \varphi^+ - \varphi^-\big)dx \le \lambda_k C_2 \|\omega+\varphi\|_{2\alpha}^{\alpha} \cdot \|\varphi^+ - \varphi^-\|_2 \\
&\le C\big(\|\omega\|_{2\alpha}^{\alpha} + \|\varphi\|_{2\alpha}^{\alpha}\big) \cdot \big(\|\varphi^+\|_2 + \|\varphi^-\|_2\big) \le C\|\omega\|^{\alpha} \cdot \|\varphi\| + C\|\varphi\|^{\alpha+1}
\end{aligned}
$$

On the other hand,

$$
\int_M \big((D-\lambda_k I)\varphi, \varphi^+ - \varphi^-\big)dx = \|\varphi\|^2 - \|\varphi\|_2^2 \ge (1-c_k)\|\varphi\|^2
$$

since $\|\varphi\|_2^2 \le c_k\|\varphi\|^2$ by (2.1) with $c_k < 1$. From the two sides above, we have:

$$
(1-c_k)\|\varphi\| \le C\|\omega\|^{\alpha} + C\|\varphi\|^{\alpha} \tag{3.5}
$$

where $\varphi = \varphi(\lambda_k, \omega) \in X^{\perp}$ and $\varphi(\lambda_k,\omega) \ne \theta$. When $\omega \to \theta$ in $H^{1/2}$, it occurs that $\varphi = \varphi(\lambda_k,\omega) = o(\|\omega\|)$. If φ is small enough, inequality (3.5) implies $\|\varphi\| \le C\|\omega\|^{\alpha}$. Then we have:

$$
\begin{aligned}
\mathcal{J}_{\lambda_k}(\omega) &\le C\|\varphi\|^2 - \lambda_k C_1 \int_M |\omega+\varphi|^{\beta+1}dx \\
&\le C\|\omega\|^{2\alpha} - \lambda_k C_1 \|\omega+\varphi\|_{\beta+1}^{\beta+1}
\end{aligned}
$$

Since $\varphi = \varphi(\lambda_k,\omega) = o(\|\omega\|)$, given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\omega\| < \delta$ in X, $\|\varphi\| < \epsilon\|\omega\|$. Note that $H^{1/2} \hookrightarrow L^{\beta+1}$ continuously, then $\|\varphi\|_{\beta+1} \le \gamma\|\varphi\| \le \epsilon\cdot\gamma\|\omega\|$. The kernel space X is finite dimensional, so the norms $\|\cdot\|_{\beta+1}$ and $\|\cdot\|$ are equivalent. Assume $\|\omega\|_{\beta+1} \le C_3\|\omega\|$ and let ϵ small enough. We have:

$$
\|\omega+\varphi\|_{\beta+1} \ge \|\omega\|_{\beta+1} - \|\varphi\|_{\beta+1} \ge C_3\|\omega\| - \epsilon\gamma\|\omega\| \ge \frac{1}{4}C_3\|\omega\|
$$

Then

$$
\mathcal{J}_{\lambda_k}(\omega) \le C\|\omega\|^{2\alpha} - C_4\|\omega\|^{\beta+1}
$$

where C_4 is a proper constant depend on C_1, C_3 and λ_k. Since $\alpha > \beta > 1$, it is easy to see $\mathcal{J}_{\lambda_k}(\omega) < 0$ when $\omega \ne \theta$ and ω is sufficiently close to θ. Thus we obtain that θ is an isolated local maximum of the functional $\mathcal{J}_{\lambda_k}$.

When $\lambda_k < 0$, replace $\mathcal{J}_{\lambda_k}$ with $-\mathcal{J}_{\lambda_k}$. A similar procedure will induce that θ is an isolated local minimum.

Since θ is an isolated maximum or minimum point for the functional $\mathcal{J}_{\lambda_k}$ on the set $\Omega \subset X$, then 0 is an isolated critical level of $\mathcal{J}_{\lambda_k}$ when we choose

Ω nicely. Under this situation, $S_{\lambda_k} = \{\theta\}$ is a dynamically isolated critical point set.

We now turn our attention to constructing a Gromoll-Meyer pair for $S_{\lambda_k} = \{\theta\}$, the dynamically isolated critical set of $\mathcal{J}_{\lambda_k}(\cdot)$. The previous assumption that Ω is compact in $X \cong \mathbb{R}^n$ shows $\mathcal{J}_\lambda(\cdot)$ trivially satisfies the Palais-Smale condition on Ω for all $\lambda \neq 0$. According to the existence result of Gromoll-Meyer pair in [2] and [4], we give the following lemma.

Lemma 3.2. *For the dynamically isolated critical set $S_{\lambda_k} = \{\theta\}$, there is a Gromoll-Meyer pair of the functional $\mathcal{J}_{\lambda_k}(\cdot)$ with respect to the pseudo-gradient vector field $d\mathcal{J}_{\lambda_k}$.*

Proof. Choose $r > 0$ small such that θ is the unique critical point in $B_r(\theta) = \{\omega \in \Omega \mid \|\omega\| \leq r\}$. Since $\mathcal{J}_{\lambda_k}$ satisfies the (PS) condition, we define $\kappa = \inf_{\omega \in B_r \setminus B_{r/2}} \|d\mathcal{J}_k(\omega)\|$, then $\kappa > 0$; Take ξ such that $0 < \xi < \frac{\kappa}{2r}$. Let $f(\omega) = \mathcal{J}_{\lambda_k}(\omega) + \xi\|\omega\|^2$, $\omega \in B_r$. Define:

$$W = \mathcal{J}_{\lambda_k}^{-1}[-\gamma, \gamma] \cap f_\sigma, \quad W_- = \mathcal{J}_k^{-1}(-\gamma) \cap W$$

where γ, σ satisfy the following conditions. Assume $\varepsilon > 0$ such that 0 is the unique critical value of $\mathcal{J}_{\lambda_k}|_{B_r}$ in $[-\varepsilon, \varepsilon]$.

$$0 < \gamma < \min\left\{\varepsilon, \frac{3\xi r}{8}\right\}, \quad \text{and} \quad \frac{\xi r^2}{4} + \gamma < \sigma < \xi r^2 - \gamma$$

$$B_{r/2} \cap \mathcal{J}_{\lambda_k}^{-1}[-\gamma, \gamma] \subset W \subset B_r \cap \mathcal{J}_{\lambda_k}^{-1}[-\varepsilon, \varepsilon] \tag{3.6}$$

$$\mathcal{J}_{\lambda_k}^{-1}[-\gamma, \gamma] \cap f^{-1}(\sigma) \subset B_r \setminus B_{r/2} \tag{3.7}$$

$$(df(\omega), d\mathcal{J}_{\lambda_k}(\omega)) > 0, \qquad \text{for all } \omega \in B_r \setminus intB_{r/2} \tag{3.8}$$

We first claim W satisfies the mean value property. Let $\eta \in C([0,1] \times \Omega, \Omega)$ be the negative gradient flow for $\mathcal{J}_{\lambda_k}$ with respect to $d\mathcal{J}_{\lambda_k}$. Without loss of generality, let us assume $\eta(0)$, $\eta(t) \in W$. Then we wish to prove $\eta(s) \in W$ for every $s \in [0, t]$. Define:

$$T = \sup\left\{s \in [0, t] \mid \eta(\tau) \in W,\ 0 < \tau \leq s\right\}$$

Suppose $T < t$ on the contrary. Since $\mathcal{J}_{\lambda_k} \circ \eta(\cdot)$ is decreasing on $[0, t]$, we have

$$-\gamma \leq \mathcal{J}_{\lambda_k}(\eta(t)) \leq \mathcal{J}_{\lambda_k}(\eta(T)) \leq \mathcal{J}_{\lambda_k}(\eta(0)) \leq \gamma \tag{3.9}$$

By the (3.6) and the definition of T, we have $\eta(T)$ does not belong to $int(B_{r/2})$, and $\eta(T) \in B_r \setminus int(B_{r/2})$. Then by condition (3.8), we have:

$$(f \circ \eta)' = -(d\mathcal{J}_{\lambda_k}(\eta(T)), df(\eta(T))) < 0 \tag{3.10}$$

From (3.9), (3.10) and the fact $f(\eta(T)) \leq \sigma$, we know there exists a small right-side neighborhood Λ of T such that

$$\eta(\tau) \in \mathcal{J}_{\lambda_k}^{-1}[-\gamma, \gamma], \quad \text{and} \quad \eta(\tau) \in f_\sigma$$

for any $\tau \in \Lambda$ and $\tau < t$. This contradicts with the definition of T.

Next we show $W_- = \{\omega \in W \mid \eta(t, \omega) \notin W, \ for\ all \ \ t > 0\}$ is closed in W. Let us denote $W^- = \{\omega \in W \mid \eta(t, \omega) \notin W, \ for\ all \ \ t > 0\}$. It is easy to see $W^- \subset \partial W$. And we give the components of ∂W as below:

$$\partial W = W_- \cup (\mathcal{J}_{\lambda_k}^{-1}(\gamma) \cap int(f_\sigma)) \cup (f^{-1}(\sigma) \cap (W \setminus W_-)) \tag{3.11}$$

For every $\omega \in \mathcal{J}_{\lambda_k}^{-1}(\gamma) \cap int(f_\sigma)$, and ω does not belong to W^-. Thus $\omega \in \mathcal{J}_{\lambda_k}^{-1}(\gamma) \cap int(f_\sigma) \cap W^- = \emptyset$. Given $\omega \in (f^{-1}(\sigma) \cap (W \setminus W_-))$, we have $\omega \in B_r \setminus int(B_{r/2})$ by the condition (3.7). Due to (3.8) and the fact ω does not belong to W_-, we have:

$$(f \circ \eta)'(0, \omega) = -(d\mathcal{J}_{\lambda_k}(\omega), df(\omega)) < 0,$$

$$f(\eta(0, \omega)) = f(\omega) = \sigma.$$

Then $\omega \notin W^-$ and $(f^{-1}(\sigma) \cap (W \setminus W_-)) \cap W^- = \emptyset$. By (3.11), we know $W^- \subset W_-$. On the orther hand, it is obviously $W_- \subset W^-$. Thus $W^- = W_-$.

Therefore (W, W_-) is a Gromoll-Meyer pair for θ with respect to $d\mathcal{J}_{\lambda_k}$.

□

For later use, we also need to introduce the critical groups of a function J at its isolated critical point p. Let $c = J(p)$. Then the q-th critical group is defined as:

$$C_q(J, p; F) = H_q(J_c \cap \mathcal{O}_p, (J_c \backslash \{p\}) \cap \mathcal{O}_p; F).$$

Here $H_q(A, B)$ is the q-th relative singular homology group of the topological pair (A, B) with coefficients in a field F. $\mathcal{O}_p$ is a neighborhood of p which contains no other critical points.

When $\lambda_k > 0$, θ is an isolated local maximum of $\mathcal{J}_{\lambda_k}$. We have:

$$C_q(\mathcal{J}_{\lambda_k}, \theta; F) = \begin{cases} F, & q = n \\ 0, & q \neq n. \end{cases}$$

Simplify the above expression as $C_*(\mathcal{J}_{\lambda_k}, \theta; F) = \delta_{*,n}F$. When $\lambda_k < 0$, θ is an isolated local minimum and $C_*(\mathcal{J}_{\lambda_k}, \theta; F) = \delta_{*,0}F$.

Take advantage of the relationships between the Gromoll-Meyer pair for θ and the critical groups of θ. We obtain:

$$H_*(W, W_-; F) = C_*(\mathcal{J}_{\lambda_k}, \theta; F) = \begin{cases} \delta_{*,n}F & \lambda_k > 0 \\ \delta_{*,0}F & \lambda_k < 0. \end{cases}$$

To proceed further, we need to prove the following Lemma:

Lemma 3.3. *Let K_λ be the critical point set of $\mathcal{J}_\lambda$ and $S_\lambda = W \bigcap K_\lambda$. Then there exists a neighborhood $I_k \subset \mathbb{R}$ of λ_k such that for every $\lambda \in I_k \setminus \{\lambda_k\}$, (W, W_-) is also a Gromoll-Meyer pair for S_λ with respect to a certain pseudo-gradient vector field of $\mathcal{J}_\lambda$.*

Proof. Given $\varepsilon > 0$ small, there exists $\delta > 0$ such that for every $\lambda \in \mathbb{R}$ satisfying $|\lambda - \lambda_k| \leq \delta$, $\|\mathcal{J}_\lambda - \mathcal{J}_{\lambda_k}\|_{C^1(W)} \leq \varepsilon$. For ε small there exist $r_2 > r_1 > 0$ depending on λ such that we have $B(\theta, r_1) \subset B(\theta, r_2) \subset \mathrm{int}(W)$ and

$$\varpi = \inf \Big\{ \|d\mathcal{J}_\lambda(\omega)\| \ \mid \ \omega \in W \setminus B(\theta, r_1) \Big\} > 0$$

Since for $\omega \in W \setminus B(\theta, r_1)$, $\|d\mathcal{J}_\lambda(\omega)\| > 0$. It is not difficult to see that $S_\lambda = W \bigcap K_\lambda \subset B(\theta, r_1)$. Define $\rho \in C^1(\Omega, \mathbb{R})$ satisfying $0 \leq \rho \leq 1$ and

$$\rho(w) = \begin{cases} 1, & \omega \in \overline{B(\theta, r_1)} \\ 0 \leq \rho \leq 1, & \omega \in \overline{B(\theta, r_2)} \setminus B(\theta, r_1) \\ 0, & \text{others.} \end{cases}$$

Then set a vector field of $\mathcal{J}_\lambda$ on Ω as $V(\omega) = \frac{4}{3}[\rho(\omega) d\mathcal{J}_\lambda + (1 - \rho(\omega)) d\mathcal{J}_{\lambda_k}]$ Let $0 < \varepsilon < \frac{1}{4}\varpi$. For $\omega \in W \setminus B(\theta, r_1)$, we have:

$$\begin{aligned} \|V(\omega)\| \leq & \frac{4}{3}\rho(\omega)\|d\mathcal{J}_\lambda(\omega)\| + \frac{4}{3}(1 - \rho(\omega))\|d\mathcal{J}_{\lambda_k}(\omega)\| \\ \leq & \frac{4}{3} \parallel d\mathcal{J}_\lambda(\omega) \parallel + \varepsilon \ \leq 2\|d\mathcal{J}_\lambda(\omega)\| \end{aligned}$$

and

$$(V(\omega), d\mathcal{J}_\lambda(\omega)) \ \geq \frac{4}{3}\|d\mathcal{J}_\lambda(\omega)\|^2 - \frac{4}{3}\varepsilon\|d\mathcal{J}_\lambda(\omega)\| \geq \|d\mathcal{J}_\lambda(\omega)\|^2$$

If $\omega \in B(\theta, r_1)$, $V(\omega) = \frac{4}{3} d\mathcal{J}_\lambda(\omega)$. Therefore $V(\omega)$ is a pseudo-gradient vector field of $\mathcal{J}_\lambda$.

Since $V(\omega) = \frac{4}{3}d\mathcal{J}_{\lambda_k}(\omega)$ on $W \setminus B(\theta, r_2)$, (W, W_-) is a Gromoll-Meyer pair of $\mathcal{J}_{\lambda_k}$, the negative gradient flow η_λ remain the same as η_{λ_k} outside $B(\theta, r_2)$. It is clearly that W satisfies the mean value property with respect to the flow η_λ. Moreover, W_- keeps invariant when λ changes. So (W, W_-) is a Gromoll-Meyer pair of $\mathcal{J}_\lambda$ with respect to $V(\omega)$. Thus we verify the conclusion. $\square$

We now complete the proof of Theorem1.1.

Proof of Theorem 1.1. We first consider the case $\lambda_k = 1/\mu_k \in Spec(D)$ and $\lambda_k > 0$. Recall the functional expression of $\mathcal{J}_\lambda(\omega)$. Since $\varphi(\lambda, \omega) = o(\|\omega\|)$ for λ near λ_k and ω near θ, we can see that the dominating term in $\mathcal{J}_\lambda(\omega)$ near $\omega = \theta$ is $\frac{1}{2}(\lambda_k - \lambda)\|\omega\|^2$.

For $\mu > \mu_k > 0$, $\lambda = 1/\mu < 1/\mu_k = \lambda_k$, θ is an isolated minimum of $\mathcal{J}_\lambda$ and the Morse index $ind(\mathcal{J}_\lambda) = 0$. Then $C_*(\mathcal{J}_\lambda, \theta) = \delta_{*,0}F$.

For $0 < \mu < \mu_k$, $\lambda = \frac{1}{\mu} > \frac{1}{\mu_k} = \lambda_k$, θ is an isolated maximum of $\mathcal{J}_\lambda$, and $ind(\mathcal{J}_\lambda) = n$. Then $C_*(\mathcal{J}_\lambda, \theta) = \delta_{*,n}F$.

Because (W, W_-) is a Gromoll-Meyer pair of $\mathcal{J}_{\lambda_k}$ at θ, we have known that $H_*(W, W_-; F) = \delta_{*,n}F$. By Lemma 3.3, (W, W_-) is also a Gromoll-Meyer pair of $\mathcal{J}_\lambda$ for the critical set S_λ when λ close to λ_k. Then we find if $\lambda < \lambda_k$, $H_*(W, W_-; F) \neq C_*(\mathcal{J}_\lambda, \theta)$. This means that there exists a nontrivial critical point ω_λ of $\mathcal{J}_\lambda$ in W (actually in S_λ).

Since $\mathcal{J}_{\lambda_k}(\theta) = 0$ and θ is an isolated local maximum, there exists $\varepsilon > 0$ such that $\mathcal{J}_{\lambda_k}(\omega) \leq -2\varepsilon$ on $\partial B_{\hat{r}}(\theta)$, where $\hat{r} > 0$ and small enough such that $B_{\hat{r}}(\theta) \subset W$. The functionals $\mathcal{J}_\lambda$ is continuous with respect to λ. Therefore for λ near λ_k, $\mathcal{J}_\lambda(\omega) \leq -\varepsilon$ on $\partial B_{\hat{r}}(\theta)$. However, $\mathcal{J}_\lambda(\omega) = \frac{\lambda_k - \lambda}{2}\|\omega\|^2 + o(\|\omega\|^2)$ as $\omega \to \theta$. Hence, $\mathcal{J}_\lambda(\omega) > 0$ if $\lambda < \lambda_k$ and $\omega \neq \theta$ is sufficiently small. In particular, there exists a $\rho \in (0, \hat{r})$ such that

$$\mathcal{J}_\lambda(\omega) \geq \frac{\lambda_k - \lambda}{4}\rho^2 > 0 \ \text{ for } \omega \in \partial B_\rho(\theta)$$

With all these facts, we can choose ω_λ to be an isolated maximizer of $\mathcal{J}_\lambda$ in $B_{\hat{r}}(\theta) \subset W$. The critical groups $C_*(\mathcal{J}_\lambda, \omega_\lambda) = \delta_{*,n}F$. If ω_λ is not isolated, it means W contains infinitely many critical points. There is nothing to say.

Let $deg(d\mathcal{J}_\lambda, W, \theta)$ denote the Leray-Schauder degree of $d\mathcal{J}_\lambda$ on W, and $index(d\mathcal{J}_\lambda, \omega_\lambda)$ the index of the isolated zero point ω_λ of $d\mathcal{J}_\lambda$. Then by the relations between Leray-Schauder degree with Morse theory, we have the following:

$$deg(d\mathcal{J}_\lambda, W, \theta) = \sum_{q=0}^{\infty}(-1)^q \text{rank} H_q(W, W_-)$$

$$index(d\mathcal{J}_\lambda, \omega_\lambda) = \sum_{q=0}^{\infty}(-1)^q \mathrm{rank} C_q(\mathcal{J}_\lambda, \omega_\lambda).$$

Then we have

$$deg(d\mathcal{J}_\lambda, W, \theta) = (-1)^n;$$

$$index(d\mathcal{J}_\lambda, \omega_\lambda) = (-1)^n, \quad index(d\mathcal{J}_\lambda, \theta) = (-1)^0$$

Obviously, $deg(d\mathcal{J}_\lambda, W, \theta) \neq index(d\mathcal{J}_\lambda, \theta) + index(d\mathcal{J}_\lambda, \omega_\lambda)$. By the topological degree theory, we obtain a second critical point for $\mathcal{J}_\lambda$ in W. Then Eq(1.1) has at least 2 nontrivial solutions.

As for the case when $\lambda_k = 1/\mu_k \in Spec(D)$ and $\lambda_k < 0$, $H_*(W, W_-; F) \neq C_*(\mathcal{J}_\lambda, \theta)$ if $\lambda > \lambda_k$ and there exists a nontrivial critical point ω_λ of $\mathcal{J}_\lambda$ in W. By using the similar analysis method as before, we can choose ω_λ to be a minimizer. Moreover, $C_*(\mathcal{J}_\lambda, \omega_\lambda) = \delta_{*,0}F$. A simple calculation induce $deg(d\mathcal{J}_\lambda, W, \theta) \neq index(d\mathcal{J}_\lambda, \theta) + index(d\mathcal{J}_\lambda, \omega_\lambda)$. Hence there must be a second solution. The proof is completed. $\square$

4 Bifurcation from infinity

In this section, we prove Theorem 1.2. Under the assumption (h_2), equation (1.1) is of Landesman-Lazer type. We will give some a priori estimates by using a method in [11]. Then apply a bifurcation theorem from infinity to obtain our results. As in the previous section, assume $X = Ker(\mu_k D - I)$, then$H^{1/2} = X \oplus X^\perp$. Denote P, $P^\perp$ the orthogonal projectors of $H^{1/2}$ onto X, $X^\perp$, respectively.

4.1 Existing theory

Let E be a Hilbert space and Ω be a neighborhood of infinity. Consider the following operator eaquation with a parameter $\lambda \in \mathbb{R}$:

$$Lu + g(u) = \lambda u \tag{$*$}$$

where L is a bounded linear selfadjoint operator on E and $g \in C(\Omega, E)$, with $g(u) = o(\|u\|)$ as $\|u\| \to \infty$. Furthermore, we assume that g is of potential type, i.e., there exists $G \in C^1(\Omega, \mathbb{R})$ such that $dG = g$. The following theorem can be found in [14].

Theorem 4.1. *Let L and g be as above. Suppose that $I_\lambda \in C^1(\Omega, \mathbb{R})$, with*

$$dI_\lambda(u) = Lu + g(u) - \lambda u$$

If μ is an isolated eigenvalue of L of finite multiplicity, and there are only finitely many eigenvalues of L larger than (or less than) μ with all of them are *of finite multiplicity. Then (μ, ∞) is a bifurcation point for $(*)$. Moreover, if I_μ satisfies the (PS) condition, then at least one of the following alternatives occurs:*

(a) There are infinitely many solutions for equation $(*)$ with $\lambda = \mu$, say (μ, u_n), with $\|u_n\| \to \infty$ as $n \to \infty$;

(b) There is a one-sided neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \mu$, equation $(*)$ possesses at least one solution u_λ with $\|u_\lambda\| \to \infty$ as $\lambda \to \mu$.

4.2 Proof of Theorem1.2

We shall apply Theorem4.1 to prove the result we want. However, there are some difficulties to Eq(1.1). So we must make some adjustments. Since M is a compact Riemannian manifold which has no harmonic spinors, the Dirac operator D has an inverse operator T on the spinor space $H^{1/2}(M, \mathbb{S}(M))$. Let $g(\psi) = T \circ h(\psi)$. Then Eq(1.1) is equivalent with the following form of equation:

$$\mu\psi = T\psi + g(\psi) \tag{4.1}$$

Let us assume μ is an eigenvalue of T, that is in the context: $\mu D\psi = \psi$ This also means $1/\mu$ is an eigenvalue of the Dirac operator D. $Spec(T) = \{\mu \in \mathbb{R} \mid 1/\mu \in Spec(D)\} \cup \{0\} \subset \mathbb{R}$ is a bounded set. It implies T is a bounded linear operator. Now we choose a finite number $k \in \mathbb{Z}$ such that $1/\mu_k$ is the k-th eigenvalue of the Dirac operator D. Then $\mu_k \in Spec(T)$. Moreover, if $\mu_k > 0$, there are only finitely many eigenvalues of T lager than μ_k. If $\mu_k < 0$, there are only finitely many eigenvalues less than μ_k. And all these eigenvalues are of finite multiplicity. Thus we could conclude (μ, ∞) is a bifurcation point for equation(1.1).

Next in order to prove a further result, we need some a priori estimates. Let us consider the following equation:

$$T\psi + g(\psi) - \mu\psi = \varepsilon \tag{4.2}$$

where ε is a $H^{1/2}$-spinor with sufficiently small norm. $\mu \in \mathbb{R}$ and $\mu \neq 0$. First let us give a priori estimate of (4.2).

If $\psi \in H^{1/2}(M, \mathbb{S}(M))$ is a solution of (4.2) and $X = ker(\mu_k D - I)$, then according to the space decomposition $H^{1/2} = X \oplus X^\perp$, $\psi = \psi_1 + \psi_2$, where $\psi_1 \in X$ and $\psi_2 \in X^\perp$. Project $H^{1/2}$ onto $X^\perp$, we have:

$$(T - \mu I)\psi_2 + P^\perp g(\psi) = P^\perp(\varepsilon).$$

Since $T - \mu I$ is an isomorphism of $X^\perp$ onto $X^\perp$ for μ closely near μ_k, we also have:

$$\psi_2 = (T - \mu I)^{-1}(P^\perp(\varepsilon) - P^\perp g(\psi)).$$

Assume $\|\psi_1\| \neq 0,\ R = \|(T-\mu I)^{-1}\|$, then

$$
\begin{aligned}
\|\psi_2\| \leq & \|(T-\mu I)^{-1}\| \cdot \|P^{\perp}\| \cdot (\|g(\psi)\| + \|\varepsilon\|) \\
\leq & R(\|T\| \cdot \|h(\psi_1+\psi_2)\| + \|\varepsilon\|) \\
\leq & R(c(\|\psi_1\|^\alpha + \|\psi_2\|^\alpha) + d + \|\varepsilon\|) \\
\leq & R(c\|\psi_1\|^\alpha (1 + \frac{\|\psi_2\|}{\|\psi_1\|})^\alpha + d^{'}) \\
\leq & R(c\|\psi_1\|^\alpha (1 + \alpha\frac{\|\psi_2\|}{\|\psi_1\|}) + d^{'}) \\
\leq & Rc\|\psi_1\|^\alpha + Rc\|\psi_1\|^\alpha \cdot \frac{\alpha\|\psi_2\|}{\|\psi_1\|} + d^{'}R.
\end{aligned}
\tag{4.3}
$$

where we have used (h_1) and the fact that T is a bounded linear operator. Dividing (4.3) by $\|\psi_1\|^\alpha$, we obtain:

$$\frac{\|\psi_2\|}{\|\psi_1\|^\alpha} \leq c_1 + \frac{c_1\alpha}{\|\psi_1\|^{1-\alpha}} \cdot \frac{\|\psi_2\|}{\|\psi_1\|^\alpha} + \frac{c_2}{\|\psi_1\|^\alpha},$$

where c_1, c_2 are constants. Consequently, if

$$\|\psi_1\| \geq (2c_1\alpha)^{\frac{1}{1-\alpha}} = c_3,$$

we have

$$\frac{\|\psi_2\|}{\|\psi_1\|^\alpha} \leq c_1 + \frac{1}{2}\Big(\frac{\|\psi_2\|}{\|\psi_1\|^\alpha}\Big) + \frac{c_2}{c_3^\alpha}$$

or

$$\|\psi_2\| \leq 2\Big(c_1 + \frac{c_2}{c_3^\alpha}\Big)\|\psi_1\|^\alpha = c_4\|\psi_1\|^\alpha.$$

On the other hand,

$$
\begin{aligned}
\|\psi\| \leq & \|\psi_1\| + \|\psi_2\| \\
\leq & \|\psi_1\| + R\|T\| \cdot (a\|\psi_1+\psi_2\|^\alpha + d^{'}) \\
\leq & c\|\psi\|^\alpha + \|\psi_1\| + c.
\end{aligned}
$$

Since $0 < \alpha < 1$, if $\|\psi_1\| \leq c_3$, we have:

$$\|\psi\| \leq c_5 \tag{4.4}$$

Now we shall prove:

Lemma 4.2. *Suppose $\mathcal{L}_{\mu_k} \in C^1(H^{1/2}, \mathbb{R})$, with*

$$d\mathcal{L}_{\mu_k}(\psi) = T\psi + g(\psi) - \mu_k\psi.$$

Under the assumption (h_2), $\mathcal{L}_{\mu_k}$ satisfies the Palais-Smale condition.

Proof. Let $\psi_n \subset H^{1/2}$ be a $(PS)_c$-sequence of $\mathcal{L}_{\mu_k}$, i.e.,

$$\mathcal{L}_{\mu_k}(\psi_n) \to c \quad \text{as } n \to \infty$$

and

$$\|d\mathcal{L}_{\mu_k}(\psi_n)\| \to 0 \quad \text{as } n \to \infty$$

Then there exists a sequence $\varepsilon_n \subset H^{1/2}$ which converges to θ in $H^{1/2}$ and such that

$$T\psi_n - \mu_k\psi_n + g(\psi_n) = \varepsilon_n.$$

We first show that $\psi_n \subset H$ is bounded.

Suppose on the contrary $\|\psi_n\| \to \infty$ as $n \to \infty$. Let us write $\psi_n = \psi_n^1 + \psi_n^2$, where $\psi_n^1 \in X = Ker(\mu_k D - I)$, and $\psi_n^2 \in X^\perp$. Then ψ_n satisfies the equivalent equation:

$$\mu_k D\psi_n = \psi_n + h(\psi_n) + \delta_n \tag{4.5}$$

with $\delta_n \to \theta$ in $H^{-1/2}$. By using the a priori estimates (4.4), we immediately obtain $\|\psi_n^1\| \to \infty$, as $n \to \infty$. Let $t_n = \|\psi_n^1\|$ and $\omega_n = \frac{1}{t_n}\psi_n^1$. Since X is finite dimensional, we may assume that $\omega_n \to \omega \in X$. Then we have:

$$\psi_n = t_n\omega_n + \psi_n^2 \tag{4.6}$$

Substitute (4.6) into the equation (4.5), we obtain:

$$\mu_k D\psi_n^2 = \psi_n^2 + h(t_n\omega_n + \psi_n^2) + \delta_n. \tag{4.7}$$

Product (4.7) with ω on $\mathbb{S}(M)$ and then integrating the result over the manifold M, we have:

$$\int_M (h(t_n\omega_n + \psi_n^2), \omega)dx + \int_M (\delta_n, \omega)dx = 0.$$

Since $\delta_n \to 0$, it follows that the second term will tend to zero. Hence we obtain a contradiction to the assumption of (h_2). Then $\psi_n \subset H^{1/2}$ is bounded.

By the compactness of D^{-1} and the completely continuousness of h, from (4.7) it is not difficult to see that ψ_n^2 is a convergent sequence in $X^\perp$. Since X is finite dimensional, ψ_n^1 has a convergent subsequence in X because of the boundedness of ψ_n. Assume $\psi_n^1 \to \omega$ and $\psi_n^2 \to \varphi$ as $n \to \infty$, then $\psi_n = \psi_n^1 + \psi_n^2 \to \omega + \varphi$ as $n \to \infty$. Thus $\mathcal{L}_{\mu_k}$ satisfies the (PS) condition. $\square$

Next we wish to prove:

Lemma 4.3. *Suppose* (μ, ψ) *solves* (1.1), $\sigma > 0$ *is a constant,* h *is completely continuous with* $h(\psi) = o(\|\psi\|)$ *as* $\|\psi\| \to \infty$, *then for fixed* $\mu \in (\mu_k - \sigma, \mu_k]$, *where* $(\mu_k - \delta, \mu_k]$ *contains no other eigenvalues except for* μ_k, *there exists a constant* $M(\mu) > 0$ *such that* $\|\psi\| \leq M(\mu)$.

Proof. Since (μ, ψ) solves equation (1.1), ψ also satisfies the equation

$$T\psi + g(\psi) - \mu\psi = 0. \tag{4.8}$$

Write $\psi = \psi_1 + \psi_2$, where $\psi_1 \in X$ and $\psi_2 \in X^{\perp}$. By a similar procedure with a priori estimates ahead, we obtain similarly:

$$\|\psi_2\| \leq b_2\|\psi_1\|^{\alpha} \quad \text{if} \quad \|\psi_1\| \geq b_1 \tag{4.9}$$

$$\|\psi\| \leq b_3 \quad \text{if} \quad \|\psi_1\| \leq b_1 \tag{4.10}$$

where b_1, b_2, b_3 are constants.

For fixed $\mu \in (\mu_k - \delta, \mu_k]$, suppose on the contrary that there exists a sequence of solutions ψ_n and $\|\psi_n\| \to \infty$ as $n \to \infty$. Write $\psi_n = \psi_n^1 + \psi_n^2$, where $\psi_n^1 \in X$ and $\psi_n^2 \in X^{\perp}$. On one hand, (μ, ψ_n) solves (1.1), we have:

$$\mu D\psi_n^2 = \Big(1 - \frac{\mu}{\mu_k}\Big)\psi_n^1 + \psi_n^2 + h(\psi_n^1 + \psi_n^2). \tag{4.11}$$

On the other hand, (μ, ψ_n) satisfies (4.8) and then has the estimates (4.9) and (4.10). Therefore we may write:

$$\psi_n^1 = t_n\omega_n,$$

where $t_n \subset \mathbb{R}$ is an unbounded sequence of positive numbers, and $\omega_n \subset X$ with $\| \omega_n \| = 1$. Assume $\omega_n \to \omega \in X$. Producting (4.11) with ω and then integrating the result over M. We have:

$$\int_M \Big(1 - \frac{\mu}{\mu_k} t_n\omega_n,\, \omega\Big)dx + \int_M (h(t_n\omega_n + \psi_n^2), \omega)dx = 0.$$

The first term in the above sum on the left-side is nonnegative and we then get a contradiction since (h_2). Therefore there exists $M(\mu) > 0$ such that $\|\psi\| \leq M(\mu)$. □

We now complete the proof of Theorem1.2.

Proof of Theorem 1.2. We have know that (μ_k, ∞) is a bifurcation point for equation(1.1). By Lemma 4.2 and Lemma 4.3 and applying Theorem 4.1, we can easily obtain a right-neighborhood Λ of μ_k such that for all $\mu \in \Lambda \backslash \mu_k$, equation (1.1) posesses at least one solution ψ_μ with $\| \psi_\mu \| \to \infty$ as $\mu \to \mu_k$. The proof is completed. □

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References

[1] B. Ammann: A variational problem in conformal spin geometry, Habilitationsschift Universit$\ddot{a}$t Hamburg, 2003.

[2] K.C. Chang: Infinite Dimentional Morse Theory and Multiple Solutions, Birkh$\ddot{a}$user, 1993.

[3] K.C. Chang and Z.Q. Wang: Notes on the bifurcation theorem, J. fixed point theory appl. **1**(2007), 195-208.

[4] K.C. Chang and N. Ghoussoub: The conley index and the critical groups via an existence of Gromoll-Meyer theory, Topological Methhods in Nonliear Analysis, vol.7, 1996, 77-93.

[5] T. Friedrich: Dirac operators in Riemannian Geometry, Grad. Stud. Math., vol.25, Amer. Math. Soc, Providence, RI, 2000.

[6] N.Ginoux: The Dirac Spectrum, Lecture Notes in Math., vol.1976, Springer, 2009.

[7] Emmand Hebey: Sobolev Spaces on Riemannian Manifolds, Lecture notes in mathematics 1635, Springer, 1996.

[8] Takeshi Isobe: Nonlinear Dirac equations with critical nonlinearities on compact spin manifolds, J. Funct. Anal., **260** (2011), 253-307.

[9] Takeshi Isobe: Existence results for solutions to nonlinear Dirac equations on compact spin manifolds, manuscripta math., **135** (2011), 329-360.

[10] H.B. Lawson and M.L. Michelson: Spin Geometry, Princeton University Press, 1989.

[11] J. Mawhin and K. Schmitt: Landesman-Lazer type problems at an eigenvalue of odd multiplicity, Results in Math., **14** (1988), 138-146.

[12] P.H. Rabinowitz: On bifurcation from infinity, J. Diff. Eqs., **14** (1973), 462-275.

[13] P.H. Rabinowitz: A bifurcation theorem for potential operators, J. Funct. Anal., **25** (1977), 412-424.

[14] K. Schmitt and Z.Q. Wang: On bifurcation from infinity for potential operators, Differential and Integral Equations, **4** (1991), 933-943.