Zeta Functions and the Log-behavior of Combinatorial Sequences

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Abstract. In this paper, we use the Riemann zeta function $\zeta(x)$ and the Bessel zeta function $\zeta_\mu(x)$ to study the log-behavior of combinatorial sequences. We prove that $\zeta(x)$ is log-convex for $x > 1$. As a consequence, we deduce that the sequence $\{|B_{2n}|/(2n)!\}_{n\geq 1}$ is log-convex, where B_n is the n-th Bernoulli number. We introduce the function $\theta(x) = (2\zeta(x)\Gamma(x+1))^{\frac{1}{x}}$, where $\Gamma(x)$ is the gamma function, and we show that $\log\theta(x)$ is strictly increasing for $x \geq 6$. This confirms a conjecture of Sun stating that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 1}$ is strictly increasing. Amdeberhan, Moll and Vignat defined the numbers $a_n(\mu) = 2^{2n+1}(n+1)!(\mu+1)_n\zeta_\mu(2n)$ and conjectured that the sequence $\{a_n(\mu)\}_{n\geq 1}$ is log-convex for $\mu = 0$ and $\mu = 1$. By proving that $\zeta_\mu(x)$ is log-convex for $x > 1$ and $\mu > -1$, we show that the sequence $\{a_n(\mu)\}_{n\geq 1}$ is log-convex for any $\mu > -1$. We introduce another function $\theta_\mu(x)$ involving $\zeta_\mu(x)$ and the gamma function $\Gamma(x)$ and we show that $\log\theta_\mu(x)$ is strictly increasing for $x > 8e(\mu+2)^2$. This implies that $\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}$ for $n > 4e(\mu+2)^2$. Based on Dobinski's formula, we prove that $\sqrt[n]{B_n} < \sqrt[n+1]{B_{n+1}}$ for $n \geq 1$, where B_n is the n-th Bell number. This confirms another conjecture of Sun. We also establish a connection between the increasing property of $\{\sqrt[n]{B_n}\}_{n\geq 1}$ and Hölder's inequality in probability theory.

Keywords: log-convexity, Riemann zeta function, Bernoulli number, Bell number, Bessel zeta function, Narayana number, Hölder's inequality

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1 Introduction

The objective of this paper is to present an analytic approach to the log-behavior of combinatorial sequences.

Let B_n denote the n-th Bernoulli number, see [11] and [14]. Recall that $B_{2n+1} = 0$ for $n \geq 1$ and B_{2n} alternate in sign for $n \geq 1$. We consider the log-behavior of the sequence $\{|B_{2n}|\}_{n\geq 1}$. A sequence $\{a_n\}_{n\geq 1}$ of real numbers is said to be log-convex if for $n \geq 2$,

$$a_n^2 \leq a_{n-1}a_{n+1}.$$

It is well-known that

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \tag{1.1}$$

where

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is the Riemann zeta function. By proving that $\zeta(x)$ is log-convex for $x > 1$, we establish the log-convexity of the sequence $\{|B_{2n}|/(2n)!\}_{n\geq 1}$. Consequently, the sequence $\{|B_{2n}|\}_{n\geq 1}$ is log-convex. Moreover, we introduce the function

$$\theta(x) = (2\zeta(x)\Gamma(x+1))^{\frac{1}{x}}, \tag{1.2}$$

where $\Gamma(x)$ is the gamma function. We show that $\log\theta(x)$ is strictly increasing for $x \geq 6$. From relation (1.1), it can be seen that

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2}\theta^2(2n).$$

So we reach the assertion that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 1}$ is strictly increasing. This confirms a conjecture of Sun [15], which has been independently proved by Luca and Stănică [9]. We conjecture that $(\log\theta(x))'' < 0$ for $x \geq 6$.

Our approach also applies to the sequence of generalized Lasalle numbers. Let C_n denote the nth Catalan number, that is,

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

and let $N_r(z)$ denote the r-th Narayana polynomial as given by

$$N_r(z) = \sum_{k=1}^{r} \frac{1}{r}\binom{r}{k-1}\binom{r}{k} z^k.$$

Lasalle [8] derived the recurrence relation

$$(z+1)N_r(z) - N_{r+1}(z) = \sum_{n\geq 1}(-z)^n\binom{r-1}{2n-1}A_n N_{r-2n+1}(z),$$

where the numbers A_n satisfy the recurrence relation

$$(-1)^{n-1}A_n = C_n + \sum_{j=1}^{n-1}(-1)^j \binom{2n-1}{2j-1} A_j C_{n-j}. \tag{1.3}$$

Let

$$a_n = \frac{2A_n}{C_n}.$$

Lasalle [8] showed that $\{a_n\}_{n\geq 1}$ is an increasing sequence of positive integers. Amdeberhan, Moll and Vignat [2] established a connection between a_n and the Bessel zeta functions $\zeta_\mu(x)$. Recall that for a real number μ, the Bessel function $J_\mu(z)$ of the first kind of order μ is defined by

$$J_\mu(z) = \left(\frac{z}{2}\right)^\mu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\mu+k+1)k!} \left(\frac{z}{2}\right)^{2k}.$$

For $\mu \geq -1$, $J_\mu(z)$ has infinitely many positive real zeros $j_{\mu,n}$, where we assume that

$$0 < j_{\mu,1} < j_{\mu,2} < j_{\mu,3} < \cdots,$$

see [3, Sect. 4.14]. The Bessel zeta functions $\zeta_\mu(x)$ are defined by

$$\zeta_\mu(x) = \sum_{n=1}^{\infty} \frac{1}{j_{\mu,n}^x}. \tag{1.4}$$

Amdeberhan, Moll and Vignat [2] found the following relation

$$a_n = 2^{2n+1}(n+1)!(n-1)!\zeta_1(2n). \tag{1.5}$$

They also gave the following generalization of a_n for $\mu \geq -1$,

$$a_n(\mu) = 2^{2n+1}(n-1)!(\mu+1)_n\zeta_\mu(2n), \tag{1.6}$$

where $(\mu+1)_n = (\mu+1)(\mu+2)\cdots(\mu+n)$.

It is easily seen that $a_n = a_n(1)$. Setting $\mu = 0$ in (1.6), Amdeberhan, Moll and Vignat defined the sequence $\{b_n\}_{n\geq 1}$ as given by

$$b_n = \frac{1}{2}a_n(0) = 2^{2n}n!(n-1)!\zeta_0(2n). \tag{1.7}$$

Note that this sequence has been studied by Carlitz [6]. It is listed as Sequence A002190 in [10].

Amdeberhan, Moll and Vignat conjectured that the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are log-convex. We show that $\zeta_\mu(x)$ is log-convex for $x > 1$. This implies that the sequence $\{a_n(\mu)\}_{n\geq 1}$ is log-convex for any $\mu > -1$.

This confirms the above conjectures, which have been independently proved by Wang and Zhu [16].

Moreover, we define the following function

$$\theta_\mu(x) = \left(\frac{2}{\mu!}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\mu+1\right)\zeta_\mu(x)\right)^{\frac{1}{x}}. \tag{1.8}$$

It can be easily checked that

$$4\theta_\mu^2(2n) = \sqrt[n]{a_n(\mu)}. \tag{1.9}$$

We show that $\log\theta_\mu(x)$ is strictly increasing for $x > 8e(\mu+2)^2$. This leads to the increasing property that

$$\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}, \tag{1.10}$$

for $n > 4e(\mu+2)^2$. We note that for $\mu = 0$ and $\mu = 1$ the above relation (1.10) has been independently proved by Wang and Zhu [16].

Owing to the formula of Dobinski, we may use our analytic approach to study the log-behavior of Bell numbers. Let B_n be the n-th Bell number, that is, the number of partitions of $\{1,\ 2,\ \ldots,\ n\}$, see [5] and [12]. Notice that we have adopted the same notation B_n for both Bell numbers and Bernoulli numbers. Recall that Dobinski's formula for the Bell numbers states that

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

For $x > 0$, we define

$$B(x) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^x}{k!}, \tag{1.11}$$

so that we have $B_n = B(n)$ whenever n is a nonnegative integer.

We show that $\log B(x)^{1/x}$ is increasing for $x \geq 1$. This implies that the sequence $\{\sqrt[n]{B_n}\}_{n\geq 1}$ is increasing, as conjectured by Sun [15]. We conjecture that $(\log B(x)^{1/x})'' < 0$ for $x \geq 1$. In the last section, we give a probabilistic proof of the increasing property of the sequence $\{\sqrt[n]{B_n}\}_{n\geq 1}$ by using Hölder's inequality.

2 The log-convexity of Bernoulli numbers

To prove the log-convexity of Bernoulli numbers, we consider the log-behavior of the Riemann zeta function $\zeta(x)$ for $x > 1$. Recall that a positive function

f is called log-convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \le f(x)^{\lambda} f(y)^{1-\lambda}, \tag{2.12}$$

see, for example, Artin [4]. It is known that a positive function f is log-convex if and only if $(\log f(x))'' \ge 0$. So, if

$$(\log \zeta(x))'' > 0, \tag{2.13}$$

for $x > 1$, then we can deduce that $\zeta(x)$ is log-convex for $x > 1$.

Lemma 2.1. *The Riemann zeta function $\zeta(x)$ is log-convex for $x > 1$.*

Proof. Clearly, condition (2.13) is equivalent to

$$\zeta(x) \cdot \zeta''(x) - (\zeta'(x))^2 > 0. \tag{2.14}$$

Since $\zeta(x)$ converges for $x > 1$, we find that for $x > 1$,

$$\begin{aligned} &\zeta(x)\zeta''(x) - (\zeta'(x))^2 \\ &= \sum_{m=1}^{\infty} \frac{1}{m^x} \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^x} - \sum_{m=1}^{\infty} \frac{\log m}{m^x} \sum_{n=1}^{\infty} \frac{\log n}{n^x} \\ &= \sum_{n>m\ge 1} \frac{(\log n)^2 + (\log m)^2 - 2\log m \log n}{(mn)^x} \\ &= \sum_{n>m\ge 1} \frac{(\log n - \log m)^2}{(mn)^x}, \end{aligned}$$

which is positive. This completes the proof. ∎

The log-convexity of $\zeta(x)$ enables us to deduce the following property of Bernoulli numbers.

Theorem 2.2. *The sequence $\left\{\frac{|B_{2n}|}{(2n)!}\right\}_{n\ge 1}$ is log-convex.*

Proof. Since $\zeta(x)$ is log-convex, setting $x = 2n-2$, $y = 2n+2$ and $\lambda = 1/2$ in the defining relation (2.12), we find that

$$\zeta(2n-2)\zeta(2n+2) \ge \zeta(2n)^2. \tag{2.15}$$

Invoking relation (1.1) between $\zeta(x)$ and B_n, we obtain that

$$\left(\frac{|B_{2n}|}{(2n)!}\right)^2 \le \frac{|B_{2n-2}|}{(2n-2)!} \cdot \frac{|B_{2n+2}|}{(2n+2)!}.$$

This completes the proof. ∎

Since $((2n)!)^2 < (2n-2)! \cdot (2n+2)!$ for $n \ge 1$, the above theorem implies the following property.

Corollary 2.3. *The sequence $\{|B_{2n}|\}_{n\ge 1}$ is log-convex.*

3 The log-behavior of $\theta(x)$

In this section, we consider the log-behavior of the function

$$\theta(x) = (2\zeta(x)\Gamma(x+1))^{\frac{1}{x}}.$$

We begin with the following monotone property of $\log\theta(x)$.

Theorem 3.1. $\log\theta(x)$ *is strictly increasing for* $x \geq 6$.

Proof. To prove that $\log\theta(x)$ is increasing for $x \geq 6$, we aim to show that

$$(\log\theta(x))' > 0, \tag{3.16}$$

for $x \geq 6$. Let

$$g(x) = 2\zeta(x)\Gamma(x+1).$$

Then we have

$$\theta(x) = g(x)^{1/x}$$

and

$$(\log\theta(x))' = \frac{1}{x}\left(\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x}\right).$$

Thus (3.16) can be rewritten as

$$\frac{g'(x)}{g(x)} > \frac{\log g(x)}{x},$$

for $x \geq 6$. Since $\zeta(x)$ and $\Gamma(x)$ are continuous and differentiable on $(1,\infty)$, so is $g(x)$ on $(1,\infty)$. Applying the mean value theorem to $\log g(x)/x$, it can be shown that there exists t in $(2,x)$ such that

$$\frac{g(t)'}{g(t)} > \frac{\log g(x)}{x}. \tag{3.17}$$

Since $\zeta(2) = \frac{\pi^2}{6}$ and $\Gamma(3) = 2$, we find that

$$\log g(2) = \log(2\zeta(2)\Gamma(3)) = \log\frac{2\pi^2}{3} < 2. \tag{3.18}$$

On the other hand, for $x \geq 6$, it is easily seen that $\zeta(x) > 1$ and $\Gamma(x+1) > e^x$. It follows that

$$\log g(x) = \log 2 + \log\zeta(x) + \log\Gamma(x+1) > x. \tag{3.19}$$

In view of (3.18) and (3.19), we deduce that for $x \geq 6$,

$$\frac{\log g(x)}{x} = \frac{(1-2/x)\log g(x)}{(1-2/x)x} < \frac{\log g(x) - 2}{x-2} < \frac{\log g(x) - \log g(2)}{x-2}. \tag{3.20}$$

Applying the mean value theorem to $\log g(x)$, we see that there exists $t \in (2, x)$ such that

$$(\log g(t))' = \frac{\log g(x) - \log g(2)}{x - 2}, \tag{3.21}$$

that is,

$$\frac{g'(t)}{g(t)} = \frac{\log g(x) - \log g(2)}{x - 2}. \tag{3.22}$$

Combining (3.20) and (3.22), we get (3.17).

Now we proceed to show that

$$\frac{g(x)'}{g(x)} > \frac{g(t)'}{g(t)}. \tag{3.23}$$

Clearly, (3.23) is equivalent to

$$\left(\frac{g'(y)}{g(y)}\right)' > 0. \tag{3.24}$$

By the definition of $g(x)$, we have

$$\left(\frac{g'(y)}{g(y)}\right)' = (\log g(y))'' = (\log \Gamma(y+1))'' + (\log \zeta(y))''.$$

It is known that $(\log \Gamma(y+1))'' > 0$ for $y > 1$, see Andrews, Askey and Roy [3, Theorem. 1.2.5]. On the other hand, in the proof of Lemma 2.1, we have shown that $(\log \zeta(y))'' > 0$. This proves (3.24). In other words, $\frac{g'(y)}{g(y)}$ is strictly increasing for $y > 1$. Thus for $2 < t < x$, inequality (3.23) holds.

Combining (3.17) and (3.23), we deduce that for $x \geq 6$,

$$\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x} > \frac{g'(x)}{g(x)} - \frac{g'(t)}{g(t)} > 0.$$

Thus $(\log \theta(x))' > 0$ for $x \geq 6$. This completes the proof. ∎

From the log-behavior of $\theta(x)$, we are led to an affirmative answer to a conjecture of Sun [15].

Corollary 3.2. *The sequence* $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 1}$ *is strictly increasing.*

Proof. From relation (1.1), we see that for $n \geq 1$,

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2} \sqrt[n]{2\zeta(2n)(2n)!} = \frac{1}{4\pi^2}\theta^2(2n). \tag{3.25}$$

Since $\log \theta(x)$ is strictly increasing for $x \geq 6$, we find that $\theta(x)$ is also strictly increasing for $x \geq 6$. It follows from (3.25) that $\sqrt[n]{|B_{2n}|}$ is strictly increasing for $n \geq 3$. On the other hand, it is easily checked that

$$|B_2| < \sqrt{|B_4|} < \sqrt[3]{|B_6|}.$$

This completes the proof. ∎

The conjecture of Sun was independently proved by Luca and Stănică [9]. In fact, they proved that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 1}$ is log-concave, which was also conjectured by Sun [15].

We pose the following conjecture concerning the function $\theta(x)$. If it is true, then it implies that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 1}$ is log-concave.

Conjecture 3.3. *The function $\theta(x)$ is log-concave for $x \geq 6$, that is, for $x \geq 6$, $(\log f(x))'' < 0$.*

4 The log-behavior of the sequence $\{a_n(\mu)\}_{n\geq 1}$

In this section, we study the log-behavior of the sequence $\{a_n(\mu)\}_{n\geq 1}$. We begin with the log-behavior of the Bessel zeta functions $\zeta_\mu(x)$.

Lemma 4.1. *For $\mu > -1$, the Bessel zeta function $\zeta_\mu(x)$ is log-convex for $x > 1$.*

Proof. We proceed to show that for $x > 1$,

$$(\log \zeta_\mu(x))'' > 0,$$

or equivalently,

$$\zeta_\mu(x)\zeta_\mu''(x) - (\zeta_\mu'(x))^2 > 0. \tag{4.26}$$

By the convergence of $\zeta_\mu(x)$, it is easily seen that

$$\zeta_\mu'(x) = -\sum_{n=1}^{\infty} \frac{\log j_{\mu,n}}{j_{\mu,n}^x}$$

and

$$\zeta_\mu''(x) = \sum_{n=1}^{\infty} \frac{(\log j_{\mu,n})^2}{j_{\mu,n}^x}.$$

Hence

$$\begin{aligned} &\zeta_\mu(x)\zeta_\mu''(x) - (\zeta_\mu'(x))^2 \\ &\quad= \sum_{m=1}^{\infty} \frac{1}{j_{\mu,m}^x} \sum_{n=1}^{\infty} \frac{(\log j_{\mu,n})^2}{j_{\mu,n}^x} - \sum_{m=1}^{\infty} \frac{\log j_{\mu,m}}{j_{\mu,m}^x} \sum_{n=1}^{\infty} \frac{\log j_{\mu,n}}{j_{\mu,n}^x} \\ &\quad= \sum_{n>m\geq 1} \frac{(\log j_{\mu,m})^2 + (\log j_{\mu,n})^2 - 2(\log j_{\mu,m})(\log j_{\mu,n})}{j_{\mu,m}^x j_{\mu,n}^x} \\ &\quad= \sum_{n>m\geq 1} \frac{(\log j_{\mu,m} - \log j_{\mu,n})^2}{j_{\mu,m}^x j_{\mu,n}^x}, \end{aligned}$$

which is positive. This completes the proof. ∎

Setting $f(x) = \zeta_\mu(x)$, $x = 2n-2$, $y = 2n+2$ and $\lambda = 1/2$ in the defining relation (2.12) of a log-convex function, we obtain that for $\mu > -1$,

$$\zeta_\mu(2n-2)\zeta_\mu(2n+2) > \zeta_\mu(2n)^2. \tag{4.27}$$

This yields that the sequence $\{\zeta_\mu(2n)\}_{n\geq 1}$ is log-convex for $\mu > -1$. On the other hand, it is easily checked that the sequence $\{2^{2n+1}(n+1)!(\mu+1)_n\}_{n\geq 1}$ is log-convex for $\mu > -1$. It is well-known that for two positive log-convex sequences $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$, the sequence $\{u_n v_n\}_{n\geq 1}$ is also log-convex. So we arrive at the following property.

Theorem 4.2. *The sequence $\{a_n(\mu)\}_{n\geq 1}$ is log-convex for $\mu > -1$.*

For $\mu = 0$ and $\mu = 1$, Theorem 4.2 gives affirmative answers to the two conjectures of Amdeberhan, Moll and Vignat [2] on the log-convexity of the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, where $a_n = a_n(1)$ and $b_n = \frac{1}{2}a_n(0)$.

Next we consider the monotone property of the sequence $\{\sqrt[n]{a_n(\mu)}\}_{n\geq 1}$ for $\mu > 0$.

Theorem 4.3. *For $\mu > 0$, the sequence $\{\sqrt[n]{a_n(\mu)}\}_{n\geq 1}$ is increasing for $n > 4e(\mu+2)^2$*

To prove the above theorem, we introduce the function

$$\theta_\mu(x) = \left(\frac{2}{\mu!}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\mu+1\right)\zeta_\mu(x)\right)^{\frac{1}{x}},$$

which has the following monotone property.

Theorem 4.4. *For $\mu \geq 0$, the function $\log\theta_\mu(x)$ is strictly increasing for $x > 8e(\mu+2)^2$.*

Proof. Assume that $\mu \geq 0$. To prove the monotone property in the theorem, we aim to show that for $x > 8e(\mu+2)^2$,

$$(\log\theta_\mu(x))' > 0. \tag{4.28}$$

Let

$$h(x) = \frac{2}{\mu!}\Gamma(x/2)\Gamma(x/2+\mu+1)\zeta_\mu(x). \tag{4.29}$$

Recalling the definition of $\theta_\mu(x)$ as given by (1.8), we have

$$\theta_\mu(x) = h(x)^{\frac{1}{x}}$$

and

$$\log \theta_\mu(x) = \frac{1}{x} \log h(x).$$

It follows that

$$(\log \theta_\mu(x))' = \frac{1}{x} \left(\frac{h'(x)}{h(x)} - \frac{\log h(x)}{x} \right). \tag{4.30}$$

Since $\zeta_\mu(x)$ and $\Gamma(x)$ are continuous and differentiable on $(1, \infty)$, so is $h(x)$. We shall apply the mean value theorem to $\log h(x)$ on $[2, x]$, where $x > 8e(\mu+2)^2$ and $\mu > -1$. To this end, we need to show that $h(2) < 1$ and $h(x) > 1$ for $\mu > -1$ and $x > 8e(\mu+2)^2$.

Recalling the definition of $h(x)$ as given by (4.29), we get

$$h(2) = \frac{2}{\mu!} \Gamma(1)\Gamma(\mu+2)\zeta_\mu(2),$$

where

$$\zeta_\mu(2) = \frac{1}{4(\mu+1)},$$

$\Gamma(1) = 1$ and $\Gamma(\mu+2) = (\mu+1)!$. Then

$$h(2) = \frac{2}{\mu!} \cdot (\mu+1)! \cdot \frac{1}{4(\mu+1)}, \tag{4.31}$$

so $h(2) < 1$.

It remains to show that $h(x) > 1$ for $\mu > -1$ and $x > 8e(\mu+2)^2$. Recall that

$$j_{\mu,1} < (\mu+1)^{\frac{1}{2}} \left((\mu+2)^{\frac{1}{2}} + 1 \right), \tag{4.32}$$

for $\mu > -1$, see Chamber [7]. It follows that for $\mu > -1$,

$$j_{\mu,1} < 2(\mu+2). \tag{4.33}$$

Therefore, we obtain that for $\mu > -1$,

$$\zeta_\mu(x) = \sum_{n=1}^{\infty} \frac{1}{j_{\mu,n}^x} > \frac{1}{j_{\mu,1}^x} > \frac{1}{2^x (\mu+2)^x}. \tag{4.34}$$

On the other hand, it is known that for $x \geq 0$,

$$\Gamma(x) > \sqrt{2\pi x} \left(\frac{x}{e} \right)^x, \tag{4.35}$$

see Alzer [1]. Combining (4.34) and (4.35), we deduce that for $x > 2$ and $\mu > -1$,

$$2\Gamma\left(\frac{x}{2}\right) \zeta_\mu(x) > 2\sqrt{\pi x} \left(\frac{x}{8e(\mu+2)^2} \right)^{\frac{x}{2}}.$$

Consequently, for $\mu > -1$ and $x > 8e(\mu+2)^2$, we obtain that

$$2\Gamma\left(\frac{x}{2}\right)\zeta_\mu(x) > 2\sqrt{\pi x} > 1. \tag{4.36}$$

Clearly, for $x > 0$ we have

$$\frac{\Gamma(x/2+\mu+1)}{\mu!} > 1. \tag{4.37}$$

In view of (4.36) and (4.37), we find that for $\mu > -1$ and $x > 8e(\mu+2)^2$,

$$h(x) = \frac{2}{\mu!}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\mu+1\right)\zeta_\mu(x) > 1, \tag{4.38}$$

as claimed.

Next we proceed to prove that there exists t in $(2, x)$ such that

$$\frac{h'(t)}{h(t)} > \frac{\log h(x)}{x}. \tag{4.39}$$

By the mean value theorem applied to $\log h(x)$ on $[2, x]$, there exists $t \in (2, x)$ such that

$$\frac{h'(t)}{h(t)} = (\log h(t))' = \frac{\log h(x) - \log h(2)}{x-2}. \tag{4.40}$$

On the other hand, we have shown that $h(2) < 1$ and $h(x) > 1$ for $\mu > -1$ and $x > 8e(\mu+2)^2$. Consequently, we have $\log h(2) < 0$ and $\log h(x) > 0$. Note that for $\mu > -1$ and $x > 8e(\mu+2)^2$, we have $x > 2$. Hence

$$\frac{\log h(x)}{x} < \frac{\log h(x) - \log h(2)}{x-2}. \tag{4.41}$$

Combining (4.40) and (4.41), we obtain (4.39).

Moreover, it can be shown that

$$\frac{h'(x)}{h(x)} > \frac{h'(t)}{h(t)}. \tag{4.42}$$

We claim that for $y > 2$,

$$\left(\frac{h'(y)}{h(y)}\right)' > 0. \tag{4.43}$$

By the definition of $h(x)$ as given by (4.29), we have

$$\begin{aligned}\left(\frac{h'(y)}{h(y)}\right)' &= (\log h(y))'' \\ &= (\log\Gamma(y/2))'' + (\log\Gamma(y/2+\mu+1))'' + (\log\zeta_\mu(x))''.\end{aligned}$$

It is known that $(\log \Gamma(y))'' > 0$ for $y > 1$, see [3, Theorem 1.2.5]. Thus, $(\log \Gamma(y/2))'' > 0$ and $(\log \Gamma(y/2 + \mu + 1))'' > 0$ for $y > 2$. Moreover, in the proof of Lemma 4.1, we have shown that $(\log \zeta_\mu(y))'' > 0$. This proves (4.43). In other words, $\frac{h'(y)}{h(y)}$ is strictly increasing for $y > 2$. Thus for $2 < t < x$, (4.42) holds.

Combining (4.39) and (4.42), for $\mu > -1$ and $x > 8e(\mu+2)^2$, we find that

$$\frac{h'(x)}{h(x)} - \frac{\log h(x)}{x} > \frac{h'(x)}{h(x)} - \frac{h'(t)}{h(t)} > 0.$$

Hence (4.28) follows from (4.30). This completes the proof. ∎

In view of relation (1.6), it can be checked that

$$\sqrt[n]{a_n(\mu)} = 4\theta_\mu(2n)^2. \tag{4.44}$$

Thus Theorem 4.4 implies that for any $\mu \geq 0$ and $n > 4e(\mu + 2)^2$, we have $\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}$.

For $\mu = 1$, it can be verified that $\sqrt[n]{a_n(1)} < \sqrt[n+1]{a_{n+1}(1)}$ for $2 \leq n \leq 108$. In the meantime, for $\mu = 1$, Theorem 4.4 states that $\sqrt[n]{a_n(1)} < \sqrt[n+1]{a_{n+1}(1)}$ for $n > 101$. Thus we have the following assertion.

Theorem 4.5. *The sequence* $\{\sqrt[n]{a_n}\}_{n\geq 2}$ *is strictly increasing.*

For $\mu = 0$, it can be verified that $\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)}$ for $2 \leq n \leq 48$. Meanwhile, for $\mu = 0$, Theorem 4.4 states that $\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)}$ for $n > 45$. So we have $\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)}$ for $n \geq 2$. Since $b_n = \frac{1}{2}a_n(0)$, we have for $n \geq 2$,

$$\sqrt[n]{b_n} = \frac{\sqrt[n]{a_n(0)}}{\sqrt[n]{2}} < \frac{\sqrt[n+1]{a_{n+1}(0)}}{\sqrt[n+1]{2}} = \sqrt[n+1]{b_{n+1}}.$$

Thus we have the following monotone property.

Theorem 4.6. *The sequence* $\{\sqrt[n]{b_n}\}_{n\geq 2}$ *is strictly increasing.*

Note that Wang and Zhu [16] independently proved the log-convexity of $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ and the increasing properties of $\{\sqrt[n]{a_n}\}_{n\geq 1}$ and $\{\sqrt[n]{b_n}\}_{n\geq 1}$.

5 The log-behavior of Bell numbers

In this section, we consider the log-behavior of Bell numbers, which are also denoted by B_n. Recall that the function $B(x)$ is defined by

$$B(x) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^x}{k!}.$$

Lemma 5.1. *The function $B(x)$ is log-convex for $x > 1$.*

Proof. We proceed to show that

$$(\log B(x))'' > 0,$$

that is,

$$B(x)B''(x) - (B'(x))^2 > 0. \tag{5.45}$$

For $x \geq 1$, by the convergence of $B(x)$, we have

$$B'(x) = \frac{1}{e}\sum_{n=0}^{\infty} \frac{n^x \log n}{n!}$$

and

$$B''(x) = \frac{1}{e}\sum_{n=0}^{\infty} \frac{n^x (\log n)^2}{n!}.$$

Thus, for $x > 1$, we have

$$\begin{aligned}
& B(x)B''(x) - (B'(x))^2 \\
= \;& \frac{1}{e^2}\sum_{m=0}^{\infty} \frac{m^x}{m!} \sum_{n=0}^{\infty} \frac{n^x(\log n)^2}{n!} - \frac{1}{e^2}\sum_{m=0}^{\infty} \frac{m^x \log m}{m!} \sum_{n=0}^{\infty} \frac{n^x \log n}{n!} \\
= \;& \frac{1}{e^2}\sum_{n>m\geq 0} \frac{m^x n^x}{m!n!}((\log m)^2 + (\log n)^2 - 2\log m \log n) \\
= \;& \frac{1}{e^2}\sum_{n>m\geq 0} \frac{m^x n^x}{m!n!}(\log n - \log m)^2,
\end{aligned}$$

which is positive. This completes the proof. ∎

We now turn to the log-behavior of the function $B(x)^{1/x}$.

Theorem 5.2. $\log B(x)^{1/x}$ *is strictly increasing for $x \geq 1$.*

Proof. To prove that $\log B(x)^{1/x}$ is strictly increasing, we aim to show that

$$(\log B(x)^{1/x})' > 0. \tag{5.46}$$

Since

$$(\log B(x)^{1/x})' = \frac{1}{x}\left(\frac{B'(x)}{B(x)} - \frac{\log B(x)}{x}\right),$$

(5.46) can be rewritten as

$$\frac{B'(x)}{B(x)} > \frac{\log B(x)}{x}. \tag{5.47}$$

We claim that there exists t in $(1, x)$ such that

$$\frac{B'(t)}{B(t)} > \frac{\log B(x)}{x}. \tag{5.48}$$

Since $B(1) = 1$ and $B(x) > 1$ for $x > 1$, by the mean value theorem with respect to $\log B(x)$ on $[1, x]$, there exists $t \in (1, x)$ such that

$$\frac{B'(t)}{B(t)} = \frac{\log B(x) - \log B(1)}{x - 1} = \frac{\log B(x)}{x - 1}. \tag{5.49}$$

Since $x > 1$, we have

$$\frac{\log B(x)}{x - 1} > \frac{\log B(x)}{x}. \tag{5.50}$$

Combining (5.49) and (5.50), we obtain (5.48).

Next we show that for $x > t > 1$,

$$\frac{B'(x)}{B(x)} > \frac{B'(t)}{B(t)}. \tag{5.51}$$

In fact, by Lemma 5.1, we see that for $y \geq 1$,

$$\left(\frac{B'(y)}{B(y)}\right)' = (\log B(y))'' > 0.$$

This implies that $\frac{B'(y)}{B(y)}$ is strictly increasing for $y > 1$. This proves (5.51).

Combining (5.48) and (5.51), we obtain (5.47). This completes the proof. ∎

Since $B(n) = B_n$ whenever n is a positive integer, Theorem 5.2 implies the following monotone property conjectured by Sun [15].

Corollary 5.3. *The sequence* $\{\sqrt[n]{B_n}\}_{n\geq 1}$ *is strictly increasing.*

The above property was independently obtained by Wang and Zhu [16] via a different approach. Furthermore, we pose the following conjecture which implies the conjecture of Sun [15] stating that the sequence $\{\sqrt[n]{B_n}\}_{n\geq 1}$ is log-concave.

Conjecture 5.4. *The function* $B(x)^{1/x}$ *is log-concave for* $x \geq 1$*, that is,* $(\log B(x)^{1/x})'' < 0$ *for* $x > 1$.

6 A connection to Hölder's inequality

In this section, we give a derivation of the monotone property of the function $B(x)^{1/x}$ as given in Theorem 5.2 by applying Hölder's inequality in probability theory. In fact, it can be shown the condition $1 < x < y$ in Theorem 5.2 can be relaxed to $0 < x < y$.

Let Z be the discrete random variable with Possion distribution as given by

$$P(Z = k) = \frac{1}{e}\frac{1}{k!}.$$

From Dobinski's formula, it is easily checked that $B(x) = E[Z^x]$. Hölder's inequality states that for real-valued random variables U, V and positive numbers p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$E[|UV|] \leq E[|U|^p]^{1/p} E[|V|^q]^{1/q},$$

and the equality holds if and only if either there exist constants $\alpha, \beta > 0$ such that $\alpha|U|^p = \beta|V|^q$ or $E[|U|^p] = 0$ or $E[|V|^q] = 0$, see, for example, Sachkov [13]. For $0 < x < y$, we set $p = y/x$, and set $U = Z^x$ and $V = 1$. It is not hard to see that in this case Hölder inequality is strict. Hence we obtain that

$$E[Z^x]^{1/x} < E[Z^y]^{1/y},$$

which can be restated as follows.

Theorem 6.1. *For $0 < x < y$, we have $B(x)^{1/x} < B(y)^{1/y}$.*

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