Scattering of solutions to the defocusing energy sub-critical semi-linear wave equation in 3D*

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Abstract

In this paper we consider a semi-linear, energy sub-critical, defocusing wave equation $\partial_t^2 u - \Delta u = -|u|^{p-1}u$ in the 3-dimensional space with $p \in [3,5)$. We prove that if initial data (u_0, u_1) are radial so that $\|\nabla u_0\|_{L^2(\mathbb{R}^3;d\mu)}, \|u_1\|_{L^2(\mathbb{R}^3;d\mu)} < \infty$, where $d\mu = (|x|+1)^{1+2\varepsilon}$ with $\varepsilon > 0$, then the corresponding solution u must exist for all time $t \in \mathbb{R}$ and scatter. The key ingredients of the proof include a transformation $\mathbf{T}$ so that $v = \mathbf{T}u$ solves the equation $v_{\tau\tau} - \Delta_y v = -\left(\frac{|y|}{\sinh |y|}\right)^{p-1} e^{-(p-3)\tau}|v|^{p-1}v$ with a finite energy, and a Morawetz-type estimate regarding a solution v as above.

1 Introduction

The defocusing semi-linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = -|u|^{p-1}u, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u(\cdot,0) = u_0; \\ u_t(\cdot,0) = u_1 \end{cases} \qquad (CP1)$$

has been extensively studied in the past few decades. This problem is locally well-posed if initial data (u_0, u_1) are contained in the critical Sobolev space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ with $s_p \doteq 3/2 - 2/(p-1)$. Please see [16] for more details on the local theory. Suitable solutions also satisfy an energy conservation law:

$$E(u,u_t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u(\cdot,t)|^2 + \frac{1}{2}|u_t(\cdot,t)|^2 + \frac{1}{p+1}|u(\cdot,t)|^{p+1}\right) dx = \text{Const.}$$

The problem of global existence and scattering is much more difficult. In the energy critical case $p = 5$, M. Grillakis [8] proved that any solution with initial data in the space $\dot{H}^1 \times L^2(\mathbb{R}^3)$ must scatter in both two time directions. In other words, the asymptotic behaviour of any solution mentioned above resembles that of a free wave. It is conjectured that a similar result holds for other exponents p as well: Any solution to (CP1) with initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ must exist for all time $t \in \mathbb{R}$ and scatter in both two time directions. This conjecture has not been proved yet, as far as the author knows, in spite of some progress:

- It has been proved that if a radial solution u with a maximal lifespan I satisfies an a priori estimate
$$\sup_{t\in I} \|(u(\cdot,t), u_t(\cdot,t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} < +\infty, \qquad (1)$$

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then u is a global solution in time and scatters. The proof uses the standard compactness-rigidity argument, where the radial assumption plays a crucial role in the rigidity part. The details can be found in [14] for $p > 5$, [17] for $3 < p < 5$ and [3] for $1 + \sqrt{2} < p \leq 3$. The author would also like to mention that the same result still holds in the non-radial case if $p > 5$, see [15]. Please note that our assumption (1) is automatically true in the energy critical case $p = 5$, thanks to the conservation law of energy. When p is other than 5, however, nobody has ever found a way to actually prove this a priori estimate without additional assumptions on initial data.

- In the energy sub-critical case $3 \leq p < 5$, the scattering result can be proved via conformal conservation laws if initial data satisfy an additional regularity-decay condition

$$\int_{\mathbb{R}^3} \left[(|x|^2+1)(|\nabla u_0(x)|^2 + |u_1(x)|^2) + |u_0(x)|^2\right] dx < \infty. \tag{2}$$

The proof consists of three steps. First of all, our assumption guarantees that the solution has a finite energy. A combination of a local theory and the energy conservation law immediately gives the global existence. In addition, these solutions satisfy the conformal conservation law

$$\frac{d}{dt} Q(t, u, u_t) = \frac{4(3-p)t}{p+1} \int_{\mathbb{R}^3} |u(x,t)|^{p+1}\, dx.$$

Here $Q(t,\varphi,\psi) = Q_0(t,\varphi,\psi) + Q_1(t,\varphi)$ is called the conformal charge with

$$Q_0(t,\varphi,\psi) = \|x\psi + t\nabla\varphi\|_{L^2(\mathbb{R}^3)}^2 + \left\|(t\psi + 2\varphi)\frac{x}{|x|} + |x|\nabla\varphi\right\|_{L^2(\mathbb{R}^3)}^2$$

$$Q_1(t,\varphi) = \frac{2}{p+1}\int_{\mathbb{R}^3} (|x|^2 + t^2)|u(x,t)|^{p+1} dx.$$

Please see [6, 9] for more details. It immediately follows that

$$\sup_{t\in\mathbb{R}} Q_1(t, u(\cdot,t)) \leq \sup_{t\in\mathbb{R}} Q(t,u,u_t) = Q(0,u_0,u_1) < +\infty.$$

Therefore we have a space-time integral estimate

$$\int_{|t|>1}\int_{\mathbb{R}^3} |u(x,t)|^{p+1}\, dxdt < +\infty.$$

Finally, one can prove the scattering of solutions by using the integral estimate above and following a similar argument as in Section 3.4 below. The assumption (2) is essential to guarantee the finiteness of the conformal charge $Q(t,u,u_t)$ as defined above. One advantage of this argument is that the radial assumption is not necessary.

Main Result In this work we still deal with the energy subcritical equation with $p \in [3,5)$ as the conformal conservation law method does. We assume that initial data are radial but satisfy a weaker decay condition than (2) and prove the same scattering result. Due to the weaker decay assumption on initial data, we no longer have access to the powerful conformal conservation law. Instead we introduce a transformation that maps radial solutions of (CP1) to solutions of another nonlinear wave equation $v_{\tau\tau} - \Delta_y v = -(|y|/\sinh|y|)^{p-1} e^{-(p-3)\tau}|v|^{p-1}v$, which turns out to be relatively easier to work on.

Theorem 1.1. *Assume that A, ε are positive constants and $3 \leq p < 5$. Let $(u_0,u_1) \in \dot{H}^1 \times L^2$ be radial initial data so that*

$$\|\nabla u_0\|_{L^2(\mathbb{R}^3;d\mu)}, \|u_1\|_{L^2(\mathbb{R}^3;d\mu)} \leq A, \qquad d\mu = (|x|+1)^{1+2\varepsilon}dx.$$

Then the corresponding solution u to (CP1) scatters in both two time directions with

$$\|u\|_{L^{2(p-1)}L^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)} \leq C(A,\varepsilon) < \infty.$$

Here the upper bound $C(A,\varepsilon)$ are solely determined by the values of A and ε.

Here are some remarks regarding the initial data in the main theorem.

Remark 1.2. *The initial data (u_0, u_1) satisfy the inequality*

$$\begin{aligned}\int_{\mathbb{R}^3} \left(|\nabla u_0|^{\frac{3}{2}} + |u_1|^{\frac{3}{2}}\right) dx \leq & 2\left[\int_{\mathbb{R}^3} \left(|\nabla u_0|^2 + |u_1|^2\right)(1+|x|)^{1+2\varepsilon}\, dx\right]^{3/4} \left[\int_{\mathbb{R}^3} (1+|x|)^{-3-6\varepsilon}\, dx\right]^{1/4} \\ \leq & C(A,\varepsilon) < \infty.\end{aligned}$$

In other words we have $(u_0,u_1) \in \dot{W}^{1,3/2} \times L^{3/2}$. It immediately follows that $(u_0,u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ by the Sobolev embedding $\dot{W}^{1,3/2} \times L^{3/2} \hookrightarrow \dot{H}^{1/2} \times \dot{H}^{-1/2}$ and an interpolation.

Remark 1.3. *The radial assumption implies that the initial data (u_0, u_1) satisfy*

$$\int_0^\infty \left(|\partial_r u_0(r)|^2 + |u_1(r)|^2\right) r^{3+2\varepsilon} dr \leq (1/4\pi)A^2.$$

Remark 1.4. *Any pair (u_0, u_1) as in Theorem 1.1 comes with a finite energy*

$$E(u_0,u_1) = \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u_0(x)|^2 + \frac{1}{2}|u_1(x)|^2 + \frac{1}{p+1}|u_0(x)|^{p+1}\right] dx \leq C(A) < \infty.$$

In addition, u_0 satisfies a point-wise estimate $|u_0(x)| \leq A|x|^{-1-\varepsilon}$.

Proof. By Remark 1.3 we have $(0 < r_1 < r_2 < \infty)$

$$\begin{aligned}|u_0(r_1) - u_0(r_2)| \leq & \int_{r_1}^{r_2} |\partial_r u_0(r)| dr \leq \left(\int_{r_1}^{r_2} |\partial_r u_0(r)|^2 r^{3+2\varepsilon} dr\right)^{1/2} \left(\int_{r_1}^{r_2} r^{-3-2\varepsilon} dr\right)^{1/2} \\ \leq & A r_1^{-1-\varepsilon}. \end{aligned} \quad (3)$$

Next we recall the point-wise estimate for radial $\dot{H}^1$ functions $|u_0(x)| \leq C\|u_0\|_{\dot{H}^1}|x|^{-1/2}$ (Please refer to, for instance, Lemma 3.2 of [14]), make $r_2 \to \infty$ in the inequality (3) above and obtain a point-wise estimate $|u_0(x)| \leq A|x|^{-1-\varepsilon}$. Furthermore, we can combine this point-wise estimate with the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ to conclude $\|u_0\|_{L^{p+1}(\mathbb{R}^3)} \leq C(A)$. This immediately gives a finite upper bound on the energy. □

The idea Wave equations are time-invertible, thus it is sufficient to consider the positive time direction. Since the initial data come with a finite energy, Energy-subcriticality leads to the global existence of the corresponding solution u. In order to obtain the scattering result, we also need to show the critical space-time norm $\|u\|_{L^{2(p-1)}L^{2(p-1)}([0,\infty)\times\mathbb{R}^3)}$ is finite. In fact, we can choose a suitable positive number R and split the space-time $\mathbb{R}^3 \times [0,\infty)$ into two parts

$$\Omega_1 = \{(x,t) \in \mathbb{R}^3 \times [0,\infty) : |x| < t + R\}; \qquad \Omega_2 = \{(x,t) \in \mathbb{R}^3 \times [0,\infty) : |x| \geq t + R\}.$$

A combination of the small-data scattering result and the finite speed of propagation of wave equation gives a finite upper bound on the norm $\|u\|_{L^{2(p-1)}L^{2(p-1)}(\Omega_2)}$ as long as R is sufficiently large. The majority of this paper is devoted to the control of the norm $\|u\|_{L^{2(p-1)}L^{2(p-1)}(\Omega_1)}$. There are three major steps.

- We show that the function $v = \mathbf{T}u$ defined by

$$v(y,\tau) = \frac{\sinh|y|}{|y|} e^{\tau} u\left(e^{\tau}\frac{\sinh|y|}{|y|}\cdot y, t_0 + e^{\tau}\cosh|y|\right), \quad (y,\tau)\in\mathbb{R}^3\times\mathbb{R}$$

solves the non-linear wave equation

$$v_{\tau\tau} - \Delta_y v = -\left(\frac{|y|}{\sinh|y|}\right)^{p-1} e^{-(p-3)\tau}|v|^{p-1}v. \quad \text{(CP2)}$$

Here the number t_0 is a large negative number so that the image Ω of the set $\mathbb{R}^3\times\mathbb{R}^+$ under the geometric transformation $(y,\tau)\to(ye^{\tau}\sinh|y|/|y|, t_0+e^{\tau}\cosh|y|)$ contains the region Ω_1. Please pay attention that this transformation works for radial solutions only. In addition, we prove that v has a finite energy E at $\tau=0$ by a few decay properties of v for large y and an space-time integral estimate for small y.

- We show that the solution v satisfies a few space-time integral estimates. In particular we have a Morawetz estimate

$$\int_0^{\infty}\int_{\mathbb{R}^3} e^{-(p-3)\tau}\frac{|y|^{p-1}\cosh|y|}{\sinh^p|y|}|v(y,\tau)|^{p+1}dy\,d\tau \lesssim E < \infty.$$

Next we use the radial assumption again, apply the pointwise estimate $|v(y,\tau)|\lesssim \|v(\cdot,\tau)\|_{\dot{H}^1}\cdot |y|^{-1/2}\lesssim E^{1/2}\cdot|y|^{-1/2}$ and obtain

$$I' \doteq \int_0^{\infty}\int_{\mathbb{R}^3} e^{-(p-3)\tau}\left(\frac{|y|}{\sinh|y|}\right)^{p-1}|v(y,\tau)|^{2(p-1)}dy < \infty.$$

- We use the fact $\Omega_1\subset\Omega$, rewrite the integral of $|u|^{2(p-1)}$ over Ω in term of v via a change of variables formula and finally conclude

$$\begin{aligned}\iint_{\Omega_1}|u|^{2(p-1)}dx\,dt &\leq \iint_{\Omega}|u|^{2(p-1)}dx\,dt\\ &= \int_0^{\infty}\int_{\mathbb{R}^3} e^{-2(p-3)\tau}\left(\frac{|y|}{\sinh|y|}\right)^{2p-4}|v(y,\tau)|^{2(p-1)}dy\,d\tau \leq I' < \infty.\end{aligned}$$

Construction of T The transformation $v=\mathbf{T}u$ as defined above is one of the key ingredients of our proof. Its validity can be verified by a basic calculation, as given in Section 5. The author would also like to mention that the transformation can be constructed via two different routes:

Route 1 We can write $\mathbf{T}=\mathbf{T}_2\circ\mathbf{T}_1$. Here $\mathbf{T}_1$ is a transformation from the set of functions defined on the forward light cone $\{(x,t): t-t_0>|x|\}$ to the set of functions defined on $\mathbb{H}^3\times\mathbb{R}$, whose formula has been given by D. Tataru in the work [20]:

$$(\mathbf{T}_1 u)(s,\Theta,\tau) = e^{\tau}u(e^{\tau}\sinh s\cdot\Theta, t_0+e^{\tau}\cosh s).$$

Here $(s,\Theta)\in[0,\infty)\times\mathbb{S}^2$ are polar coordinates on the hyperbolic space $\mathbb{H}^3$. One can demonstrate the importance of this transformation by the fact

$$(\partial_\tau^2 - \Delta_{\mathbb{H}^3} - 1)\circ\mathbf{T}_1 = e^{2\tau}\mathbf{T}_1\circ(\partial_t^2-\Delta).$$

As a result, if u is a solution to (CP1), then the function $v_1=\mathbf{T}_1u$ solves the non-linear shifted wave equation on $\mathbb{H}^3$ (See [1, 18, 19] for Strichartz estimates and local theory on this type of equations)

$$\partial_\tau^2 v_1 - (\Delta_{\mathbb{H}^3}+1)v_1 = -e^{-(p-3)\tau}|v_1|^{p-1}v_1. \tag{4}$$

Next we introduce the second transformation[1] $(\mathbf{T}_2 v_1)(y,\tau) = \frac{\sinh|y|}{|y|} v_1(|y|,\tau)$, whose domain is the set of radial functions on $\mathbb{H}^3 \times \mathbb{R}$ and whose range is the set of radial functions on $\mathbb{R}^3 \times \mathbb{R}$. This transformation satisfies $(\partial_\tau^2 - \Delta_y) \circ \mathbf{T}_2 = \mathbf{T}_2 \circ (\partial_\tau^2 - \Delta_{\mathbb{H}^3} - 1)$. A basic calculation shows that if v_1 solves (4), then $v = \mathbf{T}_2 v_1$ satisfies (CP2).

Route 2 We have another decomposition $\mathbf{T} = \mathbf{T}_3^{-1} \circ \mathbf{T}_4 \circ \mathbf{T}_3$, where

$$(\mathbf{T}_3 u)(|x|,t) = |x| u(x,t); \qquad (\mathbf{T}_4 w)(s,\tau) = w(e^\tau \sinh s, t_0 + e^\tau \cosh s).$$

Both $\mathbf{T}_3 u$ and $\mathbf{T}_4 w$ are functions defined on $[0,\infty) \times \mathbb{R}$. Please pay attention that $\mathbf{T}_3 u$ is only defined for radial functions u. These two transformations satisfy the commutator identities

$$(\partial_t^2 - \partial_r^2) \circ \mathbf{T}_3 = \mathbf{T}_3 \circ (\partial_t^2 - \Delta_x); \qquad (\partial_\tau^2 - \partial_s^2) \circ \mathbf{T}_4 = e^{2\tau} \mathbf{T}_4 \circ (\partial_t^2 - \partial_r^2).$$

As a result, if u is a radial solution to (CP1), then $w = \mathbf{T}_3 u$ and $w_1 = \mathbf{T}_4 w$ solve the non-linear wave equations $\partial_t^2 w - \partial_r^2 w = -\frac{1}{r^{p-1}} |w|^{p-1} w$ and $\partial_\tau^2 w_1 - \partial_s^2 w_1 = -e^{-(p-3)\tau} \frac{1}{\sinh^{p-1} s} |w_1|^{p-1} w_1$, respectively.

The structure of this paper This paper is organized as follows. In section 2 we collect notations, recall the Strichartz estimates and introduce a local theory for a class of wave equations in the form of $\partial_t^2 v - \Delta v = -\phi(x) e^{-\kappa t} |v|^{p-1} u$ with a function $\phi : \mathbb{R}^3 \to [-1,1]$ and a constant $\kappa \geq 0$. In particular we can combine this with the energy conservation law to conclude that any solution to (CP1) with a finite energy is defined for all time. Next in Section 3 we discuss the global behaviour of solutions to the wave equation above with a suitable coefficient function $\phi : \mathbb{R}^3 \to [0,1]$. More precisely we have

- An energy monotonicity (conservation) law holds, i.e. the energy defined by

$$E(t) = \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla_x v(x,t)|^2 + \frac{1}{2} |v_t(x,t)|^2 + e^{-\kappa t} \phi(x) \frac{|v(x,t)|^{p+1}}{p+1} \right] dx$$

  is either a constant if $\kappa = 0$, or a nonincreasing function of time t if $\kappa > 0$. A direct corollary follows that any solution with a finite energy is defined for all $t \in [0,\infty)$.

- A Morawetz-type inequality holds if in addition $(p-1)\phi - x \cdot \nabla \phi \geq 0$ for all $x \in \mathbb{R}^3$.

$$\int_{t_0}^{\infty} \int_{\mathbb{R}^3} e^{-\kappa t} \cdot \frac{(p-1)\phi - x \cdot \nabla \phi}{|x|} \cdot |v|^{p+1} dx\, dt \lesssim_1 E(t_0).$$

  Here t_0 is an arbitrary time in the lifespan of solution v.

After all of these preparation work is finished, we prove the main theorem in the last three sections. In Section 4 we start by assuming that u is a solution as in the main theorem and proving a few preliminary estimates regarding u in the region $\Omega_2 = \{(x,t) \in \mathbb{R}^3 \times [0,\infty) : |x| > t + R\}$ for a suitable $R > 0$. Here we need to apply an "channel of energy" argument. Next in Section 5 we apply the transformation $\mathbf{T}$ and show that $v = \mathbf{T} u$ is indeed a solution to (CP2). In the final section we verify that v has a finite energy by the estimates obtained in Section 4, take advantage of the Morawetz estimate, rewrite the obtained integral estimates in term of u and eventually finish the proof.

[1] we need to use the radial assumption on v_1 in the definition.

2 Preliminary Results

2.1 Notations

The $\lesssim$ symbol We use the notation $A \lesssim B$ if there exists a constant c, so that the inequality $A \leq cB$ always holds. In addition, a subscript of the symbol $\lesssim$ indicates that the constant c is determined by the parameter(s) mentioned in the subscript but nothing else. In particular, $\lesssim_1$ means that the constant c is an absolute constant.

Radial functions Let $u(x,t)$ be a spatially radial function. By convention $u(r,t)$ represents the value of $u(x,t)$ when $|x| = r$.

Linear wave propagation Given a pair of initial data (u_0, u_1), we define $\mathbf{S}_{L,0}(t)(u_0,u_1)$ to be the solution u of the free linear wave equation $u_{tt} - \Delta u = 0$ with initial data $(u, u_t)|_{t=0} = (u_0, u_1)$. If we are also interested in the velocity u_t, we can use the notation

$$\mathbf{S}_L(t)(u_0,u_1) \doteq (u(\cdot,t), u_t(\cdot,t)), \qquad \mathbf{S}_L(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \doteq \begin{pmatrix} u(\cdot,t) \\ u_t(\cdot,t) \end{pmatrix}.$$

2.2 Local theory

In this subsection we consider the local theory of the equation

$$\left\{\begin{array}{ll} \partial_t^2 v - \Delta v = -\phi(x) e^{-\kappa t} |v|^{p-1} v, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ v(\cdot, t_0) = v_0 \in \dot{H}^1(\mathbb{R}^3); & \\ v_t(\cdot,t_0) = v_1 \in L^2(\mathbb{R}^3). & \end{array}\right. \tag{5}$$

Here $\phi : \mathbb{R}^3 \to [-1,1]$ is a measurable function, κ is a nonnegative constant and $p \in [3,5)$. This covers both equations (CP1) and (CP2).

Definition 2.1. *We say that a solution v solves the equation* (5) *in a time interval I containing t_0, if v satisfies*

- $(v(\cdot,t), v_t(\cdot,t)) \in C(I; \dot{H}^1 \times L^2(\mathbb{R}^3))$;
- *The norm* $\|v\|_{L^{2p/(p-3)} L^{2p}(J \times \mathbb{R}^3)}$ *is finite for any bound closed interval* $J \subseteq I$;
- *The integral equation*

$$v(\cdot,t) = \mathbf{S}_{L,0}(t-t_0)(v_0,v_1) + \int_{t_0}^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} G(\cdot, \tau, v(\cdot,\tau)) d\tau$$

  holds for all $t \in I$, here $G(x,t,v) = -\phi(x) e^{-\kappa t} |v|^{p-1} v$.

Strichartz estimates The basis of our local theory is the following generalized Strichartz estimates. (Please see Proposition 3.1 of [7], here we use the Sobolev version in $\mathbb{R}^3$)

Proposition 2.2. *Let $2 \leq q_1, q_2 \leq \infty$, $2 \leq r_1, r_2 < \infty$ and $\rho_1, \rho_2, s \in \mathbb{R}$ with*

$$1/q_i + 1/r_i \leq 1/2, \quad i = 1,2;$$
$$1/q_1 + 3/r_1 = 3/2 - s' + \rho_1; \qquad 1/q_2 + 3/r_2 = 1/2 + s' + \rho_2.$$

Let v be the solution of the following linear wave equation ($t_0 \in I$)

$$\left\{\begin{array}{ll} \partial_t^2 v - \Delta v = F(x,t), & (x,t) \in \mathbb{R}^3 \times I; \\ (v, v_t)|_{t=t_0} = (v_0, v_1) \in \dot{H}^{s'}(\mathbb{R}^3) \times \dot{H}^{s'-1}(\mathbb{R}^3). & \end{array}\right. \tag{6}$$

Then there exists a constant independent of I and initial data (u_0,u_1), so that

$$\begin{aligned} &\|(v(\cdot,t), v_t(\cdot,t))\|_{C(I;\dot{H}^{s'} \times \dot{H}^{s'-1})} + \|D_x^{\rho_1} v\|_{L^{q_1} L^{r_1}(I \times \mathbb{R}^3)} \\ &\quad \leq C \left(\|(v_0,v_1)\|_{\dot{H}^{s'} \times \dot{H}^{s'-1}} + \|D_x^{-\rho_2} F(x,t)\|_{L^{\bar{q}_2} L^{\bar{r}_2}(I \times \mathbb{R}^3)} \right). \end{aligned}$$

A fixed-point argument We first choose specific coefficients $\rho_1 = \rho_2 = 0$, $s' = 1$, $(q_1, r_1) = (2p/(p-3), 2p)$, $(q_2, r_2) = (\infty, 2)$ in the Strichartz estimates

$$\begin{aligned}\|(v(\cdot,t), v_t(\cdot,t))\|_{C([t_1,t_2];\dot{H}^1\times L^2)} + \|v\|_{L^{\frac{2p}{p-3}} L^{2p}([t_1,t_2]\times\mathbb{R}^3)} \\ \leq C_p \left[\|(v(\cdot,t_1), v_t(\cdot,t_1))\|_{\dot{H}^1\times L^2} + \|(\partial_t^2 - \Delta)v\|_{L^1 L^2([t_1,t_2]\times\mathbb{R}^3)}\right],\end{aligned}$$

and observe the inequalities

$$\|G(\cdot,\cdot,v)\|_{L^1L^2([t_1,t_2]\times\mathbb{R}^3)} \leq e^{-\kappa t_1}(t_2 - t_1)^{\frac{5-p}{2}} \|v\|^p_{L^{\frac{2p}{p-3}} L^{2p}([t_1,t_2]\times\mathbb{R}^3)};$$

$$\begin{aligned}\|G(\cdot,\cdot,v_1) - G(\cdot,\cdot,v_2)\|_{L^1L^2([t_1,t_2]\times\mathbb{R}^3)} \leq \left[\|v_1\|^{p-1}_{L^{\frac{2p}{p-3}} L^{2p}([t_1,t_2]\times\mathbb{R}^3)} + \|v_2\|^{p-1}_{L^{\frac{2p}{p-3}} L^{2p}([t_1,t_2]\times\mathbb{R}^3)}\right] \\ \times e^{-\kappa t_1}(t_2 - t_1)^{\frac{5-p}{2}} \|v_1 - v_2\|_{L^{\frac{2p}{p-3}} L^{2p}([t_1,t_2]\times\mathbb{R}^3)}.\end{aligned}$$

A fixed-point argument then shows (Our argument is similar to a lot of earlier works. See [10, 16], for instance.)

Theorem 2.3 (Local solution)**.** *Given a time t_0 and a pair $(v_0, v_1) \in \dot{H}^1 \times L^2$, then there is a maximal time interval $(t_0 - T_-(v_0, v_1, t_0), t_0 + T_+(v_0, v_1, t_0))$ in which the equation* (5) *with the initial condition $(v, v_t)|_{t=t_0} = (v_0, v_1)$ has a unique solution $v(x,t)$. In addition we have*

$$T_+(v_0, v_1, t_0) > T_1 \doteq C_1(p) e^{2\kappa t_0/(5-p)} \|(v_0, v_1)\|^{-2(p-1)/(5-p)}_{\dot{H}^1\times L^2(\mathbb{R}^3)};$$

$$\|v(x,t)\|_{L^{2p/(p-3)} L^{2p}([t_0, t_0+T_1]\times\mathbb{R}^3)} \leq C_2(p) \|(v_0, v_1)\|_{\dot{H}^1\times L^2(\mathbb{R}^3)}.$$

Remark 2.4. *If v is a solution to* (5)*, then we have $\|D^{1/2} v\|_{L^4L^4([a,b]\times\mathbb{R}^3)} < +\infty$ for any finite bounded interval $[a, b]$ contained in the maximal lifespan of v by the Strichartz estimates.*

Proposition 2.5. *Any solution u to (CP1) with a finite energy is global in time, i.e. it has a maximal lifespan $\mathbb{R}$.*

Proof. The conservation law of energy guarantees that the norm $\|(u(\cdot,t), u_t(\cdot,t))\|_{\dot{H}^1\times L^2} \lesssim E^{1/2}$ is uniformly bounded for all time t in the maximal lifespan of u. The combination of this fact and Theorem 2.3 implies that u is well-defined for all $t > 0$. Since (CP1) is time-invertible, we are able to conclude that the maximal lifespan of u must be $\mathbb{R}$. □

Perturbation theory Next let us consider the continuous dependence of the solutions to (5) on the initial data. The special case with $\phi(x) \equiv 1$ and $\kappa = 0$ has been proved in Appendix of [17]. We can prove the general case in exactly the same way.

Theorem 2.6. *Let $\tilde{v}$ be a solution of equation* (5) *in a bounded time interval I with initial data $(\tilde{v}_0, \tilde{v}_1)$, so that*

$$\|(\tilde{v}_0, \tilde{v}_1)\|_{\dot{H}^1\times L^2} < \infty; \qquad \|\tilde{v}\|_{L^{2p/(p-3)} L^{2p}(I\times\mathbb{R}^3)} < M.$$

There exist two constants $\varepsilon_0(I, M), C(I, M) > 0$, such that if $(v_0, v_1) \in \dot{H}^1 \times L^2$ satisfy

$$\|(v_0 - \tilde{v}_0, v_1 - \tilde{v}_1)\|_{\dot{H}^1\times L^2} < \varepsilon_0(I, M),$$

then the corresponding solution v of (5) *with initial data (v_0, v_1) is well-defined in I so that*

$$\|v - \tilde{v}\|_{L^{2p/(p-3)} L^{2p}(I\times\mathbb{R}^3)} \leq C(I, M) \|(v_0 - \tilde{v}_0, v_1 - \tilde{v}_1)\|_{\dot{H}^1\times L^2};$$

$$\left\|\begin{pmatrix} v(\cdot,t) \\ v_t(\cdot,t)\end{pmatrix} - \begin{pmatrix} \tilde{v}(\cdot,t) \\ \tilde{v}_t(\cdot,t)\end{pmatrix}\right\|_{C(I;\dot{H}^1\times L^2)} \leq C(I, M) \|(v_0 - \tilde{v}_0, v_1 - \tilde{v}_1)\|_{\dot{H}^1\times L^2}.$$

2.3 A Pointwise Estimate on Radial $\dot{H}^1$ Functions

Lemma 2.7. *There exists a constant C, so that the inequality $|f(x)| \leq C\|f\|_{\dot{H}^1(\mathbb{R}^3)} \cdot |x|^{-1/2}$ holds for any radial $\dot{H}^1(\mathbb{R}^3)$ function f and any $x \in \mathbb{R}^3 \setminus \{0\}$.*

This is a special case of Lemma 3.2 in [14]. A smooth approximation argument immediately gives

Corollary 2.8. *If $u(\cdot,t) \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^3))$, then $u(x,t)$ is continuous in $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$.*

3 A Wave Equation with a Time Dependent Nonlinearity

In this section we discuss the global behaviour of the solutions to the equation

$$\begin{cases} v_{tt} - \Delta u = -\phi(x) e^{-\kappa t} |v|^{p-1} v, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ v(\cdot, t_0) = v_0 \in \dot{H}^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3; \phi(x)dx); \\ v_t(\cdot, t_0) = v_1 \in L^2(\mathbb{R}^3). \end{cases} \tag{7}$$

Here we assume that $p \in [3,5)$, $\kappa \geq 0$ are constants and $\phi : \mathbb{R}^3 \to [0,1]$ is a measurable function. The equation (CP2) corresponds to the case with $\kappa = p - 3$ and $\phi(x) = \left(\frac{|x|}{\sinh |x|}\right)^{p-1}$. In this case the parameter $\kappa > 0$ whenever $p > 3$.

3.1 Monotonicity of the Energy

Now let us consider the "energy" defined by

$$E(t) = \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla_x v(x,t)|^2 + \frac{1}{2}|v_t(x,t)|^2 + e^{-\kappa t}\phi(x)\frac{|v(x,t)|^{p+1}}{p+1}\right] dx.$$

If u is sufficiently smooth and decays sufficiently fast near infinity, we can differentiate and obtain

$$\begin{aligned} E'(t) &= \int_{\mathbb{R}^3} \left[\nabla v \nabla v_t + v_t v_{tt} + e^{-\kappa t}\phi(x)|v|^{p-1} v v_t - \kappa e^{-\kappa t}\phi \frac{|v|^{p+1}}{p+1}\right] dx \\ &= \int_{\mathbb{R}^3} v_t \left(-\Delta v + v_{tt} + e^{-\kappa t}\phi|v|^{p-1}v\right) dx - \frac{\kappa}{p+1}\int_{\mathbb{R}^3} e^{-\kappa t}\phi|v|^{p+1} dx \\ &= -\frac{\kappa}{p+1}\int_{\mathbb{R}^3} e^{-\kappa t}\phi(x)|v(x,t)|^{p+1} dx \leq 0. \end{aligned}$$

One can verify that this formula of $E'(t)$ works for general solutions v of the Cauchy problem (7) as well by standard smooth approximation and cut-off techniques. Therefore we have

Proposition 3.1. *Let v be a solution to the Cauchy problem (7) in a time interval $[t_0, t_0 + T_+)$ with $E(t_0) < \infty$.*

- *If $\kappa > 0$, then $E(t)$ is a non-increasing function of $t \in [t_0, t_0 + T_+)$. In addition, we have the integral estimate*

$$\int_{t_0}^{t_0+T_+}\int_{\mathbb{R}^3} e^{-\kappa t}\phi(x)|v(x,t)|^{p+1} dx\, dt \leq \frac{p+1}{\kappa} E(t_0).$$

- *If $\kappa = 0$, then $E(t)$ is a constant independent of t.*

3.2 Global behaviour in the positive time direction

Assume that v is a solution to the Cauchy problem (7) with a maximal lifespan (t_0-T_-, t_0+T_+). Given any $t \in I_+ \doteq [t_0, t_0+T_+)$, Proposition 3.1 implies

$$\|(v(\cdot,t), v_t(\cdot,t))\|_{\dot{H}^1\times L^2} \leq [2E(t)]^{1/2} \leq [2E(t_0)]^{1/2}.$$

According to Theorem 2.3, this means that there are two positive constants T_1 and N_1, such that if $t \in I_+$, then we have $[t, t+T_1] \subseteq I_+$ and $\|v\|_{L^{2p/(p-3)}L^{2p}([t,t+T_1])} \leq N_1$. It immediately follows that $T_+ = +\infty$. Namely the solution u is defined for all time $t > t_0$. Furthermore, if $\kappa > 0$ we have

$$\begin{aligned}
\|G(x,t,v)\|_{L_t^1L_x^2([t_0,\infty)\times\mathbb{R}^3)} &= \sum_{j=0}^{\infty} \left\|e^{-\kappa t}\phi(x)|v|^{p-1}v\right\|_{L^1L^2([t_0+jT_1,t_0+(j+1)T_1]\times\mathbb{R}^3)}\\
&\leq \sum_{j=0}^{\infty} e^{-\kappa t_0 - j\kappa T_1} T_1^{(5-p)/2} \|v\|^p_{L^{2p/(p-3)}L^{2p}([t_0+jT_1,t_0+(j+1)T_1]\times\mathbb{R}^3)}\\
&= \sum_{j=0}^{\infty} e^{-\kappa t_0 - j\kappa T_1} T_1^{(5-p)/2} N_1^p < \infty.
\end{aligned}$$

Recalling the Strichartz estimates and the fact that the linear wave propagation preserves the $\dot{H}^1 \times L^2$ norm, we obtain

$$\begin{aligned}
&\lim_{t_1,t_2\to+\infty} \left\| \mathbf{S}_L(-t_1)\begin{pmatrix} v(\cdot,t_1)\\ v_t(\cdot,t_1)\end{pmatrix} - \mathbf{S}_L(-t_2)\begin{pmatrix} v(\cdot,t_2)\\ v_t(\cdot,t_2)\end{pmatrix}\right\|_{\dot{H}^1\times L^2}\\
=& \lim_{t_1,t_2\to+\infty} \left\| \mathbf{S}_L(t_2-t_1)\begin{pmatrix} v(\cdot,t_1)\\ v_t(\cdot,t_1)\end{pmatrix} - \begin{pmatrix} v(\cdot,t_2)\\ v_t(\cdot,t_2)\end{pmatrix}\right\|_{\dot{H}^1\times L^2}\\
\leq& \lim_{t_1,t_2\to+\infty} \|G(x,t,v)\|_{L_t^1L_x^2([t_1,t_2]\times\mathbb{R}^3)} = 0.
\end{aligned}$$

As a result, the pair $\mathbf{S}_L(-t)(v(\cdot,t), v_t(\cdot,t))$ converges in the space $\dot{H}^1 \times L^2$ as $t \to \infty$. Let us assume $\mathbf{S}_L(-t)(v(\cdot,t), v_t(\cdot,t)) \to (v_0^+, v_1^+)$. This is equivalent to saying

$$\lim_{t\to+\infty} \left\|(v(\cdot,t), v_t(\cdot,t)) - \mathbf{S}_L(t)(v_0^+, v_1^+)\right\|_{\dot{H}^1\times L^2} = 0.$$

We summarize our results below

Theorem 3.2 (Global behaviour). *Let v be a solution to the Cauchy problem* (7) *with a finite energy $E(t_0) < \infty$. Then v is well-defined for all $t \geq t_0$. If we also have $\kappa > 0$, then there exists a pair $(v_0^+, v_1^+) \in \dot{H}^1 \times L^2$ so that*

$$\lim_{t\to\infty} \left\|(v(\cdot,t), v_t(\cdot,t)) - \mathbf{S}_L(t)(v_0^+, v_1^+)\right\|_{\dot{H}^1\times L^2} = 0.$$

A combination of Theorem 3.2 and Proposition 3.1 immediately gives

Corollary 3.3. *Let v be a solution to the Cauchy problem* (7) *with $\kappa > 0$ and a finite energy $E(t_0) < \infty$. Then we have*

$$\int_{t_0}^{\infty}\int_{\mathbb{R}^3} e^{-\kappa t}\phi(x)|v(x,t)|^{p+1}dx\,dt \leq \frac{p+1}{\kappa}E(t_0).$$

3.3 A Morawetz-type Inequality

Proposition 3.4. *Let v be a solution to the Cauchy problem (7) in a time interval $[t_0, t_0+T_+)$ so that*

(I) $E(t_0) < \infty$;

(II) The inequalities $0 \leq \phi(x) \leq 1$ and $(p-1)\phi - x\cdot\nabla\phi \geq 0$ hold for all $x \in \mathbb{R}^3$.

Then we have the following Morawetz-type inequality

$$\int_{t_0}^{t_0+T_+}\int_{\mathbb{R}^3} e^{-\kappa t}\cdot\frac{(p-1)\phi - x\cdot\nabla\phi}{|x|}\cdot|v|^{p+1}dx\,dt \lesssim_1 E(t_0).$$

Remark 3.5. *According to Theorem 3.2, we can substitute the upper limit of integral by $+\infty$.*

Outline of the proof Let us consider a function $a(x) = |x|$ and define

$$M(t) = \int_{\mathbb{R}^3} v_t(x,t)\left(\nabla v(x,t)\cdot\nabla a(x) + \frac{1}{2}\Delta a(x) v(x,t)\right)dx.$$

A basic calculation shows

$$\nabla a = \frac{x}{|x|}, \qquad \Delta a = \frac{2}{|x|}, \qquad \mathbf{D}^2 a \geq 0, \qquad \Delta\Delta a \leq 0.$$

As a result, we obtain an upper bound on $|M(t)|$ by Hardy's inequality $\|v/|x|\|_{L^2} \lesssim \|\nabla v\|_{L^2}$:

$$|M(t)| \leq \|v_t(\cdot,t)\|_{L^2}\left(\|\nabla v(\cdot,t)\|_{L^2} + \|v(x,t)/|x|\|_{L^2_x(\mathbb{R}^3)}\right) \lesssim_1 E(t). \tag{8}$$

Next we calculate the derivative $M'(t)$ informally

$$\begin{aligned} M'(t) =& \int_{\mathbb{R}^3} v_{tt}\left(\nabla v\cdot\nabla a + \frac{1}{2}v\Delta a\right)dx + \int_{\mathbb{R}^3} v_t\left(\nabla v_t\cdot\nabla a + \frac{1}{2}v_t\Delta a\right)dx \\ =& \int_{\mathbb{R}^3}\Delta v\left(\nabla v\cdot\nabla a + \frac{1}{2}v\Delta a\right)dx - \int_{\mathbb{R}^3}\phi(x)e^{-\kappa t}|v|^{p-1}v\left(\nabla v\cdot\nabla a + \frac{1}{2}v\Delta a\right)dx \\ &+ \int_{\mathbb{R}^3} v_t\left(\nabla v_t\cdot\nabla a + \frac{1}{2}v_t\Delta a\right)dx \\ =& I_1 + I_2 + I_3. \end{aligned}$$

Let us start with I_1. For simplicity we use lower indices to represent partial derivatives.

$$\begin{aligned} I_1 =& \int_{\mathbb{R}^3}\left(\sum_{i,j=1}^3 v_{ii}v_j a_j\right)dx - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2\Delta a\,dx - \frac{1}{2}\int_{\mathbb{R}^3} v\nabla v\cdot\nabla\Delta a\,dx \\ =& -\int_{\mathbb{R}^3}\left(\sum_{i,j=1}^3 a_{ij}v_i v_j\right)dx - \int_{\mathbb{R}^3}\left(\sum_{i,j=1}^3 a_j v_i v_{ij}\right)dx - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2\Delta a\,dx + \frac{1}{4}\int_{\mathbb{R}^3}|v|^2\Delta\Delta a\,dx \\ \leq& -\frac{1}{2}\int_{\mathbb{R}^3}\nabla a\cdot\nabla(|\nabla v|^2)dx - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2\Delta a\,dx \\ =& 0. \end{aligned}$$

Here we use the facts $\mathbf{D}^2 a \geq 0$ and $\Delta\Delta a \leq 0$. In addition we have

$$\begin{aligned}
I_2 =& -\frac{1}{p+1}\int_{\mathbb{R}^3} \phi(x) e^{-\kappa t} \nabla(|v|^{p+1})\cdot \nabla a\, dx - \frac{1}{2}\int_{\mathbb{R}^3} \phi(x) e^{-\kappa t}|v|^{p+1}\Delta a\, dx\\
=& \frac{1}{p+1}\int_{\mathbb{R}^3} e^{-\kappa t}|v|^{p+1}\nabla\phi\cdot\nabla a dx + \left(\frac{1}{p+1}-\frac{1}{2}\right)\int_{\mathbb{R}^3} e^{-\kappa t}|v|^{p+1}\phi\Delta a\, dx\\
=& \frac{1}{p+1}\int_{\mathbb{R}^3} e^{-\kappa t}|v|^{p+1}\left(\nabla\phi\cdot\nabla a - \frac{p-1}{2}\phi\Delta a\right) dx\\
=& \frac{-1}{p+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} e^{-\kappa t}\cdot\frac{(p-1)\phi - x\cdot\nabla\phi}{|x|}\cdot|v|^{p+1} dx\, dt.
\end{aligned}$$

Finally

$$I_3 = \frac{1}{2}\int_{\mathbb{R}^3}\nabla(|\partial_t v|^2)\cdot\nabla a\, dx + \frac{1}{2}\int_{\mathbb{R}^3}|\partial_t v|^2\Delta a\, dx = 0.$$

Now we collect all the terms above and then integrate from $t = t_1$ to $t = t_2$:

$$M(t_2) - M(t_1) \leq \frac{-1}{p+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} e^{-\kappa t}\cdot\frac{(p-1)\phi - x\cdot\nabla\phi}{|x|}\cdot|v|^{p+1} dx\, dt.$$

We plug the upper bound on $|M(t)|$ as given in (8) into the left hand side above, recall the monotonicity of $E(t)$ and finally complete our proof.

Remark 3.6. *The argument above works only for solutions v that satisfies certain regularity conditions. However, Proposition 3.4 still holds for all solutions v with a finite energy $E(t_0) < \infty$. This can be proved via standard smooth approximation and cut-off techniques. Please refer to Section 4 of [18] for more details about this type of argument.*

3.4 An Equivalent Condition of Scattering

Let us start by a technical result.

Proposition 3.7. *Let v be a solution to the Cauchy problem (7) in a bounded closed time interval $I = [a, b]$ with initial data $(v_0, v_1) \in (\dot{H}^1 \cap \dot{H}^{s_p}) \times (L^2 \cap \dot{H}^{s_p - 1})$. Then we have $(v(\cdot, t), v_t(\cdot, t)) \in C(I; \dot{H}^{s_p} \times \dot{H}^{s_p-1})$ and*

$$\|D^{s_p - 1/2} v\|_{L^4 L^4([a,b]\times\mathbb{R}^3)} < +\infty.$$

Proof. Let us recall the Strichartz estimate

$$\begin{aligned}
&\|(v(\cdot,t), v_t(\cdot,t))\|_{C(I;\dot{H}^{s_p}\times\dot{H}^{s_p-1})} + \|D^{s_p-1/2}v\|_{L^4L^4([a,b]\times\mathbb{R}^3)}\\
&\qquad\qquad \lesssim \|(v_0, v_1)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} + \|(\partial_t^2 - \Delta)v\|_{L^{\frac{2}{1+s_p}}L^{\frac{2}{2-s_p}}(I\times\mathbb{R}^3)}.
\end{aligned}$$

As a result, it suffices to show

$$\left\|-e^{-\kappa t}\phi(x)|v|^{p-1}v\right\|_{L^{\frac{2}{1+s_p}}L^{\frac{2}{2-s_p}}(I\times\mathbb{R}^3)} < \infty \Leftrightarrow \left\|\phi^{1/p}v\right\|_{L^{\frac{4(p-1)p}{5p-9}}L^{\frac{4(p-1)p}{p+3}}(I\times\mathbb{R}^3)} < \infty. \tag{9}$$

A combination of the Sobolev embeddings $\dot{H}^{s_p} \hookrightarrow L^{3(p-1)/2}$, $\dot{H}^1 \hookrightarrow L^6$ and the interpolation $L^{3(p-1)/2} \cap L^6 \hookrightarrow L^{p+1}$ implies that $u_0 \in L^{p+1}$. Therefore the initial data come with a finite energy $E(0) < \infty$. Two space-time estimates regarding $\phi^{1/p}v$ immediately follow: On one hand, the monotonicity of $E(t)$ gives

$$\sup_{t\in I}\int_{\mathbb{R}^3} e^{-\kappa t}\phi(x)|v(x,t)|^{p+1} dx\, dt < \infty \Rightarrow \left\|\phi^{1/p}v\right\|_{L^\infty L^{p+1}(I\times\mathbb{R}^3)} < \infty.$$

On the other hand, the Strichartz estimates give

$$\|v\|_{L^5L^{10}(I\times\mathbb{R}^3)} < \infty \Longrightarrow \left\|\phi^{1/p}v\right\|_{L^5L^{10}(I\times\mathbb{R}^3)} < \infty.$$

We combine these two inequalities via an interpolation (with ratio $(5-p)(2p+3)(p+1) : 5(p-3)(3p+1)$) to obtain

$$\left\|\phi^{1/p}v\right\|_{L^{\frac{2p(p-1)(9-p)}{(p-3)(3p+1)}}L^{\frac{4(p-1)p}{p+3}}(I\times\mathbb{R}^3)} < +\infty.$$

This is a sufficient condition of (9) because I is a finite interval and $\frac{2p(p-1)(9-p)}{(p-3)(3p+1)} \geq \frac{4(p-1)p}{5p-9}$. □

Proposition 3.8 (Scattering with a finite $L^{2(p-1)}L^{2(p-1)}$ norm)**.** *Let u be a solution to (CP1) with initial data $(u_0,u_1)\in(\dot H^1\cap\dot H^{s_p})\times(L^2\cap\dot H^{s_p-1})$. If $\|u\|_{L^{2(p-1)}L^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)}<\infty$, then u scatters in both two time directions. More precisely, there exist two pairs $(u_0^\pm,u_1^\pm)\in(\dot H^1\cap\dot H^{s_p})\times(L^2\cap\dot H^{s_p-1})$, so that the following limit holds for each $s'\in[s_p,1]$*

$$\lim_{t\to\pm\infty}\left\|(u(\cdot,t),u_t(\cdot,t))-\mathbf{S}_L(t)(u_0^\pm,u_1^\pm)\right\|_{\dot H^{s'}\times\dot H^{s'-1}(\mathbb{R}^3)}=0.$$

Proof. Since the equation is time-invertible, it suffices to consider the case $t\to+\infty$. In the argument below, we temporarily assume that s' is either 1 or s_p. We start by picking up an arbitrary finite time interval $[a,b]$ and applying the Strichartz estimates

$$\begin{aligned}&\|D_x^{s'-1/2}u\|_{L^4L^4([a,b]\times\mathbb{R}^3)}\\ &\quad\leq C\|(u(\cdot,a),u_t(\cdot,a))\|_{\dot H^{s'}\times\dot H^{s'-1}}+C\|D_x^{s'-1/2}(-|u|^{p-1}u)\|_{L^{4/3}L^{4/3}([a,b]\times\mathbb{R}^3)}\\ &\quad\leq C\|(u(\cdot,a),u_t(\cdot,a))\|_{\dot H^{s'}\times\dot H^{s'-1}}+C_{s',p}\|u\|^{p-1}_{L^{2(p-1)}L^{2(p-1)}([a,b]\times\mathbb{R}^3)}\|D_x^{s'-1/2}u\|_{L^4L^4([a,b]\times\mathbb{R}^3)}.\end{aligned}$$

In the last step above, we apply the chain rule with fractional derivatives. Please see Proposition 3.1 of [2] for more details. By the assumption $\|u\|_{L^{2(p-1)}L^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)}<\infty$, we can fix a large number a, so that $C_{s',p}\|u\|^{p-1}_{L^{2(p-1)}L^{2(p-1)}([a,\infty)\times\mathbb{R}^3)}<1/2$. We plug this upper bound into the inequality above, recall the fact $\|D_x^{s'-1/2}u\|_{L^4L^4([a,b]\times\mathbb{R}^3)}<\infty$ that comes from either Remark 2.4, if $s'=1$, or Proposition 3.7, if $s'=s_p$, and obtain

$$\|D_x^{s'-1/2}u\|_{L^4L^4([a,b]\times\mathbb{R}^3)}<2C\|(u(\cdot,a),u_t(\cdot,a))\|_{\dot H^{s'}\times\dot H^{s'-1}}<\infty.$$

Here the finiteness of $\dot H^{s'}\times\dot H^{s'-1}$norm comes from either the definition of a solution, if $s'=1$, or Proposition 3.7, if $s'=s_p$. Please note that the upper bound here does not depend on the right endpoint b. A combination of this uniform upper bound with the fact that $\mathbf{S}_L(t)$ preserves the $\dot H^{s'}\times\dot H^{s'-1}$ norm implies

$$\begin{aligned}&\limsup_{t_1,t_2\to+\infty}\left\|\mathbf{S}_L(-t_2)\begin{pmatrix}u(\cdot,t_2)\\u_t(\cdot,t_2)\end{pmatrix}-\mathbf{S}_L(-t_1)\begin{pmatrix}u(\cdot,t_1)\\u_t(\cdot,t_1)\end{pmatrix}\right\|_{\dot H^{s'}\times\dot H^{s'-1}}\\ =&\limsup_{t_1,t_2\to+\infty}\left\|\begin{pmatrix}u(\cdot,t_2)\\u_t(\cdot,t_2)\end{pmatrix}-\mathbf{S}_L(t_2-t_1)\begin{pmatrix}u(\cdot,t_1)\\u_t(\cdot,t_1)\end{pmatrix}\right\|_{\dot H^{s'}\times\dot H^{s'-1}}\\ \leq&C\limsup_{t_1,t_2\to+\infty}\|D_x^{s'-1/2}(-|u|^{p-1}u)\|_{L^{4/3}L^{4/3}([t_1,t_2]\times\mathbb{R}^3)}\\ \leq&C_{s',p}\limsup_{t_1,t_2\to+\infty}\left(\|u\|^{p-1}_{L^{2(p-1)}L^{2(p-1)}([t_1,t_2]\times\mathbb{R}^3)}\|D_x^{s'-1/2}u\|_{L^4L^4([t_1,t_2]\times\mathbb{R}^3)}\right)=0.\end{aligned}$$

As a result, the pair $\mathbf{S}_L(-t)(u(\cdot,t),u_t(\cdot,t))$ converges in the space $\dot H^{s'}\times\dot H^{s'-1}(\mathbb{R}^3)$ as $t\to+\infty$. Since the argument above works for both $s'=1$ and $s'=s_p$, we know that there exists a pair $(u_0^+,u_1^+)\in(\dot H^1\cap\dot H^{s_p})\times(L^2\cap\dot H^{s_p-1})$ so that the limit

$$\lim_{t\to+\infty}\left\|\mathbf{S}_L(-t)(u(\cdot,t),u_t(\cdot,t))-(u_0^+,u_1^+)\right\|_{\dot H^{s'}\times\dot H^{s'-1}(\mathbb{R}^3)}=0$$

holds for $s' \in \{1, s_p\}$. By a basic interpolation the limit above holds for all $s' \in [s_p, 1]$. This is equivalent to our conclusion

$$\lim_{t\to+\infty} \left\|(u(\cdot,t), u_t(\cdot,t)) - \mathbf{S}_L(t)(u_0^+, u_1^+)\right\|_{\dot{H}^{s'}\times\dot{H}^{s'-1}(\mathbb{R}^3)} = 0.$$

$\square$

4 Preliminary Estimates on Solutions

Lemma 4.1. *(See also Lemma 6.12 of [18] for the 2D version) Let u be a solution to the linear wave equation*

$$\begin{cases} \partial_t^2 u - \Delta u = F(x,t), & (x,t) \in \mathbb{R}^3 \times [0,T]; \\ u|_{t=0} = u_0; \\ \partial_t u|_{t=0} = u_1; \end{cases}$$

with radial data u_0, u_1 and F. Assume that (u_0, u_1) satisfy the inequalities

$$|u_0(x)| \leq A_1 |x|^{-1-\alpha}, \text{ if } |x| > R; \qquad \int_{|x|>R} |x|^{1+2\alpha} |u_1(x)|^2 dx \leq A_1^2;$$
$$|F(x,t)| \leq B_1 |x|^{-3} (|x|-t)^{-\beta}, \text{ if } t \in [0,T], \ |x| > R+t$$

with constants $R, A_1, B_1 > 0$ and $0 < \alpha, \beta < 1/2$. Then there exists a constant $C = C(\alpha,\beta) \geq 1$ such that the solution u satisfies

$$|u(x,t)| \leq C|x|^{-1} \left[A_1(|x|-t)^{-\alpha} + B_1(|x|-t)^{-\beta}\right], \text{ if } t \in [0,T] \text{ and } |x| > R+t.$$

Remark 4.2. *In the proof of Lemma 4.1 (as well as Corollary 4.4 below) we always assume that u is sufficiently smooth. Otherwise we can apply standard smooth approximation techniques.*

Proof. Let us consider the function $w : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ defined by the formula $w(r,t) = ru(r,t)$. One can check that the function w satisfies the following wave equation defined on $\mathbb{R}^+ \times [0,T]$

$$\partial_t^2 w - \partial_r^2 w = rF(r,t).$$

An explicit formula for the solution to a one-dimensional wave equation shows that

$$\begin{aligned} w(r_0,t_0) =& \frac{1}{2}\left[w(r_0-t_0,0) + w(r_0+t_0,0)\right] + \frac{1}{2}\int_{r_0-t_0}^{r_0+t_0} \partial_t w(r,0) dr \\ &+ \frac{1}{2}\int_0^{t_0}\int_{r_0-t_0+t}^{r_0+t_0-t} rF(r,t)\, dr dt, \end{aligned} \tag{10}$$

whenever $r_0 > t_0 + R$ and $t_0 \in [0,T]$. Our assumptions on F and the initial data u_0, u_1 give the upper bounds

$$|w(r_0-t_0,0)| \leq A_1(r_0-t_0)^{-\alpha}; \quad |w(r_0+t_0,0)| \leq A_1(r_0+t_0)^{-\alpha}; \quad rF(r,t) \leq B_1 r^{-2}(r-t)^{-\beta};$$

and

$$\begin{aligned} \left|\int_{r_0-t_0}^{r_0+t_0} \partial_t w(r,0) dr\right| &= \left|\int_{r_0-t_0}^{r_0+t_0} r u_1(r) dr\right| \\ &\leq \left(\int_{r_0-t_0}^{r_0+t_0} r^{-1-2\alpha} dr\right)^{1/2} \left(\int_{r_0-t_0}^{r_0+t_0} r^{3+2\alpha}|u_1(r)|^2 dr\right)^{1/2} \\ &\lesssim_\alpha (r_0-t_0)^{-\alpha} \left(\int_{|x|>r_0-t_0} |x|^{1+2\alpha}|u_1(x)|^2 dx\right)^{1/2} \\ &\leq A_1(r_0-t_0)^{-\alpha}. \end{aligned}$$

We then plug the upper bounds above into the identity (10) and obtain

$$
\begin{aligned}
|w(r_0,t_0)| \leq &\frac{A_1}{2}\left[(r_0-t_0)^{-\alpha}+(r_0+t_0)^{-\alpha}\right]+\frac{1}{2}\left|\int_{r_0-t_0}^{r_0+t_0}\partial_t w(r,0)dr\right| \\
&+\frac{B_1}{2}\int_0^{t_0}\int_{r_0-t_0+t}^{r_0+t_0-t} r^{-2}(r-t)^{-\beta}\,drdt \\
\leq &C_\alpha A_1(r_0-t_0)^{-\alpha}+\frac{B_1}{2}\int_{r_0-t_0}^{r_0+t_0}\int_s^{(r_0+t_0+s)/2} r^{-2}s^{-\beta}\,drds \\
\leq &C_\alpha A_1(r_0-t_0)^{-\alpha}+\frac{B_1}{2}\int_{r_0-t_0}^{r_0+t_0} s^{-1-\beta}ds \\
\leq &C_\alpha A_1(r_0-t_0)^{-\alpha}+C_\beta B_1(r_0-t_0)^{-\beta}.
\end{aligned}
$$

Here we deal with the double integral by the change of variables $(r,s)=(r,r-t)$. Finally we recall $w=ru$, divide both sides of the inequality above by r_0 and finish the proof. ☐

Proposition 4.3. *Assume $3\leq p<5$. Let (u_0,u_1) and A,ε be initial data and positive constants as in Theorem 1.1. Fix any constant $\delta<\min\{\varepsilon,1/6\}$. Then there exist constants $B_1=B_1(\delta)>0$ and $R=R(\delta,\varepsilon,A)>1$, such that the solution u to (CP1) with initial data (u_0,u_1) satisfies*

$$
|u(x,t)|\leq B_1|x|^{-1}(|x|-t)^{-\delta},\quad \textit{if } t\geq 0 \textit{ and } |x|>t+R. \tag{11}
$$

Proof. Let $C=C(\delta,3\delta)$ be the constant as in the conclusion of Lemma 4.1. We can always find two small positive constants $A_1=A_1(\delta)$ and $B_1=B_1(\delta)<1$, such that

$$
B_1>C(A_1+B_1^3).
$$

By Remark 1.3, Remark 1.4 and the assumption $\delta<\varepsilon$, we can always find a large constant $R=R(A,\varepsilon,\delta)>1$, such that if $|x|>R$, then

$$
|u_0(x)|<A_1|x|^{-1-\delta}; \qquad\qquad \int_{|x|>R}|x|^{1+2\delta}|u_1(x)|^2dx<A_1^2.
$$

We claim that these constants B_1 and R work. In fact, If t_1 is sufficiently small, then the restriction of solution u to the time interval $[0,t_1]$ can be obtained by a fixed-point argument according to our local theory. More precisely, if we set $\tilde{u}_0\equiv 0$ and define

$$
\tilde{u}_{n+1}(\cdot,t)=\mathbf{S}_{L,0}(t)(u_0,u_1)+\int_0^t\frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(\tilde{u}_n(\cdot,\tau))d\tau,
$$

where $F(u)=-|u|^{p-1}u$, then we have

$$
\lim_{n\to\infty}\|\tilde{u}_n-u\|_{L^{\frac{2p}{p-3}}L^{2p}([0,t_1]\times\mathbb{R}^3)}=0.
$$

An induction argument immediately follows:

(I) The function $\tilde{u}_0$ satisfies the inequality (11) if $t\in[0,t_1]$;

(II) If $\tilde{u}_n$ satisfies (11) for $t\in[0,t_1]$, then we have

$$
|F(\tilde{u}_n(x,t))|=\left|B_1|x|^{-1}(|x|-t)^{-\delta}\right|^p\leq B_1^3|x|^{-3}(|x|-t)^{-3\delta},\quad \text{if } |x|>t+R \text{ and } 0\leq t\leq t_1.
$$

Here we can substitute p by its lower bound 3 since we have assumed $B_1<1$, $|x|>|x|-t>R>1$. As a result, we can apply Proposition 4.1 and obtain

$$
\begin{aligned}
|\tilde{u}_{n+1}(x,t)| \leq &C(\delta,3\delta)|x|^{-1}\left[A_1(|x|-t)^{-\delta}+B_1^3(|x|-t)^{-3\delta}\right] \\
\leq &C(A_1+B_1^3)|x|^{-1}(|x|-t)^{-\delta} \\
\leq &B_1|x|^{-1}(|x|-t)^{-\delta},
\end{aligned}
$$

whenever $t\in[0,t_1]$ and $|x|>t+R$.

In summary, $\tilde{u}_n$ satisfies (11) for all $n \geq 0$ and $t \in [0, t_1]$. Making $n \to \infty$, we conclude that u satisfies (11) for $t \in [0, t_1]$. Passing to a limit in $L^{2p/(p-3)} L^{2p}$ usually gives an almost everywhere inequality. In this particular case, however, we obtain a pointwise inequality by the continuity of u, thanks to Corollary 2.8. In order to generalize this result from small time to all time $t \in [0, T]$ we only need to iterate our argument above. More details about this "double induction" argument can be found in Proposition 6.16 of the author's joint work [18] with G. Staffilani. □

Corollary 4.4. *Let (u_0, u_1) be initial data as in Theorem 1.1 and A, ε, δ, B_1, R be constants associated to it as above. Then there exist a function $f : [R, \infty) \to \mathbb{R}$ with*

$$\int_R^\infty s^{1+\delta} |f(s)|^2 \, ds \lesssim_{A,\varepsilon,\delta} 1$$

so that for all $t \geq 0$ and $r > t + R$ the function $w(r,t) = ru(r,t)$ satisfies

$$|w_t(r,t) + w_r(r,t)| \leq f(r+t); \qquad |w_t(r,t) - w_r(r,t)| \leq f(r-t). \tag{12}$$

Proof. For simplicity we define $z_1(r,t) = w_t(r,t) + w_r(r,t)$ and $z_2(r,t) = w_t(r,t) - w_r(r,t)$. Since z_1, z_2 satisfy the identities

$$\begin{aligned} \frac{\partial}{\partial s}[z_1(r+t-s, s)] &= (r+t-s) F(r+t-s, s); \\ \frac{\partial}{\partial s}[z_2(r-t+s, s)] &= (r-t+s) F(r-t+s, s); \end{aligned}$$

where the function F is defined as $F(r,t) = -|u(r,t)|^{p-1} u(r,t)$, we can integrate from $s = 0$ to $s = t$ by the fundamental theorem of calculus

$$\begin{aligned} z_1(r,t) &= z_1(r+t, 0) + \int_0^t (r+t-s) F(r+t-s, s) ds; \\ z_2(r,t) &= z_2(r-t, 0) + \int_0^t (r-t+s) F(r-t+s, s) ds. \end{aligned}$$

Next we rewrite $z_1(r+t, 0), z_2(r-t, 0)$ in term of u_0, u_1 by their definition and obtain

$$\begin{aligned} z_1(r,t) &= (r+t)\left[u_1(r+t) + \partial_r u_0(r+t)\right] + u_0(r+t) + \int_0^t (r+t-s) F(r+t-s, s) \, ds; \\ z_2(r,t) &= (r-t)\left[u_1(r-t) - \partial_r u_0(r-t)\right] - u_0(r-t) + \int_0^t (r-t+s) F(r-t+s, s) \, ds. \end{aligned}$$

We claim that we can choose $f(s) = s|u_1(s)| + s|\partial_r u_0(s)| + C s^{-1-\delta}$ for a suitable constant $C = C(A, \varepsilon, \delta)$. It follows Remark 1.3, Remark 1.4 and a couple of estimates on the integrals in the expression of z_1, z_2 . For the first integral we have

$$\begin{aligned} \left| \int_0^t (r+t-s) F(r+t-s, s) \, ds \right| &\lesssim \int_0^t (r+t-s) \left[(r+t-s)^{-1} (r+t-2s)^{-\delta}\right]^3 ds \\ &\lesssim (r+t)^{-2} \int_0^t (r+t-2s)^{-\delta} \, ds \\ &\lesssim (r+t)^{-1-\delta}. \end{aligned}$$

Here we need to use the point-wise estimate $u(r,t) \lesssim r^{-1}(r-t)^{-\delta}$ and our assumption

$$t > 0, \; r - t > R > 1, \; 0 < s < t \Rightarrow \left\{ \begin{array}{l} r + t - s > r \gtrsim r + t, \\ r + t - 2s > r - t > R > 1. \end{array} \right.$$

The second integral can be dealt with in a similar way

$$\left|\int_0^t (r-t+s)F(r-t+s,s)\,ds\right| \lesssim \int_0^t (r-t+s)\left[(r-t+s)^{-1}(r-t)^{-\delta}\right]^3 ds$$
$$\lesssim (r-t)^{-\delta}\int_0^t (r-t+s)^{-2}\,ds$$
$$\lesssim (r-t)^{-1-\delta}.$$

□

Remark 4.5. *An instance of the "channel of energy" method has been used in the proof above. Please see [11] to learn more about this powerful tool for the study of wave equations.*

5 A transformation

Let $u(x,t)$ be a global and radial solution to (CP1). We consider the function $v = \mathbf{T}u$ defined by

$$v(y,\tau) = \frac{\sinh|y|}{|y|} e^{\tau} u\left(e^{\tau}\frac{\sinh|y|}{|y|}\cdot y, t_0 + e^{\tau}\cosh|y|\right), \quad (y,\tau)\in\mathbb{R}^3\times\mathbb{R}.$$

Here t_0 is a negative number to be determined later. This transformation can be rewritten in the form of $(\mathbf{T}u)(y,\tau) = \frac{\sinh|y|}{|y|}e^{\tau}u(\tilde{\mathbf{T}}(y,\tau))$, where the geometric transformation $\tilde{\mathbf{T}} : \mathbb{R}^3\times\mathbb{R} \to \{(x,t)\in\mathbb{R}^3\times\mathbb{R} : t - t_0 > |x|\}$ is defined by

$$\tilde{\mathbf{T}}(y,\tau) = \left(e^{\tau}\frac{\sinh|y|}{|y|}\cdot y, t_0 + e^{\tau}\cosh|y|\right).$$

In particular, $\tilde{\mathbf{T}}$ maps the hyperplane $\tau = \tau_0$ in the y-τ space-time to the upper sheet of the hyperboloid $(t-t_0)^2 - |x|^2 = e^{2\tau_0}$ in the x-t space-time.

Radial expression The function v is still a radial function and can be given in term of polar coordinates $(s,\Theta,\tau)\in[0,\infty)\times\mathbb{S}^2\times\mathbb{R}$ by

$$v(s,\Theta,\tau) = \frac{\sinh s}{s}e^{\tau}u(e^{\tau}\sinh s\cdot\Theta, t_0 + e^{\tau}\cosh s).$$

For simplicity we can omit Θ and write

$$v(s,\tau) = \frac{\sinh s}{s}e^{\tau}u(e^{\tau}\sinh s, t_0 + e^{\tau}\cosh s).$$

Differentiation Let us recall that the function $w(r,t) = ru(r,t)$ satisfies the equation $w_{tt} - w_{rr} = -r|u|^{p-1}u$, we can rewrite the function $sv(s,\tau)$ in the form of

$$sv(s,\tau) = w(e^{\tau}\sinh s, t_0 + e^{\tau}\cosh s).$$

A simple calculation shows

$$(sv)_{\tau} = (e^{\tau}\sinh s)w_r + (e^{\tau}\cosh s)w_t; \qquad (sv)_s = (e^{\tau}\cosh s)w_r + (e^{\tau}\sinh s)w_t. \tag{13}$$

The values of w_r and w_t here are taken at the point $(e^\tau \sinh s, t_0 + e^\tau \cosh s)$. Next we can differentiate again and obtain [2].

$$
\begin{aligned}
(sv)_{\tau\tau} =& (e^\tau \sinh s) w_r + (e^\tau \sinh s)^2 w_{rr} + (e^\tau \sinh s)(e^\tau \cosh s) w_{rt} \\
&+ (e^\tau \cosh s) w_t + (e^\tau \cosh s)(e^\tau \sinh s) w_{tr} + (e^\tau \cosh s)^2 w_{tt}; \\
(sv)_{ss} =& (e^\tau \sinh s) w_r + (e^\tau \cosh s)^2 w_{rr} + (e^\tau \cosh s)(e^\tau \sinh s) w_{rt} \\
&+ (e^\tau \cosh s) w_t + (e^\tau \sinh s)(e^\tau \cosh s) w_{tr} + (e^\tau \sinh s)^2 w_{tt}.
\end{aligned}
$$

Therefore we have (let us recall $r = e^\tau \sinh s$)

$$
\begin{aligned}
v_{\tau\tau} - v_{ss} - \frac{2}{s} v_s =& \frac{1}{s}\left[(sv)_{\tau\tau} - (sv)_{ss}\right] = \frac{e^{2\tau}}{s}\left[w_{tt} - w_{rr}\right] = -\frac{e^{2\tau}}{s} r |u|^{p-1} u \\
=& -\left(\frac{s}{\sinh s}\right)^{p-1} e^{-(p-3)\tau} \left|\frac{\sinh s}{s} e^\tau u\right|^{p-1} \frac{\sinh s}{s} e^\tau u \\
=& -\left(\frac{s}{\sinh s}\right)^{p-1} e^{-(p-3)\tau} |v|^{p-1} v.
\end{aligned}
$$

In other words, $v(y,\tau)$ satisfies the non-linear wave equation

$$
v_{\tau\tau} - \Delta_y v = -\left(\frac{|y|}{\sinh |y|}\right)^{p-1} e^{-(p-3)\tau} |v|^{p-1} v, \qquad (\tau, y) \in \mathbb{R} \times \mathbb{R}^3. \quad (CP3)
$$

Finally a basic calculation gives the following change of variables formula for integrals of radial functions

$$
dx\, dt = 4\pi r^2 dr\, dt = 4\pi e^{4\tau} \sinh^2 s\, ds\, d\tau = e^{4\tau} \left(\frac{\sinh |y|}{|y|}\right)^2 dy\, d\tau. \tag{14}
$$

6 Proof of the Main Theorem

Let us consider a solution u to (CP1) as given in Theorem 1.1 with the constants A, ε. We first fix a number $\delta = \min\{\varepsilon/2, 1/10\}$ and let B_1, R be the constants as given in Proposition 4.3. In addition, we define $\bar{t} = \sqrt{R^2 + 1} + 1$. Please note that all these constants δ, B_1, R and $\bar{t}$ are determined solely by A and ε. Next we fix a negative time $t_0 = -\bar{t}$ and perform the transformation $v = \mathbf{T} u$ as described in the previous section. We claim

Lemma 6.1. *There exists a time $\tau \in [-1, 0]$, so that the energy*

$$
\begin{aligned}
E(\tau) =& \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla_y v(y,\tau)|^2 + \frac{1}{2}|v_\tau(x,\tau)|^2 + e^{-(p-3)\tau} \left(\frac{|y|}{\sinh |y|}\right)^{p-1} \frac{|v(y,\tau)|^{p+1}}{p+1}\right] dy \\
&< C(A, \varepsilon).
\end{aligned}
$$

Here $C(A,\varepsilon)$ is a finite constant determined solely by the constants A and ε.

Remark 6.2. *This actually means that $E(0) < C(A,\varepsilon) < \infty$ by monotonicity of the energy.*

6.1 Proof of Lemma 6.1

First of all, we observe that

$$
\begin{aligned}
\int_{\mathbb{R}^3} e^{-(p-3)\tau} \left(\frac{|y|}{\sinh |y|}\right)^{p-1} \frac{|v(y,\tau)|^{p+1}}{p+1} dy \lesssim_1& \left\| \left(\frac{|y|}{\sinh |y|}\right)^{p-1} \right\|_{L^{6/(5-p)}(\mathbb{R}^3)} \|v(\cdot,\tau)\|_{L^6(\mathbb{R}^3)}^{p+1} \\
\lesssim_1& \|v(\cdot,\tau)\|_{\dot{H}^1(\mathbb{R}^3)}^{p+1}
\end{aligned}
$$

[2] Here we temporarily assume that the functions involved are sufficiently smooth. Otherwise we can apply the standard smoothing approximation techniques

Therefore it suffices to show that there exists $\tau \in [-1, 0]$ so that

$$E_0(\tau) = \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla_y v(y,\tau)|^2 + \frac{1}{2}|v_\tau(x,\tau)|^2\right] dy < C'(A,\varepsilon).$$

Next we use the fact that v is radial and rewrite $E_0(\tau)$ in term of polar coordinates

$$E_0(\tau) = \int_0^\infty 2\pi \left[|v_s(s,\tau)|^2 + |v_\tau(s,\tau)|^2\right] s^2 ds.$$

We split the integral into two parts: the integral over $(s_0(\tau), \infty)$ and the integral over $(0, s_0(\tau)]$.

$$E_0(\tau) = \int_{s_0(\tau)}^\infty + \int_0^{s_0(\tau)} \doteq E_0^{(1)}(\tau) + E_0^{(2)}(\tau).$$

The radius $s_0(\tau) \doteq \cosh^{-1}(\bar{t} \cdot e^{-\tau}) > \cosh^{-1}\sqrt{2}$ corresponds to the value of time $t = t_0 + e^\tau \cosh s_0 = e^\tau \cosh s_0 - \bar{t} = 0$.

Large radius part First of all, we observe the identities

$$r + t = t_0 + e^\tau e^s; \qquad r - t = -t_0 - e^\tau e^{-s}.$$

In the large radius part we have $s > s_0(\tau)$. This implies

(i) $t = t_0 + e^\tau \cosh s = e^\tau \cosh s - \bar{t} \geq 0$.

(ii) $r - t = -t_0 - e^\tau e^{-s} = \bar{t} - e^\tau e^{-s} \geq \bar{t} - e^\tau e^{-s_0} = \sqrt{\bar{t}^2 - e^{2\tau}} > R$.

As a result, we can apply the inequalities regrading u, w_r, w_t given in Proposition 4.3 and Corollary 4.4 to obtain

$$|w_t + w_r| \leq f(t_0 + e^\tau e^s); \qquad |w_t - w_r| \leq f(-t_0 - e^\tau e^{-s}); \tag{15}$$

$$|u| \lesssim_{A,\varepsilon} (e^\tau \sinh s)^{-1} \implies |v| \lesssim_{A,\varepsilon} s^{-1}. \tag{16}$$

Here f is the function introduced in Corollary 4.4. All the values of u, w_r and w_t above are taken at the point $(r,t) = (e^\tau \sinh s, t_0 + e^\tau \cosh s)$. We combine the identities (13) with the inequalities (15) and obtain

$$\begin{aligned} 2|(sv)_\tau| &= 2\left|(e^\tau \sinh s) w_r + (e^\tau \cosh s) w_t\right| = e^\tau \left|e^s(w_t + w_r) + e^{-s}(w_t - w_r)\right| \\ &\leq e^{\tau+s} f(t_0 + e^\tau e^s) + e^{\tau - s} f(-t_0 - e^\tau e^{-s}); \\ 2\left|(sv)_s\right| &= 2\left|(e^\tau \cosh s) w_r + (e^\tau \sinh s) w_t\right| = e^\tau \left|e^s(w_t + w_r) - e^{-s}(w_t - w_r)\right| \\ &\leq e^{\tau+s} f(t_0 + e^\tau e^s) + e^{\tau - s} f(-t_0 - e^\tau e^{-s}). \end{aligned}$$

A basic calculation shows

$$\begin{aligned} E_0^{(1)}(\tau) &= 2\pi \int_{s_0(\tau)}^\infty \left[|v_s(s,\tau)|^2 + |v_\tau(s,\tau)|^2\right] s^2 ds \\ &\leq 2\pi \int_{s_0(\tau)}^\infty \left[\left|\frac{\partial(sv)}{\partial s}(s,\tau) - v(s,\tau)\right|^2 + \left|\frac{\partial(sv)}{\partial \tau}(s,\tau)\right|^2\right] ds \\ &\leq 4\pi \int_{s_0(\tau)}^\infty \left[\left|(sv)_s\right|^2 + \left|(sv)_\tau\right|^2 + v^2\right] ds. \end{aligned}$$

We substitute $|(sv)_\tau|$, $|(sv)_s|$, $|v|$ by their corresponding upper bounds given above and obtain

$$E_0^{(1)}(\tau) \lesssim_{A,\varepsilon} \int_{s_0(\tau)}^\infty \left[e^{2\tau+2s}\left|f(t_0 + e^\tau e^s)\right|^2 + e^{2\tau-2s}\left|f(-t_0 - e^\tau e^{-s})\right|^2 + s^{-2}\right] ds.$$

Finally we recall the assumptions $\tau \in [-1,0]$, $s > s_0(\tau) > \cosh^{-1}\sqrt{2}$, observe the inequality

$$e^\tau e^s \lesssim_1 e^\tau \sinh s = r \le r + t = t_0 + e^\tau e^s$$

and obtain a universal upper bound on $E_0^{(1)}(\tau)$:

$$\begin{aligned} E_0^{(1)}(\tau) \lesssim_{A,\varepsilon} & \int_{s_0(\tau)}^\infty e^{\tau+s}(t_0+e^\tau e^s)\,|f(t_0+e^\tau e^s)|^2\,ds + \int_{s_0(\tau)}^\infty e^{\tau-s}\left|f(-t_0-e^\tau e^{-s})\right|^2 ds \\ & + \int_{s_0(\tau)}^\infty s^{-2} ds \\ \lesssim_1 & \int_R^\infty \xi_1\,|f(\xi_1)|^2\,d\xi_1 + \int_R^{-t_0} |f(\xi_2)|^2\,d\xi_2 + 1 \\ \lesssim_{A,\varepsilon} & \; 1. \end{aligned}$$

Here we need to apply the change of variables $\xi_1 = t_0 + e^\tau e^s = r + t > R$, $\xi_2 = -t_0 - e^{\tau - s} = r - t > R$. In the final step we use the assumption on the function f in Corollary 4.4

$$\int_R^\infty \xi^{1+\delta}|f(\xi)|^2 d\xi \lesssim_{A,\varepsilon} 1.$$

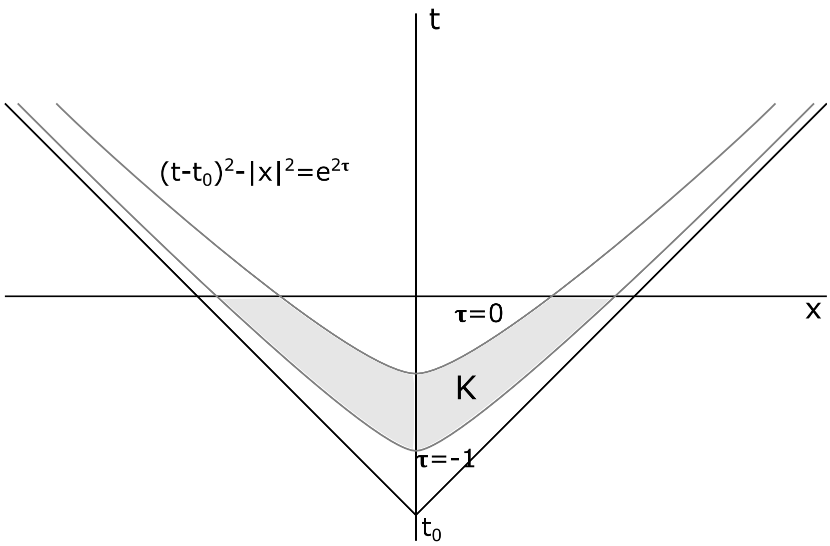

Figure 1: Illustration of region K

Small radius part Now we need to consider the upper bound of $\inf_{\tau\in[-1,0]} E_0^{(2)}(\tau)$, which can be dominated by the integral of $E_0^{(2)}(\tau)$ over the time interval $[-1,0]$:

$$\begin{aligned} \inf_{\tau\in[-1,0]} E_0^{(2)}(\tau) \le & 2\pi \int_{-1}^0 \int_0^{s_0(\tau)} \left[|v_s(s,\tau)|^2 + |v_\tau(s,\tau)|^2\right] s^2 ds\, d\tau \\ = & \frac{1}{2}\int_{-1}^0 \int_0^{s_0(\tau)} e^{-4\tau}\left(\frac{s}{\sinh s}\right)^2 \left[|v_s(s,\tau)|^2 + |v_\tau(s,\tau)|^2\right] 4\pi e^{4\tau}\sinh^2 s\, ds\, d\tau. \end{aligned}$$

Let us recall our definition of v and differentiate:

$$v_\tau = \frac{\sinh s}{s} e^\tau u + \frac{\sinh s \cosh s}{s} e^{2\tau} u_t + \frac{\sinh^2 s}{s} e^{2\tau} u_r;$$
$$v_s = \frac{s\cosh s - \sinh s}{s^2} e^\tau u + \frac{\sinh^2 s}{s} e^{2\tau} u_t + \frac{\sinh s \cosh s}{s} e^{2\tau} u_r.$$

As a result we have

$$\left|\frac{s}{\sinh s} v_\tau\right| \leq \{|u| + (t-t_0)|u_t| + r|u_r|\}_{(r,t)=(e^\tau \sinh s, t_0 + e^\tau \cosh s)};$$
$$\left|\frac{s}{\sinh s} v_s\right| \leq \{|u| + r|u_t| + (t-t_0)|u_r|\}_{(r,t)=(e^\tau \sinh s, t_0 + e^\tau \cosh s)}.$$

Our assumption $s \in (0, s_0(\tau)]$ also implies $r < t - t_0 < |t_0|$. Plugging these upper bounds in the integral above and applying the change of variables formula (14), we obtain

$$\begin{aligned}\inf_{\tau\in[-1,0]} E_0^{(2)}(\tau) &\lesssim_1 \iint_K (1+|t_0|^2)\left(|u_t|^2 + |\nabla u|^2 + |u|^2\right) dx\, dt \\ &\leq (1+|t_0|^2)\int_{t_0}^0 \int_{B(0,|t_0|)} \left(|u_t|^2 + |\nabla u|^2 + |u|^{p+1} + 1\right) dx\, dt \\ &\lesssim_1 (1+|t_0|^2)|t_0|^4 + (1+|t_0|^2)|t_0|\tilde{E} \lesssim_{A,\varepsilon} 1.\end{aligned}$$

Here the region $K = \{(x,t) : e^{-2} \leq (t-t_0)^2 - |x|^2 \leq 1, t_0 < t \leq 0\} \subseteq B(0,|t_0|) \times [t_0, 0]$, as illustrated in figure 1. The letter $\tilde{E}$ represents the energy of solution u, whose upper bound has been given in Remark 1.4. Combining the small radius part with the large radius part, we have

$$\inf_{t\in[-1,0]} E_0(\tau) \leq \sup_{t\in[-1,0]} E_0^{(1)}(\tau) + \inf_{t\in[-1,0]} E_0^{(2)}(\tau) \lesssim_{A,\varepsilon} 1,$$

thus finish the proof of Lemma 6.1.

6.2 A global integral estimate

Now v is a radial solution to (CP2) with a finite energy $E(0) \lesssim_{A,\varepsilon} 1$. Proposition 3.4 immediately gives a Morawetz inequality

$$\int_0^\infty \int_{\mathbb{R}^3} e^{-(p-3)\tau} \frac{|y|^{p-1}\cosh |y|}{\sinh^p |y|} \cdot |v(y,\tau)|^{p+1} dy\, d\tau \lesssim_1 E(0) \lesssim_{A,\varepsilon} 1 \tag{17}$$

We claim that another global space-time integral estimate holds:

$$I' \doteq \int_0^\infty \int_{\mathbb{R}^3} e^{-(p-3)\tau} \left(\frac{|y|}{\sinh |y|}\right)^{p-1} |v(y,\tau)|^{2(p-1)} dy\, d\tau \lesssim_{A,\varepsilon} 1. \tag{18}$$

Proof. First of all, the monotonicity of energy implies $E(\tau) \lesssim_{A,\varepsilon} 1$ for all $\tau \geq 0$. Since $v(\cdot,\tau)$ are radial $\dot{H}^1(\mathbb{R}^3)$ functions, we can apply Lemma 2.7 and obtain

$$|v(y,\tau)| \lesssim_1 \frac{\|v(\cdot,\tau)\|_{\dot{H}^1(\mathbb{R}^3)}}{|y|^{1/2}} \lesssim_1 \frac{(E(\tau))^{1/2}}{|y|^{1/2}} \lesssim_{A,\varepsilon} \left(\frac{\cosh |y|}{\sinh |y|}\right)^{1/2}.$$

As a result, we have (Please note that $3 \leq p < 5$)

$$|v(y,\tau)|^{p-3} \lesssim_{A,\varepsilon} \left(\frac{\cosh |y|}{\sinh |y|}\right)^{(p-3)/2} \leq \frac{\cosh |y|}{\sinh |y|}.$$

We combine this with the Morawetz inequality (17) and finally obtain

$$\begin{aligned}
I' &= \int_0^\infty \int_{\mathbb{R}^3} e^{-(p-3)\tau} \left(\frac{|y|}{\sinh|y|}\right)^{p-1} |v(y,\tau)|^{p-3} \cdot |v(y,\tau)|^{p+1} dy\, d\tau \\
&\lesssim_{A,\varepsilon} \int_0^\infty \int_{\mathbb{R}^3} e^{-(p-3)\tau} \left(\frac{|y|}{\sinh|y|}\right)^{p-1} \frac{\cosh|y|}{\sinh|y|} \cdot |v(y,\tau)|^{p+1} dy\, d\tau \lesssim_{A,\varepsilon} 1.
\end{aligned}$$

□

6.3 Completion of the proof for the main theorem

We have already known that the solution is well-defined for all time $t \in \mathbb{R}$. According to Proposition 3.8 and the time-invertible property of wave equations, it suffices to show

$$I \doteq \int_0^\infty \int_{\mathbb{R}^3} |u(x,t)|^{2(p-1)} dx\, dt \lesssim_{A,\varepsilon} 1.$$

We first break the integral into two parts

$$\begin{aligned}
I &= \int_0^\infty \int_{|x|>t+R} |u(x,t)|^{2(p-1)} dx\, dt + \int_0^\infty \int_{|x|<t+R} |u(x,t)|^{2(p-1)} dx\, dt \\
&\leq \int_0^\infty \int_{|x|>t+R} |u(x,t)|^{2(p-1)} dx\, dt + \iint_\Omega |u(x,t)|^{2(p-1)} dx\, dt \doteq I_1 + I_2.
\end{aligned}$$

Here the region $\Omega = \{(x,t) : |x|^2 < (t-t_0)^2 - 1, t > t_0\}$ satisfies (Please see figure 2. Recall the definition $t_0 = -\sqrt{R^2+1} - 1$)

- Ω contains the region $\{(x,t) : |x| < t+R, t \geq 0\}$;
- Ω corresponds to the positive-time part of the y-τ space-time. In other words we have $\Omega = \tilde{\mathbf{T}}(\{(y,\tau) : \tau > 0\})$.

It is clear that $I_1 \lesssim_{A,\varepsilon} 1$ since the inequality $u(r,t) \lesssim_{A,\varepsilon} r^{-1}(r-t)^{-\delta}$ (when $r > t+R$ and $t \geq 0$, see Proposition 4.3) implies that

$$I_1 \lesssim_{A,\varepsilon} \int_0^\infty \int_{t+R}^\infty [r^{-1}(r-t)^{-\delta}]^{2(p-1)} r^2\, drdt = \int_R^\infty \int_s^\infty r^{-2(p-2)} s^{-2(p-1)\delta}\, drds \lesssim_{A,\varepsilon} 1.$$

Here we need to use the change of variables $(s,r) = (r-t, r)$. In order to deal with I_2, we apply the change of variables formula (14).

$$\begin{aligned}
I_2 &= \int_0^\infty \int_{\mathbb{R}^3} \left(e^{-\tau} \frac{|y|}{\sinh|y|}\right)^{2(p-1)} \left|\frac{\sinh|y|}{|y|} e^\tau u(\tilde{\mathbf{T}}(y,s))\right|^{2(p-1)} \cdot e^{4\tau} \left(\frac{\sinh|y|}{|y|}\right)^2 dy\, d\tau \\
&= \int_0^\infty \int_{\mathbb{R}^3} e^{-2(p-3)\tau} \left(\frac{|y|}{\sinh|y|}\right)^{2p-4} |v(y,\tau)|^{2(p-1)} dy\, d\tau.
\end{aligned}$$

The last expression of I_2 is different from the left hand of (18) (i.e. the integral I') only in the first two exponents. A simple comparison shows that $I_2 \leq I' \lesssim_{A,\varepsilon} 1$. This finishes the proof of our main theorem.

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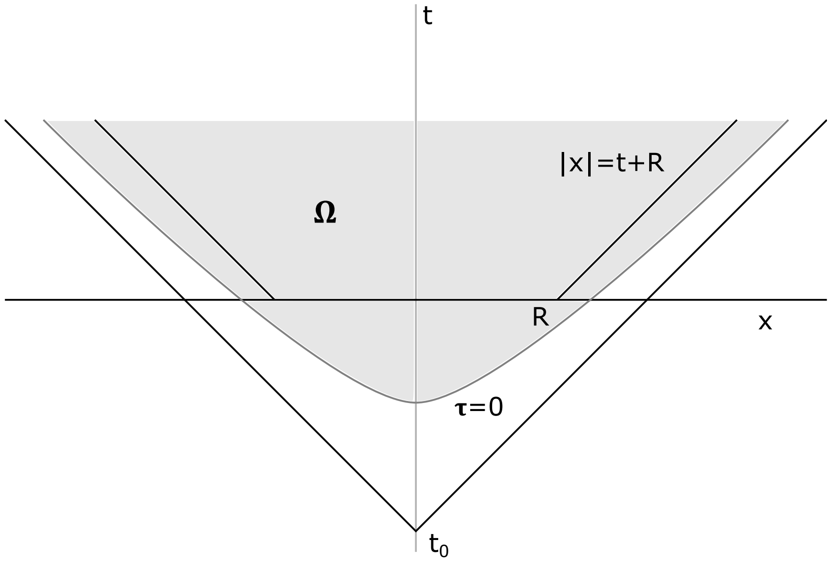

Figure 2: Illustration of the region Ω

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