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The Landis–Oleinik conjecture in the exterior domain

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ABSTRACT

In 1974, Landis and Oleinik conjectured that if a bounded solution of a parabolic equation decays fast at a time, then the solution must vanish identically before that time, provided the coefficients of the equation satisfy appropriate conditions at infinity. We prove this conjecture under some reasonable assumptions of the coefficients, which improved the earlier results.

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1. Introduction

The behavior of solutions of heat equations arose many interests in the last few decades. In 1974, Landis and Oleinik [7] proposed the following conjecture:

If $u(x,t)$ is a bounded solution of a uniformly parabolic equation

$$\sum_{i,j}\partial_i(a^{ij}(x)\partial_j u)-\partial_t u+b(x)\cdot\nabla u+c(x)u=0 \quad in \ \ \mathbb{R}^n\times[0,T],$$

and the condition

$$|u(x,T)|\le Ne^{-|x|^{2+\varepsilon}},\quad x\in\mathbb{R}^n, \tag{1}$$

holds for some positive constants N and ε, then $u(x,t)\equiv 0$ in $\mathbb{R}^n\times[0,T]$, provided that the coefficients of the equation satisfy appropriate conditions at infinity.

The original conjecture only assumes that the coefficients are time-independent and does not mention the precise conditions. However, we always consider their conjecture in the general case that the coefficients are *space–time* dependent and in the following way.

First, we denote the backward parabolic operator

$$P=\partial_t+\sum_{i,j}\partial_i(a^{ij}(x,t)\partial_j)=\partial_t+\nabla\cdot(\mathbf{A}\nabla),$$

where $\mathbf{A}(x,t)=(a^{ij}(x,t))_{i,j=1}^n$ is a real symmetric matrix such that for some $\Lambda\ge\lambda>0$,

$$\lambda|\xi|^2\le\sum_{i,j}a^{ij}(x,t)\xi_i\xi_j\le\Lambda|\xi|^2,\quad\forall\xi\in\mathbb{R}^n. \tag{2}$$

Here we work with backward parabolic operators because it is more convenient in this context. A function u satisfies that

$$|Pu|\le N(|u|+|\nabla u|)\qquad \text{in}\quad \mathbb{R}^n\times[0,T]. \tag{3}$$

We also assume that u satisfies the natural growth condition

$$|u(x,t)|\le Ne^{N|x|^2}\qquad \text{in}\quad \mathbb{R}^n\times[0,T], \tag{4}$$

and rather than (1), we assume a weaker condition

$$|u(x,0)|\le C_k e^{-k|x|^2},\quad\forall\ \ k>0,\qquad \text{in}\quad \mathbb{R}^n. \tag{5}$$

Under assumptions (2)–(5), we consider the **Landis–Oleinik conjecture**: *If $a^{ij}(x,t)$ satisfy appropriate conditions, then $u(x,t)\equiv 0$ in $\mathbb{R}^n\times[0,T]$.*

The Landis–Oleinik conjecture is closely related to many important problems. In particular, if $u(x, 0) = 0$, the conjecture is reduced to the backward uniqueness problem for parabolic equations. The backward uniqueness problem has a natural background in the control theory for PDEs, and it also appeared in the regularity theory of parabolic equations, such as the Navier–Stokes equations [3], semi-linear heat equations [10], heat flow of harmonic maps [13].

This conjecture also has a elliptic version, where probably the problem originated, the Landis conjecture, namely, if a solution of an elliptic equation decays faster than a given rate at infinity, then it is identically zero. The complex case of the Landis conjecture is solved by Meshkov [8], and a quantitative result is proved by Bourgain and Kenig [1], while the real case remains open.

There are some earlier results about the Landis–Oleinik conjecture. In the constant coefficients case, i.e., P is the backward heat operator, this conjecture was solved by Escauriaza, Kenig, Ponce and Vega [5]. They introduced some interesting Carleman estimates and proved both qualitative and quantitative results for the conjecture in the total space and half-space.

For the general case, the first result is obtained by Nguyen [11] where both qualitative and quantitative results are proved for the conjecture in the total space and half-space under the following assumptions

$$|\nabla_x a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \le M, \tag{6}$$

$$|\nabla_x a^{ij}(x,t)| \le M\langle x\rangle^{-1-\epsilon}, \tag{7}$$

$$|a^{ij}(x,t) - a^{ij}(x,s)| \le M\langle x\rangle^{-1}|t-s|^{1/2}, \tag{8}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$ and $\varepsilon > 0$. We remark that condition (6), the Lipschitz regularity assumption is reasonable, as shown in [9,12], and some decay assumptions seem necessary. However, condition (7) is not scaling invariant and we wonder if condition (8) is necessary.

Another related result is the backward uniqueness result for general parabolic equations in the half-space proved by the authors [14] under condition (6) and the decay at infinity condition:

$$|\nabla_x a^{ij}(x,t)| \le E|x|^{-1}, \quad \text{where} \quad E < E_0(n, \Lambda, \lambda). \tag{9}$$

Note that condition (9) is *scaling invariant*. Moreover, both conditions (6) and (9) are almost optimal, which could be seen from the counter examples constructed by the authors. In other words, in the particular case that $u(x, 0) = 0$, the authors in [14] proved the Landis–Oleinik conjecture in the half-space.

All these results suggest that if the Landis–Oleinik conjecture is true, certain regularity and decay at infinity assumptions of the coefficients should be required, and assumptions (6) and (9) seem to be optimal.

Now in the exterior domain, under assumptions (6) and (9), we shall prove the Landis–Oleinik conjecture. Our main result is the following.

Theorem 1.1. *Suppose $\{a^{ij}\}$ satisfy (2), and for some constants $E, M, N > 0$,*

$$|\nabla_x a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \le M, \qquad \forall\ (x,t) \in (\mathbb{R}^n \backslash B_1) \times [0,1], \tag{10}$$

and

$$|\nabla_x a^{ij}(x,t)| \le E|x|^{-1}, \qquad \forall\ (x,t) \in (\mathbb{R}^n \backslash B_1) \times [0,1]. \tag{11}$$

Assume that u satisfies

$$\begin{cases} |Pu| \le N(|u| + |\nabla u|) & \text{in } (\mathbb{R}^n \backslash B_1) \times [0,1], \\ |u(x,t)| \le Ne^{N|x|^2} & \text{in } (\mathbb{R}^n \backslash B_1) \times [0,1], \\ |u(x,0)| \le C_k e^{-k|x|^2}, \quad \forall k > 0 & \text{in } \mathbb{R}^n \backslash B_1. \end{cases} \tag{12}$$

Then there exists a constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, we have $u(x,t) \equiv 0$ in $(\mathbb{R}^n \backslash B_1) \times [0,1]$.

By the unique continuation result (see [2,6]), we immediately have the following corollary.

Corollary 1.2. *Theorem 1.1 is still valid if we replace $\mathbb{R}^n \backslash B_1$ by $\mathbb{R}^n$.*

Theorem 1.1 can be obtained immediately by the following upper bound and lower bound estimates.

Proposition 1.3 *(Upper bound). Suppose $\{a^{ij}\}$ and u are the same as above. Then there exists a constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, we have*

$$|u(x,t)| + |\nabla u(x,t)| \le e^{-k|x|^2}, \qquad \forall k > 0,$$

when $|x| \ge R_1(n, \Lambda, \lambda, M, E, N, k)$ and $0 \le t \le T_1(\Lambda, N)$.

Proposition 1.4 *(Lower bound). Suppose $\{a^{ij}\}$ are the same as above, u satisfies the first two conditions of (12), and $u(x,0) \not\equiv 0$. Then there exists a positive constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, there exists a constant $C_\star = C_\star(n, \Lambda, \lambda, M, E, N)$, such that the following estimate*

$$\frac{1}{T} \int_{T/8}^{7T/8} \int_{R-1 \le |x| \le R} (u^2 + |\nabla u|^2) dx dt \ge e^{-C_\star \frac{R^2}{T}} \tag{13}$$

holds when

$$R \ge R_2(n, \Lambda, \lambda, M, E, N, ||u(\cdot,0)||_{L^2(B(10e_1, \frac{1}{2}))})$$

and

$$0 < T \leq T_2(n, \Lambda, \lambda, M, N, ||u(\cdot, 0)||_{L^2(B(10e_1, \frac{1}{2}))}),$$

where $e_1 = (1, 0, \dots, 0)$.

Combining these two estimates together, we must have $u(x, 0) = 0$, then by the backward uniqueness result (see [14]), we have $u(x, t) \equiv 0$. Thus we proved Theorem 1.1.

Remark 1.5. This lower bound of the integration form is optimal, which can be seen from the solution of the backward heat equation $\partial_t \Gamma + \triangle \Gamma = 0$ that

$$\Gamma(x, t) = (T - t)^{-n/2} e^{-\frac{|x|^2}{4(T-t)}}.$$

The upper bound could be obtained by the following Carleman inequality.

Proposition 1.6. *Suppose* $\{a^{ij}\}$ *are the same as above. Let*

$$Q = (\mathbb{R}^n \backslash B_1) \times [0, 1], \qquad f(t) = (t + 1)^{-\beta} - 2^{-\beta}.$$

There exists a positive constant $E_0 = E_0(n, \Lambda, \lambda)$*, such that when* $E < E_0$*, for any function* $v \in C_0^\infty(Q)$ *and any* $\gamma > 0$*, we have*

$$\begin{aligned} \int\limits_Q e^{2\gamma f |x|^{3/2} - \frac{b|x|^2+\beta}{t+1}} (|v|^2 + |\nabla v|^2) dx dt \leq & \int\limits_Q e^{2\gamma f |x|^{3/2} - \frac{b|x|^2+\beta}{t+1}} |Pv|^2 dx dt \\ & + \beta \int\limits_{\mathbb{R}^n} |x|^2 e^{-\frac{b|x|^2+\beta}{2}} (|v(x, 1)|^2 + |\nabla v(x, 1)|^2) dx \\ & + \beta (1 + \gamma)^2 \int\limits_{\mathbb{R}^n} |x|^2 e^{2\gamma |x|^{3/2} - (b|x|^2+\beta)} |v(x, 0)|^2 dx, \end{aligned} \tag{14}$$

where $b = \frac{1}{16\Lambda}$*, and* $\beta = \beta(n, \Lambda, \lambda, M, E) \geq 1$.

The lower bound could be proved mainly by the following Carleman inequality.
First, let $\psi(t)$ be a cut-off function satisfying

$$\psi(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]; \\ 2, & \text{if } t \in [\frac{1}{3}, \frac{2}{3}]. \end{cases}$$

Proposition 1.7. *Suppose* $\{a^{ij}\}$ *are the same as above. Let*

$$Q_R = \{(x,t)|\ 1 < |x| < R, t \in (\frac{1}{8}, \frac{7}{8})\},$$
$$\Psi = \gamma(1-t)R^{2/3}|x|^{4/3} + \psi(t)R^2.$$

There exists a positive constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, for any function $v \in C_0^\infty(Q_R)$ and any $\gamma \geq \gamma_0(n, \Lambda, \lambda, M, E)$, we have

$$c\lambda^2 \int_{Q_R} e^{2\Psi}(\gamma^3 R^2 |v|^2 + \gamma|\nabla v|^2)dxdt \leq \int_{Q_R} e^{2\Psi}|Pv|^2 dxdt, \tag{15}$$

where c is an absolute constant.

Remark 1.8. In the argument of applying Carleman inequalities to prove the upper and lower bound, the constructions of weight functions are very crucial and technical. The ideas of constructing the two weight functions in this paper come partly from [11,14], and some new observations. Note that in the construction of Ψ, the index $\frac{4}{3}$ is optimal, happening to coincide with Meshkov's results in [8].

The paper is organized as follows. We first use the two Carleman inequalities to prove the upper and lower bound in section 2, then we prove the two Carleman inequalities in section 3.

2. Proof of upper and lower bound

In this section, we prove the upper and lower bound by assuming Proposition 1.6 and Proposition 1.7 first, and we postpone the proof of the two Carleman inequalities to the next section.

2.1. Upper bound

Proof of Proposition 1.3. We use Carleman inequalities (14) to prove the upper bound for the solution.

Step 1. By the regularity theory for solutions of parabolic equations, we have

$$|u(x,t)| + |\nabla u(x,t)| \leq C(n, \Lambda, \lambda, M, N)e^{2N|x|^2} \tag{16}$$

for $(x,t) \in (\mathbb{R}^n \backslash B_2) \times [0, \frac{1}{2}]$. Let

$$\tau = \min\{\frac{1}{2}, \frac{1}{2N}, \frac{b}{8}\}, \tag{17}$$

where b is the one in Proposition 1.6. Define

$$\tilde{u}(x,t) = u(\tau x, \tau^2 t),$$

and

$$\tilde{a}^{ij}(x,t) = a^{ij}(\tau x, \tau^2 t)$$

for $(x,t) \in (\mathbb{R}^n \backslash B_{\frac{2}{\tau}}) \times [0,1]$. Then it is easy to see

$$|\nabla_x \tilde{a}^{ij}| + |\partial_t \tilde{a}^{ij}| \leq \tau M \leq M, \quad |\nabla_x \tilde{a}^{ij}| \leq E|x|^{-1}.$$

We denote

$$\tilde{P}\tilde{u} = \partial_t \tilde{u} + \sum_{ij} \partial_i (\tilde{a}^{ij} \partial_j \tilde{u}),$$

then by (12) we have

$$|\tilde{P}\tilde{u}| \leq \tau N(|\tilde{u}| + |\nabla \tilde{u}|) \leq \frac{1}{2}(|\tilde{u}| + |\nabla \tilde{u}|). \tag{18}$$

By (16) and (17), we have

$$|\tilde{u}(x,t)| + |\nabla \tilde{u}(x,t)| \leq C(n,\Lambda,\lambda,M,N)e^{2N\tau^2|x|^2} \leq C(n,\Lambda,\lambda,M,N)e^{\frac{b}{8}|x|^2}. \tag{19}$$

We keep in mind that

$$|u(x,0)| \leq C_k e^{-k|x|^2}, \quad \forall k > 0, \tag{20}$$

and we always take k large enough.

Step 2. In order to apply Carleman inequality (14), we define a cut-off function θ satisfying

$$\theta(|x|) = \begin{cases} 0, & \text{if } |x| < R \text{ or } |x| > r_1 + 1; \\ 1, & \text{if } R+1 \leq |x| \leq r_1, \end{cases}$$

where $R > \frac{2}{\tau}$ and $r_1 > k^2 R$.

Let $v = \theta \tilde{u}$, then by (18) we have

$$\begin{aligned} |\tilde{P}v| &= |\theta \tilde{P}\tilde{u} + \tilde{u}\tilde{P}\theta + 2\tilde{a}^{ij}\partial_i \theta \partial_j \tilde{u}| \\ &\leq \frac{1}{2}\theta(|\tilde{u}| + |\nabla \tilde{u}|) + C(n,\Lambda,M)\chi(|\tilde{u}| + |\nabla \tilde{u}|)(|\nabla \theta| + |\nabla^2 \theta|) \\ &\leq \frac{1}{2}(|v| + |\nabla v|) + C(n,\Lambda,M)\chi_\Omega(|\tilde{u}| + |\nabla \tilde{u}|), \end{aligned} \tag{21}$$

where χ is the characteristic function and

$$\begin{aligned}\Omega &= \{0 < \theta < 1, t \in [0,1]\}\\ &= \{R < |x| < R+1, t \in [0,1]\} \cup \{r_1 < |x| < r_1+1, t \in [0,1]\}.\end{aligned}$$

Step 3. We apply Carleman inequality (14) for v, then

$$\begin{aligned}J \equiv &\int\limits_Q e^{2\gamma f|x|^{3/2} - \frac{b|x|^2+\beta}{t+1}}(|v|^2 + |\nabla v|^2)dxdt\\ \le &\int\limits_Q e^{2\gamma f|x|^{3/2} - \frac{b|x|^2+\beta}{t+1}}|\tilde{P}v|^2 dxdt\\ &+ \beta \int\limits_{\mathbb{R}^n} |x|^2 e^{-\frac{b|x|^2+\beta}{2}}(|v(x,1)|^2 + |\nabla v(x,1)|^2)dx\\ &+ \beta(1+\gamma)^2 \int\limits_{\mathbb{R}^n} |x|^2 e^{2\gamma|x|^{3/2} - (b|x|^2+\beta)}|v(x,0)|^2 dx.\end{aligned}$$

By (21) we have

$$\begin{aligned}J \le \frac{3}{4}J + C(n,\Lambda,M) &\int\limits_Q e^{2\gamma f|x|^{3/2} - \frac{b|x|^2+\beta}{t+1}}\chi_\Omega(|\tilde{u}| + |\nabla\tilde{u}|)^2 dxdt\\ + \beta &\int\limits_{\mathbb{R}^n} |x|^2 e^{-\frac{b|x|^2+\beta}{2}}(|v(x,1)|^2 + |\nabla v(x,1)|^2)dx\\ + \beta(1+\gamma)^2 &\int\limits_{\mathbb{R}^n} |x|^2 e^{2\gamma|x|^{3/2} - (b|x|^2+\beta)}|v(x,0)|^2 dx,\end{aligned}$$

thus

$$\begin{aligned}J \le C(n,\Lambda,M) &\int\limits_\Omega e^{2\gamma f|x|^{3/2} - \frac{b|x|^2+\beta}{t+1}}(|\tilde{u}| + |\nabla\tilde{u}|)^2 dxdt\\ + c\beta &\int\limits_{|x|\ge R} |x|^2 e^{-\frac{b|x|^2+\beta}{2}}(|\tilde{u}(x,1)|^2 + |\nabla\tilde{u}(x,1)|^2)dx\\ + 4\beta(1+\gamma)^2 &\int\limits_{|x|\ge R} |x|^2 e^{2\gamma|x|^{3/2} - (b|x|^2+\beta)}|\tilde{u}(x,0)|^2 dx\\ \equiv I_1 + I_2 + I_3.&\end{aligned}$$

Step 4. Now we estimate both sides of the above inequality. We will see how the weight function $e^{2\gamma f(t)|x|^{3/2} - \frac{b|x|^2+\beta}{t+1}}$ works in these estimates. Actually, we use $e^{-\frac{b|x|^2}{t+1}}$ to control the growth of u. As for I_2, it doesn't involve γ-term because we set $f(1) = 0$, and thus we can control I_2 easily. As for I_3, we use the fast decay condition of $u(x,0)$, namely (20), to control it.

Next we estimate I_2 and I_3 first, then I_1, at last J.

Estimate of I_2.

By (19),

$$I_2 \leq c\beta e^{-\frac{\beta}{2}} \int\limits_{|x|\geq R} |x|^2 e^{-\frac{b|x|^2}{4}} dx \leq C(n,b) = C(n,\Lambda). \tag{22}$$

Estimate of I_3.

Recall (20), then

$$|\tilde{u}(x,0)| = |u(\tau x,0)| \leq C(\frac{bk}{\tau^2}) e^{-\frac{bk}{\tau^2}|\tau x|^2} = C(\Lambda,N,k)e^{-bk|x|^2},$$

and thus

$$\begin{aligned} I_3 &\leq 4\beta e^{-\beta}(1+\gamma)^2 C(\Lambda,N,k) \int\limits_{|x|\geq R} |x|^2 e^{-2bk|x|^2+2\gamma|x|^{3/2}-b|x|^2} dx \\ &\leq C(\Lambda,N,k)(1+\gamma)^2 \int\limits_{|x|\geq R} |x|^2 e^{-2bk|x|^2+2\gamma|x|^{3/2}-b|x|^2} dx. \end{aligned}$$

Now we choose

$$\gamma = \frac{bk}{16} R^{1/2}. \tag{23}$$

In the region $\{|x| \geq R\}$,

$$2\gamma|x|^{3/2} = \frac{bk}{8} R^{1/2}|x|^{3/2} \leq \frac{bk}{8}|x|^2,$$

then

$$\begin{aligned} I_3 &\leq C(\Lambda,N,k)k^2 R \int\limits_{|x|\geq R} |x|^2 e^{-bk|x|^2-b|x|^2} dx \\ &\leq C(\Lambda,N,k)k^2 R e^{-bkR^2} \int\limits_{|x|\geq R} |x|^2 e^{-b|x|^2} dx \\ &\leq C(n,\Lambda,N,k)k^2 R e^{-bkR^2} \leq 1, \end{aligned} \tag{24}$$

if $R \geq R_0(n,\Lambda,N,k)$ large enough.

Estimate of I_1.

$$I_1 \le C(n,\Lambda,M)\int_\Omega e^{2\gamma|x|^{3/2}-\frac{b|x|^2}{2}}(|\tilde{u}|+|\nabla\tilde{u}|)^2dxdt.$$

Using (19) again, we obtain

$$\begin{aligned}I_1 &\le C(n,\Lambda,\lambda,M,N)\int_\Omega e^{2\gamma|x|^{3/2}-\frac{b|x|^2}{4}}dxdt\\ &\le C(n,\Lambda,\lambda,M,N)(\int_{r_1<|x|<r_1+1}+\int_{R<|x|<R+1})e^{2\gamma|x|^{3/2}-\frac{b|x|^2}{4}}dx\\ &\equiv I_{1,1}+I_{1,2}.\end{aligned}$$

We can control $I_{1,1}$ easily since we set r_1 large enough.

Actually, in the region $\{r_1<|x|<r_1+1\}$,

$$2\gamma|x|^{3/2}=\frac{bk}{8}R^{1/2}|x|^{3/2}\le\frac{b}{8}|x|^2$$

since $r_1>k^2R$. Then

$$I_{1,1}\le C(n,\Lambda,\lambda,M,N)\int_{r_1<|x|<r_1+1}e^{-\frac{b|x|^2}{8}}dx\le C(n,\Lambda,\lambda,M,N).$$

In $\{R<|x|<R+1\}$,

$$2\gamma|x|^{3/2}=\frac{bk}{8}R^{1/2}|x|^{3/2}\le\frac{bk}{8}|x|^2,$$

then

$$\begin{aligned}I_{1,2}&\le C(n,\Lambda,\lambda,M,N)\int_{R<|x|<R+1}e^{\frac{bk}{8}|x|^2-\frac{b|x|^2}{4}}dx\\ &\le C(n,\Lambda,\lambda,M,N)e^{\frac{bk}{8}(R+1)^2}\int_{R<|x|<R+1}e^{-\frac{b|x|^2}{4}}dx\\ &\le C(n,\Lambda,\lambda,M,N)e^{\frac{bk}{2}R^2}.\end{aligned}$$

Thus we have

$$I_1\le C(n,\Lambda,\lambda,M,N)e^{\frac{bk}{2}R^2}. \tag{25}$$

Combining (22), (24) and (25), we have that when $R\ge R_0(n,\Lambda,N,k)$,

$$J \leq C(n, \Lambda, \lambda, M, N) e^{\frac{bk}{2} R^2}. \tag{26}$$

Next we estimate a lower bound for J.

Estimate of J.
If $k \geq 4^{\beta+5}$, then $\{4^{\beta+2}R \leq |x| \leq 4^{\beta+3}R\} \subset \{\theta = 1\}$, and thus

$$J \geq \int_0^{1/2} \int_{4^{\beta+2}R\leq|x|\leq 4^{\beta+3}R} e^{2\gamma f |x|^{3/2} - \frac{b|x|^2+\beta}{t+1}} (|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dxdt.$$

Notice that when $t \in [0, \frac{1}{2}]$, $f(t) \geq f(\frac{1}{2}) \geq 2^{-\beta-2}$, then

$$J \geq e^{-\beta} \int_0^{1/2} \int_{4^{\beta+2}R\leq|x|\leq 4^{\beta+3}R} e^{2^{-\beta-1}\gamma|x|^{3/2} - b|x|^2} (|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dxdt.$$

In the region $\{4^{\beta+2}R \leq |x| \leq 4^{\beta+3}R\}$,

$$2^{-\beta-1}\gamma|x|^{3/2} = 2^{-\beta-5}bkR^{1/2}|x|^{3/2} \geq 2^{-\beta-5}bk(4^{-\beta-3}|x|)^{1/2}|x|^{3/2} = 4^{-\beta-4}bk|x|^2,$$

then

$$2^{-\beta-1}\gamma|x|^{3/2} - b|x|^2 \geq (4^{-\beta-4}k - 1)b|x|^2.$$

Notice that $k \geq 4^{\beta+5}$, then

$$4^{-\beta-4}k - 1 \geq 4^{-\beta-5}k,$$

and

$$2^{-\beta-1}\gamma|x|^{3/2} - b|x|^2 \geq 4^{-\beta-5}bk|x|^2 \geq 4^{-\beta-5}bk(4^{\beta+2}R)^2 = 4^{\beta-1}bkR^2 \geq bkR^2.$$

Thus

$$J \geq e^{-\beta+bkR^2} \int_0^{1/2} \int_{4^{\beta+2}R\leq|x|\leq 4^{\beta+3}R} (|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dxdt.$$

Recall that $\tilde{u}(x,t) = u(\tau x, \tau^2 t)$ and $\tau < 1$, then

$$J \geq \tau^{-n} e^{-\beta+bkR^2} \int_0^{\tau^2/2} \int_{\tau 4^{\beta+2}R\leq|x|\leq \tau 4^{\beta+3}R} (|u|^2 + |\nabla u|^2) dxdt. \tag{27}$$

Combining (26) and (27) together, we have

$$\int_0^{\tau^2/2} \int_{\tau 4^{\beta+2} R \le |x| \le \tau 4^{\beta+3} R} (|u|^2 + |\nabla u|^2) dx dt \le C(n, \Lambda, \lambda, M, E, N) e^{-\frac{bk}{2} R^2} \le e^{-\frac{bk}{4} R^2}$$

when $R \ge R_0(n, \Lambda, \lambda, M, E, N, k)$.

We replace $\tau 4^{\beta+2} R$ by R, and let

$$T_1 = \frac{\tau^2}{4} = \frac{1}{16} \min\{1, \frac{1}{N^2}, \frac{b^2}{16}\},$$

then we obtain

$$\int_0^{2T_1} \int_{R \le |x| \le 4R} (|u|^2 + |\nabla u|^2) dx dt \le e^{-CkR^2}.$$

Finally, by the regularity theory for solutions of parabolic equations, we obtained our upper bound estimate.

2.2. Lower bound

The lower bound can be proved by the following two lemmas. The first one is due to Escauriaza, Fernández and Vessella (see [4]), and we copy it here.

Lemma 2.1. *There is a constant $C = C(n, \Lambda, \lambda, M, N)$ such that the inequalities*

$$C \log(C\Theta_\rho) \ge 2 \quad \text{and} \quad C \int_{B_{2\rho}} u^2(x, t) dx \ge \int_{B_\rho} u^2(x, 0) dx \tag{28}$$

hold when $0 < t \le \rho^2 / C \log(C\Theta_\rho)$ and $0 < \rho \le 1$. Here

$$\Theta_\rho = \frac{\int_0^1 \int_{B_4} u^2(x, t) dx dt}{\rho^2 \int_{B_\rho} u^2(x, 0) dx}.$$

The second one is derived from Carleman inequality (15).

Lemma 2.2. *Suppose $\{a^{ij}\}$ are the same as above, u satisfies the first two conditions of (12). Then there exists a positive constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, there exists $C_\star = C_\star(n, \Lambda, \lambda, M, E, N)$, such that the following estimate*

$$\begin{aligned} &e^{C_\star \frac{R^{2/3}}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9\le |x| \le 11} |u|^2 dxdt \\ &\le 1 + e^{C_\star \frac{R^2}{T}} \frac{1}{T} \int_{T/8}^{7T/8} \int_{R-1\le |x|\le R} (|u|^2 + |\nabla u|^2) dxdt, \end{aligned} \tag{29}$$

holds when $R \ge R_3(n,N)$ *and* $0 < T \le \frac{1}{2}$.

In the following, we prove Lemma 2.2 first, then we use the two lemmas to prove the lower bound.

Proof of Lemma 2.2. We use Carleman inequality (15) to prove Lemma 2.2. We again divided the proof into several steps.

Step 1. For any $0 < T \le \frac{1}{2}$, we define

$$\begin{aligned} &\tilde{u}(x,t) = u(\sqrt{T}x, Tt), \\ &\tilde{a}^{ij}(x,t) = a^{ij}(\sqrt{T}x, Tt), \\ &\tilde{P}\tilde{u} = \partial_t \tilde{u} + \sum_{ij} \partial_i(\tilde{a}^{ij}\partial_j \tilde{u}), \end{aligned}$$

for $(x,t) \in (\mathbb{R}^n \backslash B_{\frac{2}{\sqrt{T}}}) \times [0,1]$. Similarly, we have

$$|\tilde{u}(x,t)| + |\nabla \tilde{u}(x,t)| \le C(n,\Lambda,\lambda,M,N) e^{2NT|x|^2}, \tag{30}$$

and

$$|\tilde{P}\tilde{u}| \le \sqrt{T} N(|\tilde{u}| + |\nabla \tilde{u}|) \le N(|\tilde{u}| + |\nabla \tilde{u}|). \tag{31}$$

Step 2. In order to apply Carleman inequality (15), we choose two smooth cut-off functions. Let

$$\eta_1(|x|) = \begin{cases} 0, & \text{if } |x| \le \frac{2}{\sqrt{T}} \quad \text{or} \quad |x| \ge \gamma^{-3/4}R; \\ 1, & \text{if } \frac{3}{\sqrt{T}} \le |x| \le \gamma^{-3/4}R - \frac{1}{\sqrt{T}}, \end{cases}$$

where γ and R are the parameters in Carleman inequality (15), and

$$\gamma^{-3/4} R \ge \frac{20}{\sqrt{T}}. \tag{32}$$

We always take both γ and R large enough. Let

$$\eta_2(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{8}] \bigcup [\frac{7}{8}, 1]; \\ 1, & \text{if } t \in [\frac{1}{4}, \frac{3}{4}]. \end{cases}$$

Let $\eta(x,t) = \eta_1(|x|)\eta_2(t)$ and $v = \eta\tilde{u}$. Then *supp* $\eta \subset Q_R$ and so *supp* $v \subset Q_R$.

By (31) we have

$$\begin{aligned} |\tilde{P}v| &= |\eta\tilde{P}\tilde{u} + \tilde{u}\tilde{P}\eta + 2\tilde{a}^{ij}\partial_i\eta\partial_j\tilde{u}| \\ &\leq N\eta(|\tilde{u}| + |\nabla\tilde{u}|) + C(n,\Lambda,M)(|\tilde{u}| + |\nabla\tilde{u}|)(|\partial_t\eta| + |\nabla\eta| + |\nabla^2\eta|) \\ &\leq N(|v| + |\nabla v|) + C(n,\Lambda,M,N)(|\tilde{u}| + |\nabla\tilde{u}|)\chi_{\{0<\eta<1\}}. \end{aligned} \tag{33}$$

Step 3. We apply Carleman inequality (15) for v, then we get

$$c\lambda^2 \int\limits_{Q_R} e^{2\Psi}(\gamma^3 R^2|v|^2 + \gamma|\nabla v|^2)dxdt \leq \int\limits_{Q_R} e^{2\Psi}|\tilde{P}v|^2 dxdt.$$

By (33), we have

$$\begin{aligned} c\lambda^2 \int\limits_{Q_R} e^{2\Psi}(\gamma^3 R^2|v|^2 + \gamma|\nabla v|^2)dxdt &\leq 4N^2 \int\limits_{Q_R} e^{2\Psi}(|v|^2 + |\nabla v|^2)dxdt \\ &\quad + C \int\limits_{\{0<\eta<1\}} e^{2\Psi}(|\tilde{u}|^2 + |\nabla\tilde{u}|^2)dxdt. \end{aligned}$$

In the above inequality, if we take $\gamma = \gamma(n,\Lambda,\lambda,M,E,N)$ large enough, then the first term of the right hand side can be absorbed by the term of the left hand side, thus we obtain

$$\int\limits_{Q_R} e^{2\Psi}(|v|^2 + |\nabla v|^2)dxdt \leq C\gamma^{-1} \int\limits_{\{0<\eta<1\}} e^{2\Psi}(|\tilde{u}|^2 + |\nabla\tilde{u}|^2)dxdt.$$

Denote that

$$\Omega_1 = \{(x,t)|\ \frac{9}{\sqrt{T}} \leq |x| \leq \frac{11}{\sqrt{T}},\ t \in [\frac{1}{3}, \frac{2}{3}]\},$$

then $\Omega_1 \subset \{\eta = 1\}$ and thus

$$\int\limits_{\Omega_1} e^{2\Psi}(|\tilde{u}|^2 + |\nabla\tilde{u}|^2)dxdt \leq C\gamma^{-1} \int\limits_{\{0<\eta<1\}} e^{2\Psi}(|\tilde{u}|^2 + |\nabla\tilde{u}|^2)dxdt. \tag{34}$$

We divide the set $\{0 < \eta < 1\}$ into three parts:

$$\{0<\eta<1\}=\{0<\eta_1<1,\eta_2>0\}\cup\{\eta_1=1,0<\eta_2<1\}$$
$$\equiv\Omega_2\cup\Omega_3\cup\Omega_4,$$

where

$$\begin{aligned}\Omega_2&=\{(x,t)|\ \frac{2}{\sqrt{T}}<|x|<\frac{3}{\sqrt{T}},\ t\in(\frac{1}{8},\frac{7}{8})\},\\ \Omega_3&=\{(x,t)|\ \gamma^{-3/4}R-\frac{1}{\sqrt{T}}<|x|<\gamma^{-3/4}R,\ t\in(\frac{1}{8},\frac{7}{8})\},\\ \Omega_4&=\{(x,t)|\ \frac{3}{\sqrt{T}}\le|x|\le\gamma^{-3/4}R-\frac{1}{\sqrt{T}},\ t\in(\frac{1}{8},\frac{1}{4})\cup(\frac{3}{4},\frac{7}{8})\}.\end{aligned}\tag{35}$$

If we denote that

$$J_i=\int_{\Omega_i}e^{2\Psi}(|\tilde{u}|^2+|\nabla\tilde{u}|^2)dxdt,\quad i=1,2,3,4,$$

then we rewrite (34) as

$$J_1\le C\gamma^{-1}(J_2+J_3+J_4).\tag{36}$$

Step 4. We estimate them respectively. From the construction of the weight function $\Psi=\gamma(1-t)R^{2/3}|x|^{4/3}+\psi(t)R^2$, we could see that Ψ is smaller on Ω_2 than on Ω_1 and thus J_2 is controlled by J_1. Similarly, J_4 is also controlled by J_1 because $\psi(t)$ is smaller on Ω_4 than on Ω_1. Finally, only J_3, which can't be controlled by J_1, remains on the right hand side of (36).

Estimate of J_1.
In Ω_1, $\psi(t)=2$, and

$$\Psi\ge\frac{\gamma}{3}R^{2/3}(\frac{9}{\sqrt{T}})^{4/3}+2R^2\ge6\gamma(\frac{R}{T})^{2/3}+2R^2,$$

then

$$\begin{aligned}J_1&\ge\exp\{12\gamma(\frac{R}{T})^{2/3}+4R^2\}\int_{\Omega_1}|\tilde{u}|^2dxdt\\ &=T^{-\frac{n}{2}-1}\exp\{12\gamma(\frac{R}{T})^{2/3}+4R^2\}\int_{T/3}^{2T/3}\int_{9\le|x|\le11}|u|^2dxdt.\end{aligned}\tag{37}$$

Estimate of J_2.
In Ω_2,

$$\Psi \leq \gamma R^{2/3}(\frac{3}{\sqrt{T}})^{4/3} + 2R^2 \leq 5\gamma(\frac{R}{T})^{2/3} + 2R^2,$$

and by (30),

$$|\tilde{u}| + |\nabla \tilde{u}| \leq C(n, \Lambda, \lambda, M, N)e^{18N} \leq C,$$

thus

$$J_2 \leq CT^{-\frac{n}{2}} \exp\{10\gamma(\frac{R}{T})^{2/3} + 4R^2\}. \tag{38}$$

Estimate of J_3.
In Ω_3,

$$\Psi \leq \gamma R^{2/3}(\gamma^{-3/4}R)^{4/3} + 2R^2 = 3R^2,$$

then

$$\begin{aligned} J_3 &\leq \exp\{6R^2\} \int\limits_{\Omega_4} (|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dxdt \\ &\leq T^{-n/2-1} \exp\{6R^2\} \int\limits_{T/8}^{7T/8} \int\limits_{\gamma^{-3/4}\sqrt{T}R-1<|x|<\gamma^{-3/4}\sqrt{T}R} (|u|^2 + |\nabla u|^2) dxdt. \end{aligned} \tag{39}$$

Estimate of J_4.
In Ω_4, $\psi(t) = 0$,

$$\Psi \leq \gamma R^{2/3}(\gamma^{-3/4}R)^{4/3} = R^2,$$

and by (30),

$$|\tilde{u}| + |\nabla \tilde{u}| \leq C \exp\{2NT(\gamma^{-3/4}R)^2\} \leq C \exp\{2N(\gamma^{-3/4}R)^2\},$$

then we have

$$J_4 \leq C(\gamma^{-3/4}R)^n \exp\{2R^2 + 4N(\gamma^{-3/4}R)^2\}.$$

Notice that if $\gamma^{-3/4}R > C(n, N)$, then

$$(\gamma^{-3/4}R)^n \leq \exp\{N(\gamma^{-3/4}R)^2\},$$

hence

$$J_4 \leq C \exp\{2R^2 + 5N(\gamma^{-3/4}R)^2\} \leq C \exp\{3R^2\}. \tag{40}$$

Now we combine (36)–(40) together, then we have

$$\exp\{12\gamma(\frac{R}{T})^{2/3}+4R^2\}\frac{1}{T}\int\limits_{T/3}^{2T/3}\int\limits_{9\le|x|\le 11}|u|^2dxdt$$
$$\le C\gamma^{-1}[\exp\{10\gamma(\frac{R}{T})^{2/3}+4R^2\}$$
$$+\exp\{6R^2\}\frac{1}{T}\int\limits_{T/8}^{7T/8}\int\limits_{\gamma^{-3/4}\sqrt{T}R-1<|x|<\gamma^{-3/4}\sqrt{T}R}(|u|^2+|\nabla u|^2)dxdt].$$

In the above inequality, we divide both sides by $\exp\{10\gamma(\frac{R}{T})^{2/3}+4R^2\}$, and take $\gamma=\gamma(n,\Lambda,\lambda,M,E,N)$ large enough, then we have

$$\exp\{2\gamma(\frac{R}{T})^{2/3}\}\frac{1}{T}\int\limits_{T/3}^{2T/3}\int\limits_{9\le|x|\le 11}|u|^2dxdt$$
$$\le 1+\exp\{2R^2\}\frac{1}{T}\int\limits_{T/8}^{7T/8}\int\limits_{\gamma^{-3/4}\sqrt{T}R-1<|x|<\gamma^{-3/4}\sqrt{T}R}(|u|^2+|\nabla u|^2)dxdt,$$

when $\gamma^{-3/4}R\ge\frac{C(n,N)}{\sqrt{T}}$. If we replace $\gamma^{-3/4}\sqrt{T}R$ by R, we rewrite the above inequality as

$$\exp\{2\gamma^{3/2}\frac{R^{2/3}}{T}\}\frac{1}{T}\int\limits_{T/3}^{2T/3}\int\limits_{9\le|x|\le 11}|u|^2dxdt$$
$$\le 1+\exp\{2\gamma^{3/2}\frac{R^2}{T}\}\frac{1}{T}\int\limits_{T/8}^{7T/8}\int\limits_{R-1<|x|<R}(|u|^2+|\nabla u|^2)dxdt,$$

provided $\gamma=\gamma(n,\Lambda,\lambda,M,E,N)$ large enough, and $R\ge C(n,N)$. Thus we proved Lemma 2.2. □

Proof of Proposition 1.4. Since $u(x,0)\ne 0$, then by the unique continuation result (see [2,6]), we must have $u(x,0)\ne 0$ in $B(10e_1,\frac{1}{2})$, and thus $\|u(\cdot,0)\|_{L^2(B(10e_1,\frac{1}{2}))}>0$.

Now we apply Lemma 2.1 for $\rho=\frac{1}{2}$ and the ball $B(10e_1,\frac{1}{2})$, then we have

$$C\int\limits_{B(10e_1,1)}u^2(x,t)dx\ge\int\limits_{B(10e_1,\frac{1}{2})}u^2(x,0)dx \tag{41}$$

when

$$0 < t \le 1/C \log(C\Theta_{1/2}).$$

Notice that

$$\Theta_{1/2} \le C(N)/||u(\cdot,0)||^2_{L^2(B(10e_1,\frac{1}{2}))},$$

and

$$1/C \log(C\Theta_{1/2}) \ge C(n,\Lambda,\lambda,M,N,||u(\cdot,0)||_{L^2(B(10e_1,\frac{1}{2}))}) \equiv T_2,$$

then we have (41) when $0 < t \le T_2$.

For $0 < T \le T_2$, we apply Lemma 2.2, and then we get

$$\begin{aligned} & e^{C_\star \frac{R^{2/3}}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9\le|x|\le 11} u^2 dxdt \\ & \le 1 + e^{C_\star \frac{R^2}{T}} \frac{1}{T} \int_{T/8}^{7T/8} \int_{R-1\le|x|\le R} (u^2 + |\nabla u|^2) dxdt, \end{aligned} \tag{42}$$

when $R \ge R_3(n,N)$. Notice that the left hand side of (42)

$$e^{C_\star \frac{R^{2/3}}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9\le|x|\le 11} u^2 dxdt \ge e^{C_\star \frac{R^{2/3}}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{B(10e_1,1)} u^2 dxdt,$$

and by (41) we have

$$e^{C_\star \frac{R^{2/3}}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9\le|x|\le 11} u^2 dxdt \ge C e^{C_\star \frac{R^{2/3}}{T}} ||u(\cdot,0)||^2_{L^2(B(10e_1,\frac{1}{2}))}.$$

If we choose

$$R \ge R_2(n,\Lambda,\lambda,M,E,N,||u(\cdot,0)||_{L^2(B(10e_1,\frac{1}{2}))}),$$

then

$$e^{C_\star \frac{R^{2/3}}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9\le|x|\le 11} u^2 dxdt \ge e^{C_\star \frac{R^{2/3}}{2T}} \ge 2. \tag{43}$$

By (42) and (43), we have

$$\frac{1}{T}\int_{T/8}^{7T/8}\int_{R-1\le|x|\le R}(u^2+|\nabla u|^2)dxdt \ge e^{-C_\star\frac{R^2}{T}}.$$

Thus we proved the lower bound estimate.

3. Proof of Carleman inequalities

In this section, we shall prove the two Carleman inequalities. The main idea is to choose a proper weight function G. We denote

$$\tilde{\Delta}v = \partial_i(a^{ij}\partial_j v).$$

Here and in the following argument, we use the summation convention on the repeated indices. We shall make use of the following lemma which is due to Escauriaza and Fernández in [2] (see also [11]).

Lemma 3.1. *Suppose $\sigma(t):\mathbb{R}_+\to\mathbb{R}_+$ is a smooth function, F is differentiable, G is twice differentiable and $G>0$. Then the following identity holds for any $v\in C_0^\infty(\mathbb{R}^n\times[0,T])$ and any $\alpha\in\mathbb{R}$:*

$$\begin{aligned}
&2\int_{\mathbb{R}^n\times[0,T]}\frac{\sigma^{1-\alpha}}{\sigma'}|Lv|^2Gdxdt+\frac{1}{2}\int_{\mathbb{R}^n\times[0,T]}\frac{\sigma^{1-\alpha}}{\sigma'}v^2MGdxdt\\
&\quad+\int_{\mathbb{R}^n\times[0,T]}\frac{\sigma^{1-\alpha}}{\sigma'}\langle\mathbf{A}\nabla v,\nabla v\rangle[(\log\frac{\sigma}{\sigma'})'+\frac{\partial_tG-\tilde{\Delta}G}{G}-F]Gdxdt\\
&\quad+2\int_{\mathbb{R}^n\times[0,T]}\frac{\sigma^{1-\alpha}}{\sigma'}\langle\mathbf{D}_G\nabla v,\nabla v\rangle Gdxdt-\int_{\mathbb{R}^n\times[0,T]}\frac{\sigma^{1-\alpha}}{\sigma'}v\langle\mathbf{A}\nabla v,\nabla F\rangle Gdxdt \qquad (44)\\
&=2\int_{\mathbb{R}^n\times[0,T]}\frac{\sigma^{1-\alpha}}{\sigma'}LvPvGdxdt+\int_{\mathbb{R}^n}\frac{\sigma^{1-\alpha}}{\sigma'}\langle\mathbf{A}\nabla v,\nabla v\rangle Gdx|_0^T\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^n}\frac{\sigma^{1-\alpha}}{\sigma'}v^2(F-\frac{\alpha\sigma'}{\sigma})Gdx|_0^T
\end{aligned}$$

where

$$Lv=\partial_tv-\langle\mathbf{A}\nabla logG,\nabla v\rangle+\frac{1}{2}(F-\frac{\alpha\sigma'}{\sigma})v,$$

$$M=(log\frac{\sigma}{\sigma'})'F+\partial_tF+(F-\frac{\alpha\sigma'}{\sigma})(\frac{\partial_tG-\tilde{\Delta}G}{G}-F)-\langle\mathbf{A}\nabla F,\nabla logG\rangle,$$

and

$$\mathbf{D}_G^{ij} = a^{ik}\partial_{kl}(logG)a^{lj} + \frac{\partial_l(logG)}{2}(a^{ki}\partial_k a^{lj} + a^{kj}\partial_k a^{li} - a^{kl}\partial_k a^{ij}) + \frac{1}{2}\partial_t a^{ij}.$$

We first give a modification of this lemma which will be used in our proof. Let $\alpha = 0$ and $\sigma(t) = e^t$ in Lemma 3.1, then we obtain the following identity for $v \in C_0^\infty(\mathbb{R}^n \times [0,T])$

$$\begin{aligned}
&\frac{1}{2}\int\limits_{\mathbb{R}^n\times[0,T]} v^2 MGdxdt + \int\limits_{\mathbb{R}^n\times[0,T]} \langle[2\mathbf{D}_G + (\frac{\partial_t G - \tilde{\Delta}G}{G} - F)\mathbf{A}]\nabla v, \nabla v\rangle Gdxdt \\
&- \int\limits_{\mathbb{R}^n\times[0,T]} v\langle\mathbf{A}\nabla v, \nabla F\rangle Gdxdt = 2\int\limits_{\mathbb{R}^n\times(0,T)} Lv(Pv - Lv)Gdxdt \\
&+ \int\limits_{\mathbb{R}^n}\langle\mathbf{A}\nabla v, \nabla v\rangle Gdx|_0^T + \frac{1}{2}\int\limits_{\mathbb{R}^n} v^2 FGdx|_0^T.
\end{aligned}$$

If ∇F is differentiable, we can integrate by parts to obtain that

$$\begin{aligned}
&- \int\limits_{\mathbb{R}^n\times[0,T]} v\langle\mathbf{A}\nabla v, \nabla F\rangle Gdxdt \\
&= \frac{1}{2}\int\limits_{\mathbb{R}^n\times[0,T]} v^2\tilde{\Delta}FGdxdt + \frac{1}{2}\int\limits_{\mathbb{R}^n\times[0,T]} v^2\langle\mathbf{A}\nabla F, \nabla logG\rangle Gdxdt.
\end{aligned}$$

The function ∇F may not be differentiable, so we approximate F by some twice differentiable function F_0 and use the above identity with F_0 in place of F, following Nguyen's idea in [11]. Thus a direct corollary follows.

Corollary 3.2. *Suppose F is differentiable, F_0 and G is twice differentiable and $G > 0$. Then the following identity holds for any $v \in C_0^\infty(\mathbb{R}^n \times [0,T])$:*

$$\begin{aligned}
&\frac{1}{2}\int\limits_{\mathbb{R}^n\times[0,T]} v^2 M_0Gdxdt + \int\limits_{\mathbb{R}^n\times[0,T]} \langle[2\mathbf{D}_G + (\frac{\partial_t G - \tilde{\Delta}G}{G} - F)\mathbf{A}]\nabla v, \nabla v\rangle Gdxdt \\
&- \int\limits_{\mathbb{R}^n\times[0,T]} v\langle\mathbf{A}\nabla v, \nabla(F - F_0)\rangle Gdxdt = 2\int\limits_{\mathbb{R}^n\times[0,T]} Lv(Pv - Lv)Gdxdt \qquad (45)\\
&+ \int\limits_{\mathbb{R}^n}\langle\mathbf{A}\nabla v, \nabla v\rangle Gdx|_0^T + \frac{1}{2}\int\limits_{\mathbb{R}^n} v^2 FGdx|_0^T,
\end{aligned}$$

where

$$Lu = \partial_t v - \langle \mathbf{A}\nabla v, \nabla logG\rangle + \frac{Fv}{2},$$
$$M_0 = \partial_t F + F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) + \tilde{\Delta} F_0 - \langle \mathbf{A}\nabla(F - F_0), \nabla logG\rangle,$$

and

$$\mathbf{D}_G^{ij} = a^{ik}\partial_{kl}(logG)a^{lj} + \frac{\partial_l(logG)}{2}(a^{ki}\partial_k a^{lj} + a^{kj}\partial_k a^{li} - a^{kl}\partial_k a^{ij}) + \frac{1}{2}\partial_t a^{ij}.$$

Before we prove our Carleman inequalities, we need to prove a result which could be viewed as another version of Corollary 3.2.

In (45), we let $G = e^{2\Phi}$, $w = e^{\Phi}v$, and we denote

$$\mathbf{B} = 2\mathbf{D}_G + (\frac{\partial_t G - \tilde{\Delta} G}{G} - F)\mathbf{A}.$$

Then the third term of the left hand side of (45) is

$$\begin{aligned} &-\int\limits_{\mathbb{R}^n\times[0,T]} v\langle \mathbf{A}\nabla(F - F_0), \nabla v\rangle e^{2\Phi} dxdt \\ &= -\int\limits_{\mathbb{R}^n\times[0,T]} w\langle \mathbf{A}\nabla(F - F_0), \nabla w - \nabla\Phi w\rangle dxdt \\ &= -\int\limits_{\mathbb{R}^n\times[0,T]} w\langle \mathbf{A}\nabla(F - F_0), \nabla w\rangle dxdt + \int\limits_{\mathbb{R}^n\times[0,T]} \langle \mathbf{A}\nabla(F - F_0), \nabla\Phi\rangle w^2 dxdt. \end{aligned}$$

We use the above identity and rewrite (45) as

$$\begin{aligned} &\frac{1}{2}\int\limits_{\mathbb{R}^n\times[0,T]} M_1 w^2 dxdt + \int\limits_{\mathbb{R}^n\times[0,T]} \langle \mathbf{B}\nabla v, \nabla v\rangle e^{2\Phi} dxdt - \int\limits_{\mathbb{R}^n\times[0,T]} w\langle \mathbf{A}\nabla(F - F_0), \nabla w\rangle dxdt \\ &= 2\int\limits_{\mathbb{R}^n\times[0,T]} Lv(Pv - Lv)e^{2\Phi} dxdt + \int\limits_{\mathbb{R}^n} \langle \mathbf{A}\nabla v, \nabla v\rangle e^{2\Phi} dx|_0^T + \frac{1}{2}\int\limits_{\mathbb{R}^n} v^2 F e^{2\Phi} dx|_0^T \end{aligned} \tag{46}$$

where

$$\begin{aligned} M_1 &= \partial_t F + F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) + \tilde{\Delta} F_0, \\ \mathbf{B} &= 4\mathbf{A}D^2\Phi\mathbf{A} + 2\partial_l\Phi(a^{ki}\partial_k a^{lj} + a^{kj}\partial_k a^{li} - a^{kl}\partial_k a^{ij}) \\ &\quad + \partial_t a^{ij} + (\frac{\partial_t G - \tilde{\Delta} G}{G} - F)\mathbf{A}. \end{aligned} \tag{47}$$

By direct calculations we have

$$\frac{\partial_t G - \tilde{\Delta} G}{G} = 2\partial_t \Phi - 2a^{ij}\partial_{ij}\Phi - 2\partial_i a^{ij}\partial_j \Phi - 4\langle \mathbf{A}\nabla\Phi, \nabla\Phi\rangle. \tag{48}$$

Let

$$F = 2\partial_t \Phi - 2a^{ij}\partial_{ij}\Phi - 4\langle \mathbf{A}\nabla\Phi, \nabla\Phi\rangle - H, \tag{49}$$

where H is a smooth function to be determined. We choose

$$F_0 = 2\partial_t \Phi - 2a_\epsilon^{ij}\partial_{ij}\Phi - 4a_\epsilon^{ij}\partial_i\Phi\partial_j\Phi - H,$$

where

$$a_\epsilon^{ij}(x,t) = \int\limits_{\mathbb{R}^n} a^{ij}(x-y,t)\phi_\epsilon(y)dy,$$

ϕ is a mollifier, and $\epsilon = \frac{1}{2}$.

By (47)–(49), we have

$$\mathbf{B} = 4\mathbf{A}D^2\Phi\mathbf{A} + 2\partial_l\Phi(a^{ki}\partial_k a^{lj} + a^{kj}\partial_k a^{li} - a^{kl}\partial_k a^{ij} - a^{ij}\partial_k a^{kl}) + \partial_t a^{ij} + H\mathbf{A}. \tag{50}$$

Now we begin to prove our Carleman inequalities.

3.1. Proof of Proposition 1.6

Note that Carleman inequality (14) is very similar to the second Carleman inequality in [14], and their proofs are also similar.

In this part, we let

$$\Phi = \gamma f(t)|x|^{3/2} - \frac{b|x|^2 + \beta}{2(t+1)},$$

where $b = \frac{1}{16\Lambda}$ and $\beta = \beta(n, \Lambda, \lambda, M, E)$ large enough.

Step 1. Estimate matrix $\mathbf{B}$.

We need to estimate the lower bounds of the matrices in the right side of (50).

First we estimate $D^2\Phi$. Denote that

$$h = \gamma f|x|^{-1/2}.$$

By direct calculations we have

$$D^2\Phi = \frac{3}{2}h(I_n - \frac{x\cdot x^T}{2|x|^2}) - \frac{b}{t+1}I_n \geq (\frac{3}{4}h - \frac{b}{t+1})I_n,$$

and hence

$$4\mathbf{A}D^2\Phi\mathbf{A} \geq (3\lambda^2 h - \frac{C}{t+1})I_n.$$

Second, we estimate matrix $\partial_l \Phi a^{ki}\partial_k a^{lj}$ and $\partial_t a^{ij}$. For any $\xi \in \mathbb{R}^n$,

$$|\partial_l \Phi a^{ki}\partial_k a^{lj}\xi_i\xi_j| \leq n^2\Lambda\frac{E}{|x|}|\nabla\Phi|\sum_{i,j}|\xi_i||\xi_j| \leq \frac{n^3\Lambda E}{|x|}|\nabla\Phi||\xi|^2.$$

Since

$$\nabla\Phi = (\frac{3}{2}h - \frac{b}{t+1})x, \tag{51}$$

then

$$|\partial_l \Phi a^{ki}\partial_k a^{lj}\xi_i\xi_j| \leq n^3\Lambda E(\frac{3}{2}h + \frac{b}{t+1}),$$

and thus

$$\partial_l \Phi a^{ki}\partial_k a^{lj} \geq -n^3\Lambda E(\frac{3}{2}h + \frac{b}{t+1})I_n.$$

Similarly,

$$\partial_t a^{ij} \geq -nMI_n.$$

Consequently,

$$\begin{aligned}\mathbf{B} &\geq (3\lambda^2 - 12n^3\Lambda E)hI_n - \frac{C}{t+1}I_n - nMI_n + \lambda HI_n \\ &\geq 2\lambda^2 hI_n + (\lambda H - \frac{C}{t+1})I_n,\end{aligned}$$

if we take $E < E_0(n, \Lambda, \lambda)$. Now in this part, we choose

$$H = \frac{d}{t+1},$$

where $d = d(n, \Lambda, \lambda, M, E)$ large enough, then we have

$$\mathbf{B} \geq 2\lambda^2(h + \frac{1}{t+1})I_n + I_n. \tag{52}$$

Step 2. Prove the Carleman inequality.

By (52), we can estimate the second term of the left hand side of (46),

$$\begin{aligned}&\int_Q \langle \mathbf{B}\nabla v, \nabla v\rangle e^{2\Phi}dxdt\\&\geq \int_Q e^{2\Phi}|\nabla v|^2dxdt + 2\lambda^2\int_Q (h+\frac{1}{t+1})e^{2\Phi}|\nabla v|^2dxdt\\&= \int_Q e^{2\Phi}|\nabla v|^2dxdt + 2\lambda^2\int_Q (h+\frac{1}{t+1})|\nabla w|^2dxdt\\&\quad + 2\lambda^2\int_Q [(h+\frac{1}{t+1})|\nabla\Phi|^2 + \nabla h\cdot\nabla\Phi + (h+\frac{1}{t+1})\Delta\Phi]w^2dxdt.\end{aligned} \tag{53}$$

By (46), (53) and the Cauchy inequality, we have

$$\begin{aligned}&\int_Q e^{2\Phi}|\nabla v|^2dxdt + 2\lambda^2\int_Q (h+\frac{1}{t+1})|\nabla w|^2dxdt + \int_Q M_2w^2dxdt\\&-\int_Q w\langle A\nabla(F-F_0),\nabla w\rangle dxdt \leq \int_Q e^{2\Phi}|Pv|^2dxdt\\&\quad+\int_{\mathbb{R}^n}\langle \mathbf{A}\nabla v,\nabla v\rangle e^{2\Phi}dx|_0^1 + \frac{1}{2}\int_{\mathbb{R}^n} Fe^{2\Phi}v^2dx|_0^1,\end{aligned} \tag{54}$$

where

$$\begin{aligned}M_2 &= 2\lambda^2[(h+\frac{1}{t+1})|\nabla\Phi|^2 + \nabla h\cdot\nabla\Phi + (h+\frac{1}{t+1})\Delta\Phi]\\&\quad+\frac{1}{2}\partial_tF + \frac{1}{2}F(\frac{\partial_tG-\tilde{\Delta}G}{G} - F) + \frac{1}{2}\tilde{\Delta}F_0.\end{aligned} \tag{55}$$

We use inequality (54) to prove Proposition 1.6. We need some estimates which we list in the following lemma.

Lemma 3.3. *Set $b = \frac{1}{16\Lambda}$, $\beta = 20\frac{\Lambda}{\lambda}d$ and $d = d(n,\Lambda,\lambda,M,E)$ large enough. There exists $E_0 = E_0(n,\Lambda,\lambda)$, such that when $E < E_0$, for any $\gamma > 0$, we have*

$$M_2 \geq \lambda^2h^3|x|^2 + \frac{db}{8}\frac{|x|^2}{(t+1)^3}; \tag{56}$$

$$|\nabla(F-F_0)| \leq C(n)E[h^2+\frac{1}{(t+1)^2}]|x|; \tag{57}$$

$$F(x,0) \geq -2\beta(1+\gamma)^2|x|^2; \tag{58}$$

$$F(x,1) \leq \frac{\beta}{2}|x|^2. \tag{59}$$

We shall prove this lemma later.

By applying Lemma 3.3, in particular by (57), we have

$$
\begin{aligned}
|\int_Q w\langle A\nabla(F-F_0),\nabla w\rangle dxdt| &\le \Lambda\int_Q |\nabla(F-F_0)||w||\nabla w|dxdt\\
&\le C(n)\Lambda E\int_Q [h^2+\frac{1}{(t+1)^2}]|x||w||\nabla w|dxdt.
\end{aligned}
$$

Using the Cauchy inequality, we have

$$
\begin{aligned}
|\int_Q w\langle A\nabla(F-F_0),\nabla w\rangle dxdt| &\le C(n)\Lambda E\int_Q (h^3|x|^2+\frac{|x|^2}{(t+1)^3})w^2dxdt\\
&+C(n)\Lambda E\int_Q (h+\frac{1}{t+1})|\nabla w|^2dxdt.
\end{aligned}
$$

When $E < E_0(n,\Lambda,\lambda)$, we have

$$
\begin{aligned}
|\int_Q w\langle A\nabla(F-F_0),\nabla w\rangle dxdt| &\le \lambda^2\int_Q (h^3|x|^2+\frac{|x|^2}{(t+1)^3})w^2dxdt\\
&+\lambda^2\int_Q (h+\frac{1}{t+1})|\nabla w|^2dxdt.
\end{aligned}
\tag{60}
$$

Because of (54), (60) and (56), we have

$$
\begin{aligned}
&\int_Q e^{2\Phi}|\nabla v|^2dxdt+(\frac{db}{8}-C)\int_Q \frac{|x|^2}{(t+1)^3}w^2dxdt \le \int_Q e^{2\Phi}|Pv|^2dxdt\\
&+\int_{\mathbb{R}^n}\langle \mathbf{A}\nabla v,\nabla v\rangle e^{2\Phi}dx|_0^1+\frac{1}{2}\int_{\mathbb{R}^n} Fe^{2\Phi}v^2dx|_0^1.
\end{aligned}
\tag{61}
$$

Now we estimate the second term of the right hand side of (61).

$$
\begin{aligned}
\int_{\mathbb{R}^n}\langle \mathbf{A}\nabla v,\nabla v\rangle e^{2\Phi}dx|_0^1 &\le \int_{\mathbb{R}^n}\langle \mathbf{A}\nabla v,\nabla v\rangle e^{2\Phi}dx|_{t=1}\\
&\le \Lambda\int_{\mathbb{R}^n} |\nabla v(x,1)|^2e^{-\frac{b|x|^2+\beta}{2}}dxdt.
\end{aligned}
\tag{62}
$$

Finally we estimate the third term of the right hand side of (61).

$$\frac{1}{2}\int_{\mathbb{R}^n} Fe^{2\Phi}v^2dx|_0^1 = \frac{1}{2}\int_{\mathbb{R}^n} F(x,1)e^{-\frac{b|x|^2+\beta}{2}}v^2(x,1)dx - \frac{1}{2}\int_{\mathbb{R}^n} F(x,0)e^{2\gamma(1-2^{-\beta})|x|^{3/2}-(b|x|^2+\beta)}v^2(x,0)dx$$

By (58) and (59), we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n} Fe^{2\Phi}v^2dx|_0^1 &\le \frac{\beta}{4}\int_{\mathbb{R}^n} |x|^2e^{-\frac{b|x|^2+\beta}{2}}v^2(x,1)dx \\ &\quad + \beta(1+\gamma)^2\int_{\mathbb{R}^n} |x|^2e^{2\gamma(1-2^{-\beta})|x|^{3/2}-(b|x|^2+\beta)}v^2(x,0)dx \\ &\le \beta\int_{\mathbb{R}^n} |x|^2e^{-\frac{b|x|^2+\beta}{2}}v^2(x,1)dx \\ &\quad + \beta(1+\gamma)^2\int_{\mathbb{R}^n} |x|^2e^{2\gamma|x|^{3/2}-(b|x|^2+\beta)}v^2(x,0)dx.\end{aligned} \tag{63}$$

We combine (61), (62) and (63), and take d large enough, then we proved Carleman inequality (14).

At last, we just need to prove Lemma 3.3.

Step 3. Prove Lemma 3.3.

Estimate of M_2.

We estimate the terms of M_2 respectively.

Estimate of the first three terms.

By (51), we have

$$\begin{aligned}(h+\frac{1}{t+1})|\nabla\Phi|^2 \ge h|\nabla\Phi|^2 &= h(\frac{3}{2}h-\frac{b}{t+1})^2|x|^2 \\ &\ge h|x|^2[\frac{9}{8}h^2-\frac{b^2}{(t+1)^2}] \\ &= [\frac{9}{8}h^3-\frac{C}{(t+1)^2}h]|x|^2; \\ \nabla h\cdot\nabla\Phi = -\frac{1}{2}h(\frac{3}{2}h-\frac{b}{t+1}) &\ge -\frac{3}{4}h^2; \\ (h+\frac{1}{t+1})\Delta\Phi \ge -\frac{nb}{t+1}(h+\frac{1}{t+1}) &\ge -\frac{C}{t+1}h-\frac{C}{(t+1)^2}.\end{aligned}$$

Combining them together, we obtain

$$2\lambda^2(h|\nabla\Phi|^2+\nabla h\cdot\nabla\Phi+h\Delta\Phi)\geq[\frac{9}{4}\lambda^2h^3-Ch^2-\frac{C}{(t+1)^2}h]|x|^2-\frac{C}{(t+1)^3}. \qquad (64)$$

Estimate of $\frac{1}{2}\partial_t F$.
Recall (49), then

$$\frac{1}{2}\partial_t F=\partial_t^2\Phi-\partial_t a^{ij}\partial_{ij}\Phi-a^{ij}\partial_{ijt}\Phi-2\partial_t\langle A\nabla\Phi,\nabla\Phi\rangle-\frac{1}{2}\partial_t H.$$

We estimate them one by one. Keep in mind that $f'<0$.

$$\begin{aligned}
&\partial_t^2\Phi=\gamma f''|x|^{3/2}-\frac{b|x|^2+\beta}{(t+1)^3}=\frac{f''}{f}h|x|^2-\frac{b|x|^2+\beta}{(t+1)^3};\\
&-\partial_t a^{ij}\partial_{ij}\Phi=-\frac{3}{2}h(\partial_t a^{ii}-\frac{\partial_t a^{ij}x_ix_j}{2|x|^2})+\frac{b\partial_t a^{ii}}{t+1}\geq -Ch-\frac{C}{t+1};\\
&-a^{ij}\partial_{ijt}\Phi=-\frac{3f'}{2f}h(a^{ii}-\frac{a^{ij}x_ix_j}{2|x|^2})-\frac{ba^{ii}}{(t+1)^2}\geq C\frac{f'}{f}h-\frac{C}{(t+1)^2};\\
&-2\partial_t\langle A\nabla\Phi,\nabla\Phi\rangle\\
&\quad=[-9\frac{f'}{f}h^2+6b(\frac{f'}{(t+1)f}-\frac{1}{(t+1)^2})h+\frac{4b^2}{(t+1)^3}]a^{ij}x_ix_j\\
&\qquad-2(\frac{3}{2}h-\frac{b}{t+1})^2\partial_t a^{ij}x_ix_j\\
&\quad\geq[-9\lambda\frac{f'}{f}h^2+C(\frac{f'}{(t+1)f}-\frac{1}{(t+1)^2})h]|x|^2-C[h^2+\frac{1}{(t+1)^2}]|x|^2\\
&\quad\geq[(-9\lambda\frac{f'}{f}-C)h^2+C(\frac{f'}{(t+1)f}-\frac{1}{(t+1)^2})h]|x|^2-\frac{C|x|^2}{(t+1)^2};\\
&-\frac{1}{2}\partial_t H=\frac{d}{2(t+1)^2}.
\end{aligned}$$

Combining them together, we have

$$\frac{1}{2}\partial_t F\geq[(-9\lambda\frac{f'}{f}-C)h^2+(\frac{f''}{f}+\frac{Cf'}{(t+1)f}-\frac{C}{(t+1)^2})h]|x|^2-\frac{C|x|^2+\beta}{(t+1)^3}; \qquad (65)$$

Estimate of $\frac{1}{2}F(\frac{\partial_t G-\tilde{\Delta}G}{G}-F)$.
First we have

$$\begin{aligned}
\frac{1}{2}F(\frac{\partial_t G-\tilde{\Delta}G}{G}-F)&=(\partial_t\Phi-2\langle A\nabla\Phi,\nabla\Phi\rangle-a^{ij}\partial_{ij}\Phi-\frac{1}{2}H)(H-2\partial_i a^{ij}\partial_j\Phi)\\
&\equiv J_1-J_2-J_3,
\end{aligned}$$

where

$$J_1 = \partial_t\Phi(H - 2\partial_i a^{ij}\partial_j\Phi)$$
$$J_2 = 2\langle A\nabla\Phi, \nabla\Phi\rangle(H - 2\partial_i a^{ij}\partial_j\Phi)$$
$$J_3 = (a^{ij}\partial_{ij}\Phi + \frac{1}{2}H)(H - 2\partial_i a^{ij}\partial_j\Phi).$$

Before we estimate J_1, J_2 and J_3, we estimate $2\partial_i a^{ij}\partial_j\Phi$ first.

$$|2\partial_i a^{ij}\partial_j\Phi| \le \frac{2n^2E}{|x|}|\nabla\Phi| \le 2n^2E(\frac{3}{2}h + \frac{b}{t+1}) \le 3n^2Eh + \frac{C}{t+1},$$

then

$$-3n^2Eh + \frac{d-C}{t+1} \le H - 2\partial_i a^{ij}\partial_j\Phi \le 3n^2Eh + \frac{d+C}{t+1}.$$

Now we estimate J_1, J_2 and J_3 respectively.

$$\begin{aligned}
J_1 &= \frac{f'}{f}h|x|^2(H - 2\partial_i a^{ij}\partial_j\Phi) + \frac{b|x|^2+\beta}{2(t+1)^2}(H - 2\partial_i a^{ij}\partial_j\Phi)\\
&\ge \frac{f'}{f}h|x|^2(3n^2Eh + \frac{d+C}{t+1}) + \frac{b|x|^2+\beta}{2(t+1)^2}(-3n^2Eh + \frac{d-C}{t+1})\\
&\ge [3n^2E\frac{f'}{f}h^2 + (\frac{(d+C)f'}{(t+1)f} - \frac{C\beta+C}{(t+1)^2})h]|x|^2 + (\frac{d}{2} - C)\frac{(b|x|^2+\beta)}{(t+1)^3},\\
J_2 &\le 2\Lambda(\frac{3}{2}h - \frac{b}{t+1})^2|x|^2(3n^2Eh + \frac{d+C}{t+1})\\
&\le 4\Lambda[\frac{9}{4}h^2 + \frac{b^2}{(t+1)^2}]|x|^2(3n^2Eh + \frac{d+C}{t+1})\\
&\le [27n^2\Lambda Eh^3 + \frac{C}{(t+1)^2}h + \frac{9d\Lambda+C}{t+1}h^2]|x|^2 + \frac{4d\Lambda b^2+C}{(t+1)^3}|x|^2,
\end{aligned}$$

and

$$\begin{aligned}
J_3 &\le |\frac{3}{2}h(a^{ii} - \frac{a^{ij}x_ix_j}{2|x|^2}) + \frac{d/2 - ba^{ii}}{t+1}|(3n^2Eh + \frac{d+C}{t+1})\\
&\le (Ch + \frac{d}{t+1})^2 \le Ch^2 + \frac{2d^2}{(t+1)^2} \le Ch^2|x|^2 + \frac{4d^2}{(t+1)^3}.
\end{aligned}$$

Combining J_1, J_2 and J_3 together, we obtain

$$
\begin{aligned}
&\frac{1}{2}F(\frac{\partial_t G-\tilde{\Delta}G}{G}-F)\\
&\quad\geq [-27n^2\Lambda E h^3+(3n^2E\frac{f'}{f}-\frac{9\Lambda d+C}{t+1})h^2+(\frac{(d+C)f'}{(t+1)f}-\frac{C\beta+C}{(t+1)^2})h]|x|^2 \qquad (66)\\
&\qquad+(\frac{db}{2}-4d\Lambda b^2-C)\frac{|x|^2}{(t+1)^3}+\frac{(d/2-C)\beta-4d^2}{(t+1)^3}.
\end{aligned}
$$

Estimate of $\frac{1}{2}\tilde{\Delta}F_0$.

In order to estimate $\tilde{\Delta}F_0$ and $|\nabla(F-F_0)|$, we need some estimates about $\{a_\epsilon^{ij}\}$ which we will prove in the appendix.

In fact, $\{a_\epsilon^{ij}\}$ satisfy the following properties:

$$
\begin{aligned}
&i)\ \lambda|\xi|^2\leq a_\epsilon^{ij}(x,t)\xi_i\xi_j\leq\Lambda|\xi|^2,\ \forall\xi\in\mathbb{R}^n;\\
&ii)\ |\nabla a_\epsilon^{ij}(x,t)|\leq M;\quad |\nabla a_\epsilon^{ij}(x,t)|\leq\frac{2E}{|x|}\text{ when }|x|\geq 1;\\
&iii)\ |a_\epsilon^{ij}(x,t)-a^{ij}(x,t)|\leq 2\Lambda;\quad |a_\epsilon^{ij}(x,t)-a^{ij}(x,t)|\leq\frac{E}{|x|}\text{ when }|x|\geq 1; \qquad (67)\\
&iv)\ |\partial_{kl}a_\epsilon^{ij}(x,t)|\leq c(n)M;\quad |\partial_{kl}a_\epsilon^{ij}(x,t)|\leq\frac{c(n)E}{|x|}\text{ when }|x|\geq 1.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
F_0&=2\partial_t\Phi-2a_\epsilon^{ij}\partial_{ij}\Phi-4a_\epsilon^{ij}\partial_i\Phi\partial_j\Phi-H\\
&=2\gamma f'|x|^{3/2}+\frac{b|x|^2+\beta}{(t+1)^2}-\frac{d}{t+1}-2a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi),
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{1}{2}\tilde{\Delta}F_0&=\gamma f'\tilde{\Delta}(|x|^{3/2})+\frac{b}{2(t+1)^2}\tilde{\Delta}(|x|^2)-\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)]\\
&\geq C\gamma f'|x|^{-1/2}-\frac{C}{(t+1)^2}-\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)] \qquad (68)\\
&=\frac{Cf'}{f}h-\frac{C}{(t+1)^2}-\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)],
\end{aligned}
$$

and thus it remains to estimate $\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)]$.

By (67) we have that $|a_\epsilon^{ij}|$, $|\nabla a_\epsilon^{ij}|$ and $|\nabla^2 a_\epsilon^{ij}|$ are all bounded, and it is easy to verify that

$$
|\nabla^k\Phi|\leq C(n)(h+\frac{1}{t+1})|x|^{2-k},\quad k=1,2,3,4. \qquad (69)
$$

Direct calculations give us

$$\begin{aligned}&|\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)]|\\&\quad=|\partial_k a^{kl}\partial_l[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)]+a^{kl}\partial_{kl}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)]|\\&\quad\le C(|\nabla^2\Phi|+|\nabla^3\Phi|+|\nabla^4\Phi|+|\nabla\Phi|^2+|\nabla\Phi||\nabla^2\Phi|+|\nabla\Phi||\nabla^3\Phi|+|\nabla^2\Phi|^2).\end{aligned}$$

By the Cauchy inequality, we have

$$\begin{aligned}&|\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)]|\\&\quad\le C(|\nabla^2\Phi|+|\nabla^3\Phi|+|\nabla^4\Phi|+|\nabla\Phi|^2+|\nabla^2\Phi|^2+|\nabla^3\Phi|^2)\end{aligned}\tag{70}$$

then by (69),

$$|\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi+2\partial_i\Phi\partial_j\Phi)]|\le C(h+\frac{1}{t+1})^2|x|^2\le C[h^2+\frac{1}{(t+1)^2}]|x|^2.\tag{71}$$

We combine (68) and (71) and obtain that

$$\frac{1}{2}\tilde{\Delta}F_0\ge(\frac{Cf'}{f}h-Ch^2)|x|^2-\frac{C|x|^2}{(t+1)^3}.\tag{72}$$

At last, combining (64), (65), (66) and (72), we have

$$\begin{aligned}M_2\ge{}&[(\frac{9}{4}\lambda^2-27n^2\Lambda E)h^3+(-(9\lambda-3n^2E)\frac{f'}{f}-\frac{9\Lambda d+C}{t+1})h^2]|x|^2\\&+(\frac{f''}{f}+\frac{(d+C)f'}{(t+1)f}-\frac{C\beta+C}{(t+1)^2})h|x|^2\\&+(\frac{db}{2}-4d\Lambda b^2-C)\frac{|x|^2}{(t+1)^3}+\frac{(d/2-C)\beta-4d^2-C}{(t+1)^3}.\end{aligned}$$

Now we choose

$$b=\frac{1}{16\Lambda},\qquad \beta=20\frac{\Lambda}{\lambda}d,$$

and we take d large enough, then

$$\begin{aligned}&\frac{db}{2}-4d\Lambda b^2-C=\frac{db}{4}-C\ge\frac{db}{8},\\&(\frac{d}{2}-C)\beta-4d^2-C\ge\frac{d}{4}\beta-5d^2\ge 0,\end{aligned}$$

and thus when $E<E_0(n,\Lambda,\lambda)$, we have

$$\begin{aligned}M_2\ge{}&[\lambda^2h^3+(-\frac{9\lambda f'}{2f}-\frac{18\Lambda d}{t+1})h^2]|x|^2\\&+(\frac{f''}{f}+\frac{2df'}{(t+1)f}-\frac{C\beta}{(t+1)^2})h|x|^2+\frac{db}{8}\frac{|x|^2}{(t+1)^3}.\end{aligned}$$

We take into account that

$$f(t) = (t+1)^{-\beta} - 2^{-\beta},$$

then we have

$$\begin{aligned}-\frac{9\lambda f'}{2f} - \frac{18\Lambda d}{t+1} &= \frac{9\lambda\beta}{2(t+1)[1-(\frac{t+1}{2})^\beta]} - \frac{18\Lambda d}{t+1} \\ &\geq \frac{9\lambda\beta}{2(t+1)} - \frac{18\Lambda d}{t+1} = \frac{9(\lambda\beta - 4\Lambda d)}{2(t+1)} \geq 0,\end{aligned}$$

and

$$\begin{aligned}\frac{f''}{f} + \frac{2df'}{(t+1)f} - \frac{C\beta}{(t+1)^2} &= \frac{\beta(\beta+1-2d)}{(t+1)^2[1-(\frac{t+1}{2})^\beta]} - \frac{C\beta}{(t+1)^2} \\ &\geq \frac{\beta(\beta+1-2d)}{(t+1)^2} - \frac{d\beta}{(t+1)^2} \\ &= \frac{\beta(\beta+1-3d)}{(t+1)^2} \geq 0,\end{aligned}$$

thus

$$M_2 \geq \lambda^2 h^3 |x|^2 + \frac{db}{8}\frac{|x|^2}{(t+1)^3}. \tag{73}$$

Estimate of $|\nabla(F - F_0)|$.
Since

$$F - F_0 = 2(a_\epsilon^{ij} - a^{ij})(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi),$$

then

$$\begin{aligned}|\nabla(F - F_0)| &= 2|(\nabla a_\epsilon^{ij} - \nabla a^{ij})(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi) \\ &\quad + (a_\epsilon^{ij} - a^{ij})(\nabla\partial_{ij}\Phi + 4\partial_i\Phi\nabla\partial_j\Phi)| \\ &\leq 2|\nabla a_\epsilon^{ij} - \nabla a^{ij}|(|\partial_{ij}\Phi| + |\partial_i\Phi|^2 + |\partial_j\Phi|^2) \\ &\quad + 2|a_\epsilon^{ij} - a^{ij}|(|\nabla\partial_{ij}\Phi| + 2|\partial_i\Phi|^2 + 2|\nabla\partial_j\Phi|^2).\end{aligned}$$

By (67) we have

$$|\nabla a_\epsilon^{ij}| \leq \frac{2E}{|x|}, \quad |a_\epsilon^{ij} - a^{ij}| \leq \frac{E}{|x|},$$

then

$$
\begin{aligned}
|\nabla(F-F_0)| &\le \frac{6E}{|x|}(n|\nabla^2\Phi| + 2n|\nabla\Phi|^2) + \frac{2E}{|x|}(n|\nabla^3\Phi| + 2n|\nabla\Phi|^2 + 2n|\nabla^2\Phi|^2) \\
&\le \frac{nE}{|x|}(6|\nabla^2\Phi| + 2|\nabla^3\Phi| + 16|\nabla\Phi|^2 + 4|\nabla^2\Phi|^2).
\end{aligned}
\tag{74}
$$

By (69) we have

$$
|\nabla(F-F_0)| \le C(n)E[h^2 + \frac{1}{(t+1)^2}]|x|.
$$

Estimate of $F(x,0)$ and $F(x,1)$.

By (49) and direct calculations, we have

$$
\begin{aligned}
F &= 2\partial_t\Phi - 2a^{ij}\partial_{ij}\Phi - 4\langle \mathbf{A}\nabla\Phi, \nabla\Phi\rangle - \frac{d}{t+1} \\
&= -2\beta\gamma(t+1)^{-\beta-1}|x|^{3/2} - 9h^2a^{ij}x_ix_j + 3h[(\frac{1}{2|x|^2} + \frac{4b}{t+1})a^{ij}x_ix_j - a^{ii}] \\
&\quad + \frac{b|x|^2 - 4b^2a^{ij}x_ix_j}{(t+1)^2} + \frac{\beta}{(t+1)^2} + \frac{2ba^{ii} - d}{t+1},
\end{aligned}
$$

then

$$
\begin{aligned}
F(x,1) &= -2\beta\gamma 2^{-\beta-1}|x|^{3/2} + \frac{b|x|^2 - 4b^2a^{ij}(x,1)x_ix_j}{4} + \frac{\beta}{4} + \frac{2ba^{ii}(x,1) - d}{2} \\
&\le \frac{b|x|^2 + \beta}{4} \le \frac{\beta}{4}(|x|^2 + 1) \le \frac{\beta}{2}|x|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
F(x,0) &= -2\beta\gamma|x|^{3/2} - 9\gamma^2(1-2^{-\beta})^2|x|^{-1}a^{ij}(x,0)x_ix_j \\
&\quad + 3\gamma(1-2^{-\beta})|x|^{-1/2}[(\frac{1}{2|x|^2} + 4b)a^{ij}(x,0)x_ix_j - a^{ii}(x,0)] \\
&\quad + b|x|^2 - 4b^2a^{ij}(x,0)x_ix_j + \beta + 2ba^{ii} - d \\
&\ge -2\beta\gamma|x|^{3/2} - 9\gamma^2\Lambda|x| - 3\gamma\Lambda|x|^{-1/2} + (b - 4b^2\Lambda)|x|^2 \\
&\ge -2\beta\gamma|x|^{3/2}(1+\gamma+1) \ge -2\beta(1+\gamma)^2|x|^2.
\end{aligned}
$$

Thus we complete the proof of Lemma 3.3.

3.2. Proof of Proposition 1.7

In this part, we let

$$
\Phi = \Psi = \gamma(1-t)R^{2/3}|x|^{4/3} + \psi(t)R^2,
$$

and we denote by c absolute constants and $C = C(n, \Lambda, \lambda, M, E)$. We keep in mind that

$$\frac{|x|}{R} \le 1, \quad \frac{1}{8} \le t \le \frac{7}{8} \quad \text{in } Q_R.$$

Step 1. Estimate matrix **B**.

First we estimate the Hessian matrix $D^2\Phi$. Denote

$$g = \gamma(\frac{|x|}{R})^{-2/3}.$$

By direct calculations, we have

$$D^2\Phi = \frac{4}{3}(1-t)g(I_n - \frac{2x \cdot x^T}{3|x|^2}) \ge \frac{4}{9}(1-t)gI_n \ge cgI_n,$$

and hence

$$4\mathbf{A}D^2\Phi\mathbf{A} \ge c\lambda^2 gI_n.$$

Then we estimate $\partial_l\Phi a^{ki}\partial_k a^{lj}$ and $\partial_t a^{ij}$.

For any $\xi \in \mathbb{R}^n$,

$$|\partial_l\Phi a^{ki}\partial_k a^{lj}\xi_i\xi_j| \le n^2\Lambda\frac{E}{|x|}|\nabla\Phi|\sum_{i,j}|\xi_i||\xi_j| \le \frac{n^3\Lambda E}{|x|}|\nabla\Phi||\xi|^2.$$

Since

$$\nabla\Phi = \frac{4}{3}(1-t)gx,$$

then

$$\frac{1}{6}\gamma R^{2/3}|x|^{1/3} \le |\nabla\Phi| \le \frac{4}{3}\gamma R^{2/3}|x|^{1/3}, \tag{75}$$

and

$$|\partial_l\Phi a^{ki}\partial_k a^{lj}\xi_i\xi_j| \le cn^3\Lambda Eg|\xi|^2,$$

thus

$$\partial_l\Phi a^{ki}\partial_k a^{lj} \ge -cn^3\Lambda EgI_n.$$

Similarly,

$$\partial_t a^{ij} \ge -nMI_n.$$

Consequently,

$$\mathbf{B} \geq c(\lambda^2 - c_1 n^3 \Lambda E) g I_n - C I_n + H \mathbf{A}.$$

Now we take

$$H = 4n^2 \varphi(t) E g,$$

where $\varphi(t)$ is a smooth decreasing function on $[0, 1]$ satisfying

$$\begin{aligned} &\varphi(t) = 1 \quad \text{in } [0, \frac{1}{3}], \qquad \varphi(t) = -1 \text{ in } [\frac{2}{3}, 1], \\ &\varphi(t) > 0 \quad \text{in } [0, \frac{1}{2}), \qquad \varphi(t) < 0 \quad \text{in } (\frac{1}{2}, 1]. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{B} &\geq c(\lambda^2 - c_1 n^3 \Lambda E) g I_n - C I_n - 4n^2 \Lambda E g I_n \\ &\geq c(\lambda^2 - c_2 n^3 \Lambda E) g I_n - C I_n, \end{aligned}$$

when $E < E_0(n, \Lambda, \lambda)$, and we take $\gamma(n, \Lambda, \lambda, M, E)$ large enough, then

$$\mathbf{B} \geq 2c\lambda^2 g I_n. \tag{76}$$

Step 2. Prove the Carleman inequality.

By (76), we have the estimates of the second term of the left hand side of (46), in fact

$$\begin{aligned} \int_{Q_R} \langle \mathbf{B} \nabla v, \nabla v \rangle e^{2\Phi} dxdt &\geq 2c\lambda^2 \int_{Q_R} g e^{2\Phi} |\nabla v|^2 dxdt \\ &= c\lambda^2 \int_{Q_R} g e^{2\Phi} |\nabla v|^2 dxdt + c\lambda^2 \int_{Q_R} g |\nabla w|^2 dxdt \\ &\quad + c\lambda^2 \int_{Q_R} [g|\nabla \Phi|^2 + \nabla g \cdot \nabla \Phi + g \Delta \Phi] w^2 dxdt. \end{aligned} \tag{77}$$

By (46), (77) and the Cauchy inequality, we have

$$\begin{aligned} &c\lambda^2 \int_{Q_R} g e^{2\Phi} |\nabla v|^2 dxdt + c\lambda^2 \int_{Q_R} g |\nabla w|^2 dxdt + \int_{Q_R} M_2 w^2 dxdt \\ &- \int_{Q_R} w \langle A \nabla (F - F_0), \nabla w \rangle dxdt \leq \int_{Q_R} e^{2\Phi} |Pv|^2 dxdt, \end{aligned} \tag{78}$$

where

$$M_2 = c\lambda^2(g|\nabla\Phi|^2 + \nabla g\cdot\nabla\Phi + g\Delta\Phi) + \frac{1}{2}\partial_t F + \frac{1}{2}F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) + \frac{1}{2}\tilde{\Delta} F_0.$$

We use inequality (78) to prove Proposition 1.7. We also need some estimates which we list in the following lemma.

Lemma 3.4. *There exists a constant $E_0(n,\Lambda,\lambda)$, such that when $E < E_0$, for any $\gamma \geq \gamma_0(n,\Lambda,\lambda,M,E)$, we have*

$$M_2 \geq c\lambda^2\gamma^3 R^2; \tag{79}$$

$$|\nabla(F - F_0)| \leq cnE\gamma^2 R^{4/3}|x|^{-1/3}. \tag{80}$$

We will prove this lemma later.

Then by (80), we have

$$\begin{aligned} |\int\limits_{Q_R} w\langle A\nabla(F-F_0),\nabla w\rangle dxdt| \leq& \Lambda\int\limits_{Q_R} |\nabla(F-F_0)||w||\nabla w|dxdt \\ \leq& cn\Lambda E\int\limits_{Q_R} \gamma^2 R^{4/3}|x|^{-1/3}|w||\nabla w|dxdt. \end{aligned}$$

Using the Cauchy inequality,

$$\begin{aligned} &|\int\limits_{Q_R} w\langle A\nabla(F-F_0),\nabla w\rangle dxdt| \\ &\quad\leq cn\Lambda E[\int\limits_{Q_R} \gamma^3 R^2 w^2 dxdt + \int\limits_{Q_R} \gamma(\frac{|x|}{R})^{-2/3}|\nabla w|^2 dxdt]. \end{aligned}$$

When $E < E_0(n,\Lambda,\lambda)$, we have

$$\begin{aligned} &|\int\limits_{Q_R} w\langle A\nabla(F-F_0),\nabla w\rangle dxdt| \\ &\quad\leq \frac{1}{2}c\lambda^2[\int\limits_{Q_R} \gamma^3 R^2 w^2 dxdt + \int\limits_{Q_R} \gamma(\frac{|x|}{R})^{-2/3}|\nabla w|^2 dxdt] \\ &\quad\leq \frac{1}{2}\int\limits_{Q_R} M_2 w^2 dxdt + c\lambda^2\int\limits_{Q_R} g|\nabla w|^2 dxdt. \end{aligned} \tag{81}$$

Because of (78) and (81), we have

$$\begin{aligned}\int\limits_{Q_R} e^{2\Phi}|Pv|^2dxdt &\geq c\lambda^2\int\limits_{Q_R} ge^{2\Phi}|\nabla v|^2dxdt+\frac{1}{2}\int\limits_{Q_R} M_2w^2dxdt\\ &\geq c\lambda^2\int\limits_{Q_R} e^{2\Phi}(\gamma^3R^2v^2+\gamma|\nabla v|^2)dxdt.\end{aligned}$$

Thus we proved Carleman inequality (15).

Finally, we just need to prove Lemma 3.4.

Step 3. Prove Lemma 3.4.

Estimate of M_2.

We estimate the terms of M_2 respectively. The leading term of M_2 is $h|\nabla\Phi|^2$ and we need pay attention to two quantities, $\partial_t^2\Phi$ and $\partial_t\Phi(H-2\partial_i a^{ij}\partial_j\Phi)$.

Estimate of the first three terms.

By (75), we have

$$\begin{aligned}g|\nabla\Phi|^2 &\geq c\gamma^3R^2,\\ |\nabla g\cdot\nabla\Phi| &\leq |\nabla g||\nabla\Phi|\leq c\gamma^2(\frac{|x|}{R})^{-4/3},\\ g\Delta\Phi &\geq 0,\end{aligned}$$

then

$$c\lambda^2(g|\nabla\Phi|^2+\nabla g\cdot\nabla\Phi+g\Delta\Phi)\geq c\lambda^2\gamma^3R^2-c\gamma^2(\frac{|x|}{R})^{-4/3}. \tag{82}$$

Estimate of $\frac{1}{2}\partial_t F$.

Recall (49), then

$$\frac{1}{2}\partial_t F=\partial_t^2\Phi-\partial_t a^{ij}\partial_{ij}\Phi-a^{ij}\partial_{ijt}\Phi-2\partial_t\langle A\nabla\Phi,\nabla\Phi\rangle-\frac{1}{2}\partial_t H.$$

We estimate them one by one.

$$\begin{aligned}\partial_t^2\Phi &= \psi''R^2\geq -cR^2;\\ -\partial_t a^{ij}\partial_{ij}\Phi &\geq -C|\nabla^2\Phi|\geq -Cg;\\ -a^{ij}\partial_{ijt}\Phi &= \frac{4}{3}g(a^{ii}-\frac{2a^{ij}x_ix_j}{3|x|^2})\geq -Cg;\\ -2\partial_t\langle A\nabla\Phi,\nabla\Phi\rangle &= -2\partial_t a^{ij}\partial_i\Phi\partial_j\Phi-4a^{ij}\partial_i\Phi\partial_{jt}\Phi\\ &\geq -C|\nabla\Phi|^2+\frac{64}{9}\gamma^2(1-t)(\frac{|x|}{R})^{-4/3}a^{ij}x_ix_j\\ &\geq -C|\nabla\Phi|^2\geq -C\gamma^2R^{4/3}|x|^{2/3};\\ -\frac{1}{2}\partial_t H &= -2n^2\varphi'(t)Eg\geq 0.\end{aligned}$$

Combining them together, we have

$$\begin{aligned}\frac{1}{2}\partial_t F &\geq -cR^2 - Cg - C\gamma^2 R^{4/3}|x|^{2/3} \\ &\geq -cR^2 - C\gamma^2 R^{4/3}|x|^{2/3}.\end{aligned} \tag{83}$$

Estimate of $\frac{1}{2}F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F)$.

First we have

$$\frac{1}{2}F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) = (\partial_t \Phi - 2\langle A\nabla\Phi, \nabla\Phi\rangle - a^{ij}\partial_{ij}\Phi - \frac{1}{2}H)(H - 2\partial_i a^{ij}\partial_j\Phi).$$

Since

$$\partial_t \Phi = -\gamma R^{2/3}|x|^{4/3} + \psi' R^2,$$

then

$$\begin{aligned}\frac{1}{2}F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) &= \psi' R^2 (H - 2\partial_i a^{ij}\partial_j\Phi) \\ &\quad - [\gamma R^{2/3}|x|^{4/3} + 2\langle A\nabla\Phi, \nabla\Phi\rangle + a^{ij}\partial_{ij}\Phi + \frac{1}{2}H](H - 2\partial_i a^{ij}\partial_j\Phi) \\ &\equiv J_1 - J_2.\end{aligned}$$

Before we estimate J_1 and J_2, we estimate $2\partial_i a^{ij}\partial_j\Phi$ first.

$$|2\partial_i a^{ij}\partial_j\Phi| \leq \frac{2n^2 E}{|x|}|\nabla\Phi|,$$

and by (75), we have

$$|2\partial_i a^{ij}\partial_j\Phi| \leq \frac{8}{3}n^2 E g.$$

For J_1, we notice that

$$\psi'(t) = 0 \quad \text{in} \quad [0, \frac{1}{4}] \cup [\frac{1}{3}, \frac{2}{3}] \cup [\frac{3}{4}, 1],$$

so we just need to consider the case when $t \in [\frac{1}{4}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{3}{4}]$.

When $t \in [\frac{1}{4}, \frac{1}{3}]$, $\psi' \geq 0$, $\varphi(t) = 1$, $H = 4n^2 E g$, then

$$H - 2\partial_i a^{ij}\partial_j\Phi \geq 0,$$

and thus $J_1 \geq 0$.

When $t \in [\frac{2}{3}, \frac{3}{4}]$, $\psi' \le 0$, $\varphi(t) = -1$, $H = -4n^2 Eg$, then

$$H - 2\partial_i a^{ij} \partial_j \Phi \le 0,$$

and thus $J_1 \ge 0$.

Above all, we have

$$J_1 \ge 0.$$

For J_2,

$$\begin{aligned} J_2 &\le [\gamma R^{2/3}|x|^{4/3} + 2\Lambda|\nabla\Phi|^2 + C|\nabla^2\Phi| + cn^2 Eg] \cdot cn^2 Eg \\ &\le [\gamma R^{2/3}|x|^{4/3} + c\Lambda\gamma^2 R^{4/3}|x|^{2/3} + Cg] \cdot cn^2 Eg \\ &\le [c\Lambda\gamma^2 R^{4/3}|x|^{2/3} + C\gamma R^{2/3}|x|^{4/3}] \cdot cn^2 Eg \\ &= cn^2\Lambda E\gamma^3 R^2 + C\gamma^2 R^{4/3}|x|^{2/3}. \end{aligned}$$

Combining J_1 and J_2 together, we obtain

$$\frac{1}{2} F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) \ge -cn^2\Lambda E\gamma^3 R^2 - C\gamma^2 R^{4/3}|x|^{2/3}. \tag{84}$$

Estimate of $\frac{1}{2}\tilde{\Delta}F_0$.

Recall that

$$\begin{aligned} F_0 &= 2\partial_t\Phi - 2a_\epsilon^{ij}\partial_{ij}\Phi - 4a_\epsilon^{ij}\partial_i\Phi\partial_j\Phi - H \\ &= -2\gamma R^{2/3}|x|^{4/3} + 2\psi' R^2 - 4n^2\varphi(t)E\gamma R^{2/3}|x|^{-2/3} - 2a_\epsilon^{ij}(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi), \end{aligned}$$

then

$$\begin{aligned} \frac{1}{2}\tilde{\Delta}F_0 &= -\gamma R^{2/3}\tilde{\Delta}(|x|^{4/3}) - 2n^2\varphi(t)E\gamma R^{2/3}\tilde{\Delta}(|x|^{-2/3}) - \tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi)] \\ &\ge -C\gamma R^{2/3}|x|^{-2/3} - C\gamma R^{2/3}|x|^{-8/3} - \tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi)] \\ &\ge -Cg - \tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi)], \end{aligned} \tag{85}$$

and thus it remains to estimate $\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi)]$.

By (67) we have that $|a_\epsilon^{ij}|$, $|\nabla a_\epsilon^{ij}|$ and $|\nabla^2 a_\epsilon^{ij}|$ are all bounded, and it is easy to verify that

$$|\nabla^k\Phi| \le C\gamma R^{2/3}|x|^{4/3-k}, \quad k = 1, 2, 3, 4. \tag{86}$$

Similarly to (70),

$$
\begin{aligned}
&|\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi)]| \\
&\quad \leq C(|\nabla^2\Phi| + |\nabla^3\Phi| + |\nabla^4\Phi| + |\nabla\Phi|^2 + |\nabla^2\Phi|^2 + |\nabla^3\Phi|^2),
\end{aligned}
$$

then by (86),

$$
|\tilde{\Delta}[a_\epsilon^{ij}(\partial_{ij}\Phi + 2\partial_i\Phi\partial_j\Phi)]| \leq C|\nabla\Phi|^2 \leq C\gamma^2 R^{4/3}|x|^{2/3}. \tag{87}
$$

We combine (85) and (87) and obtain that

$$
\frac{1}{2}\tilde{\Delta}F_0 \geq -Cg - C\gamma^2 R^{4/3}|x|^{2/3} \geq -C\gamma^2 R^{4/3}|x|^{2/3}. \tag{88}
$$

At last, combining (82), (83), (84) and (88), we have

$$
\begin{aligned}
M_2 &\geq (c\lambda^2 - c_3 n^2 \Lambda E)\gamma^3 R^2 - C\gamma^2 R^{4/3}|x|^{2/3} - cR^2 \\
&\geq (c\lambda^2 - c_3 n^2 \Lambda E)\gamma^3 R^2 - C\gamma^2 R^2,
\end{aligned}
$$

when $E < E_0(n, \Lambda, \lambda)$, we have

$$
M_2 \geq (c\lambda^2\gamma^3 - C\gamma^2)R^2 \geq c\lambda^2\gamma^3 R^2,
$$

if $\gamma \geq \gamma_0(n, \Lambda, \lambda, M, E)$ large enough.

Estimate of $|\nabla(F - F_0)|$.
Similarly to (74),

$$
|\nabla(F - F_0)| \leq \frac{nE}{|x|}(6|\nabla^2\Phi| + 2|\nabla^3\Phi| + 16|\nabla\Phi|^2 + 4|\nabla^2\Phi|^2),
$$

then by (86) we have

$$
|\nabla(F - F_0)| \leq \frac{cnE}{|x|}|\nabla\Phi|^2 \leq cnE\gamma^2 R^{4/3}|x|^{-1/3}.
$$

Thus we complete the proof of Lemma 3.4.

4. Appendix

The properties of $\{a_\epsilon^{ij}\}$.

$a_\epsilon^{ij}(x,t) = \int_{\mathbb{R}^n} a^{ij}(x-y,t)\phi_\epsilon(y)dy$, where ϕ is a mollifier and $\epsilon = \frac{1}{2}$, then $\{a_\epsilon^{ij}\}$ satisfy:

i) $\lambda|\xi|^2 \le a_\epsilon^{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2,\ \forall \xi \in \mathbb{R}^n$;

ii) $|\nabla a_\epsilon^{ij}(x,t)| \le M$; $\quad |\nabla a_\epsilon^{ij}(x,t)| \le \dfrac{2E}{|x|}$ when $|x| \ge 1$;

iii) $|a_\epsilon^{ij}(x,t) - a^{ij}(x,t)| \le 2\Lambda$; $\quad |a_\epsilon^{ij}(x,t) - a^{ij}(x,t)| \le \dfrac{E}{|x|}$ when $|x| \ge 1$;

iv) $|\partial_{kl} a_\epsilon^{ij}(x,t)| \le c(n)M$; $\quad |\partial_{kl} a_\epsilon^{ij}(x,t)| \le \dfrac{c(n)E}{|x|}$ when $|x| \ge 1$.

Proof.

i) It is obvious.

ii)

$$|\nabla a_\epsilon^{ij}(x,t)| \le \int_{\mathbb{R}^n} |\nabla a^{ij}(x-y,t)|\phi_\epsilon(y)dy \le M \int_{\mathbb{R}^n} \phi_\epsilon(y)dy = M,$$

and when $|x| \ge 1$,

$$\begin{aligned} |\nabla a_\epsilon^{ij}(x,t)| &\le \int_{\mathbb{R}^n} |\nabla a^{ij}(x-y,t)|\phi_\epsilon(y)dy \\ &\le \int_{\mathbb{R}^n} \frac{E}{|x-y|}\phi_\epsilon(y)dy \\ &\le \int_{\mathbb{R}^n} \frac{E}{|x|-\frac{1}{2}}\phi_\epsilon(y)dy \le \frac{2E}{|x|}. \end{aligned}$$

iii) The first part is obvious. We only need to prove the second one.

$$\begin{aligned} |a_\epsilon^{ij}(x,t) - a^{ij}(x,t)| &\le \int_{\mathbb{R}^n} |a^{ij}(x-y,t) - a^{ij}(x,t)|\phi_\epsilon(y)dy \\ &\le \int_{\mathbb{R}^n} |\nabla a^{ij}(x-\theta y,t)||y|\phi_\epsilon(y)dy, \qquad (0<\theta<1) \end{aligned}$$

and when $|x| \ge 1$,

$$|a_\epsilon^{ij}(x,t) - a^{ij}(x,t)| \le \int_{\mathbb{R}^n} \frac{E}{2|x-\theta y|}\phi_\epsilon(y)dy \le \int_{\mathbb{R}^n} \frac{E}{2(|x|-\frac{1}{2})}\phi_\epsilon(y)dy \le \frac{E}{|x|}.$$

iv)

$$\begin{aligned}|\partial_{kl}a_\epsilon^{ij}(x,t)| &\leq \int\limits_{\mathbb{R}^n} |\partial_k a^{ij}(x-y,t)||\partial_l \phi_\epsilon(y)|dy \\ &\leq \epsilon^{-n-1}\int\limits_{\mathbb{R}^n} |\partial_k a^{ij}(x-y,t)||(\partial_l\phi)(\frac{y}{\epsilon})|dy \\ &\leq \frac{M}{\epsilon}||\partial_l\phi||_{L^1} \leq 2M||\nabla\phi||_{L^1},\end{aligned}$$

and when $|x| \geq 1$,

$$\begin{aligned}|\partial_{kl}a_\epsilon^{ij}(x,t)| &\leq \epsilon^{-n-1}\int\limits_{\mathbb{R}^n} |\partial_k a^{ij}(x-y,t)||(\partial_l\phi)(\frac{y}{\epsilon})|dy \\ &\leq \epsilon^{-n-1}\int\limits_{\mathbb{R}^n} \frac{E}{|x-y|}|(\partial_l\phi)(\frac{y}{\epsilon})|dy \\ &\leq \frac{2E}{\epsilon|x|}\|\partial_l\phi\|_{L^1} \leq \frac{4E\|\nabla\phi\|_{L^1}}{|x|}.\end{aligned}$$

Then we finished the proof.

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