Congruences on the number of restricted m-ary partitions

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Abstract. Andrews, Brietzke, Rødseth and Sellers proved an infinite family of congruences on the number of the restricted m-ary partitions when m is a prime. In this note, we show that these congruences hold for arbitrary positive integer m and thus confirm the conjecture of Andrews, et al.

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1 Introduction

Let $m \geq 2$ be a fixed integer. An m-ary partition of a nonnegative integer n is a partition of n such that each part is a power of m. If there is "no gaps" in the parts, i.e., whenever m^i is a part, $1, m, m^2, \ldots, m^{i-1}$ are parts, then the partition is called a restricted m-ary partition. The number of restricted m-ary partitions of n is denoted by $c_m(n)$. Notice that the generating function of $c_m(n)$ is given by

$$C_m(q) := \sum_{n=0}^{\infty} c_m(n) q^n = 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+\cdots+m^n}}{(1-q)(1-q^m)\cdots(1-q^{m^n})}.$$

In recent years, the arithmetic properties of m-ary partitions and restricted m-ary partitions have been widely studied since Churchhouse [5] initiated the study of 2-ary partitions in the late 1960s. For example, Rødseth [7] extended Churchhouse's results to include p-ary partition functions $b_p(n)$, where p is any prime. Andrews [1], Gupta [6] and Rødseth and Sellers [8] studied further the congruences for $b_m(n)$, where $m \geq 2$ is any positive integer. And later, Andrews, Fraenkel and Sellers [2,3] provided characterizations of the values $b_m(mn)$ and $c_m(mn)$ modulo m. Andrews, Brietzke, Rødseth and Sellers [4] proved that for odd prime m,

$$c_m(m^{j+2}n + m^{j+1} + \cdots + m^2) \equiv 0 \pmod{m^j},$$

for all $n \geq 0$ and $0 \leq j < m$. In this note, we will show that

Theorem 1.1 *For a fixed integer $m \geq 2$ and for all nonnegative integer n, we have*

$$c_m(m^{j+2}n + m^{j+1} + \cdots + m^2) \equiv 0 \pmod{\frac{m^j}{c_j}},$$

where $c_j = 1$ if m is odd and $c_j = 2^{j-1}$ if m is even.

This confirms the conjecture of Andrews et al. [4, Conjecture 4.1].

2 Proof of Theorem 1.1

Our proof is based on the following results and notations of Andrews et al. [4]. Suppose that

$$\binom{km+1}{j} = \sum_{i=0}^{j} s_{j,i} \binom{k}{i},$$

where $s_{j,i}$ is an integer independent of k such that $s_{j,0} = 0$ if $j > 1$ and $s_{j,0} = 1$ otherwise. And denote

$$h_j = \frac{q^j}{(1-q)^{j+1}}.$$

Lemma 2.1 (Lemma 3.9 of [4]) *Let $\overline{C_m}(q) := C_m(q) - 1$, then we have*

$$\sum_{n=0}^{\infty} c_m(m^{j+2}n + m^{j+1} + \cdots + m^2)q^n = \sum_{i=0}^{j+1}(P_{j,i} - Q_{j,i})h_i + \overline{C_m}(q) \sum_{i=1}^{j+1}(R_{j,i} - T_{j,i})h_i, \quad (2.1)$$

where, for $i > 0$, $P_{j,i}$ and $R_{j,i}$ are sums of monomials in $s_{u,v}$ of degree $j+1$ while $Q_{j,i}$ and $T_{j,i}$ are of degree j. In addition, $Q_{j,j+1} = Q_{j,0} = 0$ and $P_{j,0}$ is of degree j in the $s_{u,v}$. Furthermore, for $t > 0$, we have the following recurrence relations:

$$P_{j+1,t} = \sum_{i=t}^{j+2}(P_{j,i} + R_{j,i-1})s_{i,t}, \qquad R_{j+1,t} = \sum_{i=t}^{j+2} R_{j,i-1}s_{i,t}, \quad (2.2)$$

$$Q_{j+1,t} = \sum_{i=t}^{j+2}(Q_{j,i} + T_{j,i-1})s_{i,t}, \qquad T_{j+1,t} = \sum_{i=t}^{j+2} T_{j,i-1}s_{i,t}, \quad (2.3)$$

$$P_{j+1,0} = P_{j,1} = Q_{j+1,1}. \quad (2.4)$$

We firstly introduce an evaluation on monomials in $s_{u,v}$ by defining

$$v(s_{j,j}) = j, \quad v(s_{j,j-1}) = j-1-\varepsilon, \quad v(s_{j,i}) = 0, \ \forall\, 1 \le i < j-1,$$

and

$$v(s_{j,i}s_{j',i'}\cdots) = v(s_{j,i}) + v(s_{j',i'}) + \cdots,$$

where ε is an arbitrarily small positive real number.

Lemma 2.2 *Let $p = s_{j,i}s_{j',i'}\cdots$ be a monomial in $s_{u,v}$. If $v(p) > \ell$, then p is divisible by $\frac{m^{\ell+1}}{c_{\ell+2}}$.*

Proof. It is known that $s_{i,i} = m^i$. We will show that

$$s_{i+1,i} = \frac{i(m-1)}{2}m^i + m^i. \tag{2.5}$$

Recall that

$$\binom{km+1}{j} = \sum_{i=0}^{j} s_{j,i}\binom{k}{i}. \tag{2.6}$$

Notice that we have

$$\begin{aligned}\binom{km+1}{j} &= \frac{(km+1)km(km-1)\cdots(km-j+2)}{j!} \\ &= \frac{m^j k^j}{j!} - \frac{(j-3)}{2}\frac{m^{j-1}k^{j-1}}{(j-1)!} + \cdots,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\sum_{i=0}^{j} s_{j,i}\binom{k}{i} &= s_{j,j}\binom{k}{j} + s_{j,j-1}\binom{k}{j-1} + \cdots \\ &= s_{j,j}\frac{k^j}{j!} - s_{j,j}\frac{(j-1)}{2}\frac{k^{j-1}}{(j-1)!} + s_{j,j-1}\frac{k^{j-1}}{(j-1)!} + \cdots.\end{aligned} \tag{2.8}$$

By comparing the coefficients of k^{j-1} in (2.7) and (2.8), we get (2.5) immediately.

Now suppose that

$$p = s_{1,1}^{a_1}s_{2,2}^{a_2}\cdots s_{2,1}^{b_1}s_{3,2}^{b_2}\cdots s_{3,1}^{c_1}\cdots.$$

Let $\ell = \sum_i ia_i + \sum_j jb_j$. We have

$$\ell \ge v(p) = \sum_i ia_i + \sum_j (j-\varepsilon)b_j > \ell - 1.$$

If m is odd, p is divisible by m^ℓ. If m is even, p is divisible by

$$m^{\sum_i ia_i}\cdot\left(\frac{m}{2}\right)^{b_1}\cdot\left(\frac{m^2}{2}\right)^{b_2}\cdots = \frac{m^\ell}{2^{\sum_j b_j}}.$$

In both cases, we have $\frac{m^\ell}{c_{\ell+1}} \mid p$, which completes the proof. ∎

Now we consider the monomials in $P_{j,i}, Q_{j,i}, R_{j,i}$ and $T_{j,i}$. We will characterize the terms with minimal evaluations.

Lemma 2.3 *There exists a unique term with minimal evaluation in $P_{j,i}$ ($Q_{j,i}, R_{j,i}$ and $T_{j,i}$, respectively). Denote such a term by $\overline{P_{j,i}}$ ($\overline{Q_{j,i}}, \overline{R_{j,i}}, \overline{T_{j,i}}$, respectively). We have*

(1) *for* $1 \le i \le j+1$,
$$\overline{P_{j,i}} = \overline{R_{j,i}}, \quad v(\overline{P_{j,i+1}}) - v(\overline{P_{j,i}}) > i; \tag{2.9}$$

(2) *for* $1 \le i \le j$,
$$\overline{Q_{j,i}} = \overline{T_{j,i}}, \quad v(\overline{Q_{j,i+1}}) - v(\overline{Q_{j,i}}) > i. \tag{2.10}$$

Proof. We will show that (2.9) and (2.10) hold by induction on j. The uniqueness of the minimal term (i.e., the term with minimal evaluation) follows from the proof.

(1) When $j = 1$, by Lemma 3.7 of [4], we have
$$\overline{P_{1,1}} = \overline{R_{1,1}} = s_{1,1}s_{2,1}, \quad \overline{P_{1,2}} = \overline{R_{1,2}} = s_{1,1}s_{2,2}.$$
Thus (2.9) holds by straightforward checking.

Now we assume that (2.9) holds for positive integer j and we seek for the minimal term in $P_{j+1,i}$. From the recurrence relation (2.2), we see that for $1 \le i \le j+2$,
$$P_{j+1,i} = P_{j,i}s_{i,i} + P_{j,i+1}s_{i+1,i} + \cdots + P_{j,j+1}s_{j+1,i} + R_{j,i-1}s_{i,i} + R_{j,i}s_{i+1,i} + \cdots + R_{j,j+1}s_{j+2,i}, \tag{2.11}$$
and
$$R_{j+1,i} = R_{j,i-1}s_{i,i} + R_{j,i}s_{i+1,i} + \cdots + R_{j,j+1}s_{j+2,i}. \tag{2.12}$$
By the induction hypothesis, we have that for $i+1 \le k \le j+1$,
$$v(\overline{P_{j,k}}) - v(\overline{P_{j,i}}s_{i,i}) = v(\overline{P_{j,k}}) - v(\overline{P_{j,i}}) - i > 0,$$
and
$$v(\overline{R_{j,k}}) - v(\overline{R_{j,i}}s_{i+1,i}) = v(\overline{R_{j,k}}) - v(\overline{R_{j,i}}) - i + \varepsilon > 0.$$
Therefore, the minimal term must lie in $P_{j,i}s_{i,i}, R_{j,i-1}s_{i,i}$ or $R_{j,i}s_{i+1,i}$. If $i > 1$, then
$$v(\overline{P_{j,i}}s_{i,i}) > v(\overline{R_{j,i}}s_{i+1,i}) > v(\overline{R_{j,i-1}}s_{i,i}) + i - 1 - \varepsilon > v(\overline{R_{j,i-1}}s_{i,i}).$$
If $i = 1$, then
$$v(\overline{P_{j,i}}s_{i,i}) > v(\overline{R_{j,i}}s_{i+1,i}).$$
Hence
$$\overline{P_{j+1,i}} = \overline{R_{j+1,i}} = \begin{cases} \overline{R_{j,i-1}}s_{i,i}, & \text{if} \quad 2 \le i \le j+2, \\ \overline{R_{j,1}}s_{2,1}, & \text{if} \quad i = 1. \end{cases} \tag{2.13}$$

This implies that for $i > 1$,

$$\begin{aligned} v(\overline{P_{j+1,i+1}}) - v(\overline{P_{j+1,i}}) &= v(\overline{R_{j,i}}s_{i+1,i+1}) - v(\overline{R_{j,i-1}}s_{i,i}) \\ &= v(\overline{R_{j,i}}) - v(\overline{R_{j,i-1}}) + 1 \\ &> i - 1 + 1 \\ &= i, \end{aligned}$$

and for $i = 1$,

$$\begin{aligned} v(\overline{P_{j+1,2}}) - v(\overline{P_{j+1,1}}) &= v(\overline{R_{j,1}}s_{2,2}) - v(\overline{R_{j,1}}s_{2,1}) \\ &= v(s_{2,2}) - v(s_{2,1}) \\ &= 2 - (1 - \varepsilon) \\ &> 1. \end{aligned}$$

Thus (2.9) follows by mathematical induction.

(2) By a discussion similar to (1), we obtain (2.10). Moreover, we have the following recurrence relation:

$$\overline{Q_{j+1,i}} = \overline{T_{j+1,i}} = \begin{cases} \overline{T_{j,i-1}}s_{i,i}, & \text{if} \quad 2 \le i \le j+2, \\ \overline{T_{j,1}}s_{2,1}, & \text{if} \quad i = 1. \end{cases} \tag{2.14}$$

This completes the proof. ∎

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. By iterative use of Equation (2.13), we derive that for $1 \le i \le j+1$,

$$\overline{P_{j,i}} = \overline{R_{j,i}} = \begin{cases} s_{1,1}s_{2,2}\cdots s_{i,i}s_{2,1}^{j-i+1}, & \text{if} \quad 2 \le i \le j+1, \\ s_{1,1}s_{2,1}^{j}, & \text{if} \quad i = 1. \end{cases} \tag{2.15}$$

Similarly, by Equation (2.14), we deduce that for $1 \le i \le j+1$,

$$\overline{Q_{j,i}} = \overline{T_{j,i}} = \begin{cases} s_{1,1}s_{2,2}\cdots s_{i,i}s_{2,1}^{j-i}, & \text{if} \quad 2 \le i \le j+1, \\ s_{1,1}s_{2,1}^{j-1}, & \text{if} \quad i = 1. \end{cases} \tag{2.16}$$

It is easy to see that $v(\overline{Q_{j,i}}) < v(\overline{P_{j,i}})$ and for $2 \le i \le j+1$,

$$v(s_{1,1}s_{2,2}\cdots s_{i,i}s_{2,1}^{j-i}) - v(s_{1,1}s_{2,1}^{j-1}) = \frac{i(i-1)}{2} + \varepsilon(i-1) > 0.$$

Hence among all monomials in $P_{j,i}, Q_{j,i}, R_{j,i}$ and $T_{j,i}$, the minimal term is $s_{1,1}s_{2,1}^{j-1}$. Let p be a monomial in $P_{j,i}, Q_{j,i}, R_{j,i}$ or $T_{j,i}$. By induction we see that $s_{1,1}$ is a factor of p. Write $p = s_{1,1}p'$. Then

$$v(p') \geq v(s_{2,1}^{j-1}) > j - 2.$$

By Lemma 2.2 we derive that

$$\frac{m^{j-1}}{c_j} \mid p',$$

and hence $\frac{m^j}{c_j} \mid p$. Theorem 1.1 follows immediately from the generating function (2.1). ∎

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References

[1] G.E. Andrews, Congruence properties of the m-ary partition function, J. Number Theory, 3 (1971), 104–110.

[2] G.E. Andrews, A.S. Fraenkel and J.A. Sellers, Characterizing the number of m-ary partitions modulo m, Amer. Math. Monthly, 122 (2015), 880–885.

[3] G.E. Andrews, A.S. Fraenkel and J.A. Sellers, m-ary partitions with no gaps: A characterization modulo m, Discrete Math., 339 (2016), 283–287.

[4] G.E. Andrews, E. Brietzke, Ø.J. Rødseth and J.A. Sellers, Arithmetic properties of m-ary partitions without gaps, to appear in the Annal of Combinatorics.

[5] R.F. Churchhouse, Congruence properties of the binary partition function, Math. Proc. Cambridge Philos. Soc., 66 (1969), 371–376.

[6] H. Gupta, On m-ary partitions, Math. Proc. Cambridge Philos. Soc., 71 (1972), 343–345.

[7] Ø. Rødseth, Some arithmetical properties of m-ary partitions, Math. Proc. Cambridge Philos. Soc., 68 (1970), 447–453.

[8] Ø. Rødseth and J.A. Sellers, On m-ary partition function congruences: A fresh look at a past problem, J. Number Theory, 87 (2001), 270–281.