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HEAVY SUBGRAPHS, STABILITY AND HAMILTONICITY

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Abstract

Let G be a graph. Adopting the terminology of Broersma et al. and Čada, respectively, we say that G is 2-heavy if every induced claw ($K_{1,3}$) of G contains two end-vertices each one has degree at least $|V(G)|/2$; and G is o-heavy if every induced claw of G contains two end-vertices with degree sum at least $|V(G)|$ in G. In this paper, we introduce a new concept, and say that G is *S-c-heavy* if for a given graph S and every induced subgraph G' of G isomorphic to S and every maximal clique C of G', every non-trivial component of $G' - C$ contains a vertex of degree at least $|V(G)|/2$ in G. In terms of this concept, our original motivation that a theorem of Hu in 1999 can be stated as every 2-connected 2-heavy and N-c-heavy graph is hamiltonian, where N is the graph obtained from a triangle by adding three disjoint pendant edges. In this paper, we will characterize all connected graphs S such that every 2-connected o-heavy and S-c-heavy graph is hamiltonian. Our work results in a different proof of a stronger version of Hu's theorem. Furthermore, our main result improves or extends several previous results.

Keywords: heavy subgraphs, hamiltonian graphs, closure theory.

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1. Introduction

Throughout this paper, the graphs considered are undirected, finite and simple (without loops and parallel edges). For terminology and definition not defined here, we refer the reader to Bondy and Murty [4].

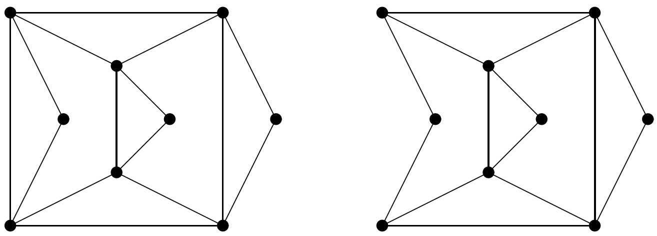
Let G be a graph and v be a vertex of G. The *neighborhood* of v in G, denoted by $N_G(v)$, is the set of neighbors of v in G; and the *degree* of v in G, denoted by $d_G(v)$, is the number of neighbors of v in G. For two vertices $u, v \in V(G)$, the *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. When there is no danger of ambiguity, we use $N(v)$, $d(v)$ and $d(u, v)$ instead of $N_G(v)$, $d_G(v)$ and $d_G(u, v)$, respectively. For a subset U of $V(G)$, we set $N_U(v) = N(v) \cap U$, and $d_U(v) = |N_U(v)|$. For a subgraph S of G such that $v \notin V(S)$, we use $N_S(v)$ and $d_S(v)$ instead of $N_{V(S)}(v)$ and $d_{V(S)}(v)$, respectively.

Let G be a graph and G' be a subgraph of G. If G' contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is an *induced subgraph* of G (or a subgraph *induced* by $V(G')$). For a given graph S, the graph G is *S-free* if G contains no induced subgraph isomorphic to S. Note that if S_1 is an induced subgraph of S_2, then an S_1-free graph is also S_2-free.

The bipartite graph $K_{1,3}$ is the *claw*. We use P_i $(i \geq 1)$ and C_i $(i \geq 3)$ to denote the path and cycle of order i, respectively. We denote by Z_i $(i \geq 1)$ the graph obtained by identifying a vertex of a C_3 with an end-vertex of a P_{i+1}; by $B_{i,j}$ $(i, j \geq 1)$ the graph obtained by identifying two vertices of a C_3 with the origins of a P_{i+1} and a P_{j+1}, respectively; and by $N_{i,j,k}$ $(i, j, k \geq 1)$ the graph obtained by identifying the three vertices of a C_3 with the origins of a P_{i+1}, a P_{j+1} and a P_{k+1}, respectively. In particular, we set $B = B_{1,1}$, $N = N_{1,1,1}$, and $W = B_{1,2}$. (These three graphs are sometimes called the *bull*, the *net* and the *wounded*, respectively.)

To find sufficient conditions for hamiltonicity of graphs is a standard topic. In particular, sufficient conditions for hamiltonicity of graphs in terms of forbidden subgraphs have received much attention from graph theorists. The following are some results in this area, where the graphs L_1 and L_2 are shown in Figure 1.

Theorem 1. *Let G be a 2-connected graph.*
(1) ([12]) If G is claw-free and N-free, then G is hamiltonian.
(2) ([6]) If G is claw-free and P_6-free, then G is hamiltonian.
(3) ([1]) If G is claw-free and W-free, then G is hamiltonian.
(4) ([15]) If G is claw-free and Z_3-free, then G is hamiltonian or $G = L_1$ or L_2.



L_1 L_2

Figure 1. Graphs L_1 and L_2.

In 1991, Bedrossian [1] characterized all pairs of forbidden subgraphs for a 2-connected graph to be hamiltonian, in his Ph.D. Thesis. In 1997, Faudree and Gould [14] extended Bedrossian's result by proving the 'only if' part based on infinite families of non-hamiltonian graphs. Before showing the result of Faudree and Gould, we first remark that the only connected graph S of order at least 3 such that the statement 'every 2-connected S-free graph is hamiltonian' holds, is P_3, see [14]. So in the following theorem, we only consider the forbidden pairs excluding P_3.

Theorem 2 [14]**.** *Let R, S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph of order $n \geq 10$. Then G being R-free and S-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, Z_3, B,$ N or W.*

Degree condition is also an important type of sufficient conditions for hamiltonicity of graphs. Let G be a graph of order n. A vertex $v \in V(G)$ is a *heavy vertex* of G if $d(v) \geq n/2$; and a pair of vertices $\{u, v\}$ is a *heavy pair* of G if $uv \notin E(G)$ and $d(u) + d(v) \geq n$. In 1952, Dirac [11] proved that every graph G of order at least 3 is hamiltonian if every vertex of G is heavy. Ore [22] improved Dirac's result by showing that every graph G of order at least 3 is hamiltonian if every pair of nonadjacent vertices is a heavy pair. Fan [13] further improved Ore's theorem by showing that every 2-connected graph G is hamiltonian if every pair of vertices at distance 2 of G contains a heavy vertex.

It is natural to relax the forbidden subgraph conditions to ones that the subgraphs are allowed, but some degree conditions are restricted to the subgraphs. Early examples of this method used in scientific papers can date back to 1990s [2, 19, 5]. In particular, Čada [10] introduced the class of o-heavy graphs by restricting Ore's condition to every induced claw of a graph. Li et al. [18] extended Čada's concept of claw-o-heavy graphs to a general one.

Let G' be an induced subgraph of G. Following [18], if G' contains a heavy pair of G, then G' is an *o-heavy subgraph* of G (or G' is *o-heavy* in G). For a given graph S, the graph G is *S-o-heavy* if every induced subgraph of G isomorphic to S is o-heavy. (It should be mentioned that Čada originally named claw-o-heavy graphs as o-heavy graphs in [10].) Note that an S-free graph is trivially S-o-heavy, and if S_1 is an induced subgraph of S_2, then an S_1-o-heavy graph is also S_2-o-heavy.

Li et al. [18] completely characterized pairs of o-heavy subgraphs for a 2-connected graph to be hamiltonian, which extends Theorem 2. The main result in [18] is given as follows.

Theorem 3 [18]**.** *Let R and S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being R-o-heavy and S-o-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ or W.*

Following [20], we introduce another type of heavy subgraph condition motivated by Fan's condition [13]. Let G be a graph and G' be an induced subgraph of G. If for each two vertices $u, v \in V(G')$ with $d_{G'}(u, v) = 2$, either u or v is heavy in G, then G' is an *f-heavy subgraph* of G (or G' is *f-heavy* in G). For a given graph S, the graph G is *S-f-heavy* if every induced subgraph of G isomorphic to S is f-heavy. A claw-f-heavy graph is also called a *2-heavy* graph (see [5]).

Note that an S-free graph is trivially S-f-heavy, but in general, an S_1-f-heavy graph is not necessarily S_2-f-heavy when S_1 is an induced subgraph of S_2. In Figure 2, we show the implication relations among the conditions being S-f-heavy for the graphs S listed in Theorem 2.

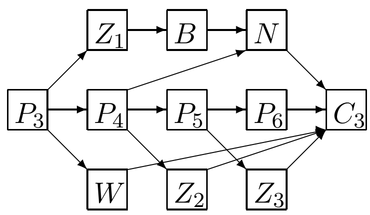


Figure 2. $S_1 \to S_2$: Being S_1-f-heavy implies being S_2-f-heavy

We remark that f-heavy conditions cannot compare with o-heavy conditions in general. For example, every P_3-o-heavy graph is P_3-f-heavy; and every claw-f-heavy graph is claw-o-heavy, but for the conditions being N-o-heavy and being N-f-heavy, no one can imply the other.

Motivated by Theorem 3, Ning and Zhang [20] characterized pairs of f-heavy subgraphs for a 2-connected graph to be hamiltonian, which not only is a new extension of Theorem 2 but also unifies some previous theorems in [2, 9, 19].

Theorem 4 [20]**.** *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph of order $n \geq 10$. Then G being R-f-heavy and S-f-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W.*

Now we will put our views to another new sufficient condition for hamiltonicity of graphs due to Hu [17]. Some previous theorems can be obtained from Hu's theorem as corollaries (see [2, 19]).

Theorem 5 [17]**.** *Let G be a 2-connected graph. If G is 2-heavy and every induced P_4 in an induced N of G contains a heavy vertex, then G is hamiltonian.*

In fact, we can see that the cases $S = Z_1, B, N$ in Theorem 4 can be deduced from Hu's theorem. This motivates us to consider the counterpart results for other subgraphs. Armed with this idea, we first propose the following definition.

Definition 1. Let G be a graph and G' be an induced subgraph of G. If for every maximal clique C of G', each nontrivial component of $G' - C$ contains a heavy vertex of G, then G' is a *clique-heavy* (or in short, *c-heavy*) subgraph of G. For a given graph S, G is *S-c-heavy* if every induced subgraph of G isomorphic to S is c-heavy.

In Figure 3, we show the implication relations of the conditions being S-c-heavy for the graphs S listed in Theorem 2.

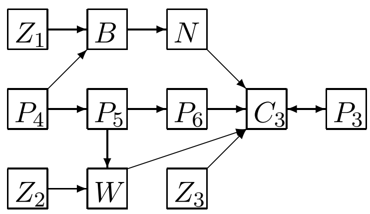


Figure 3. $S_1 \to S_2$: Being S_1-c-heavy implies being S_2-c-heavy.

So Theorem 5 can be stated as every 2-connected claw-f-heavy and N-c-heavy graph is hamiltonian. As we will show below, this can be extended to that every 2-connected claw-o-heavy and N-c-heavy graph is hamiltonian.

We remark that saying a graph is claw-c-heavy is meaningless (if we remove a maximal clique from a claw, then only isolated vertices remain). Motivated by Theorems 2, 3 and 4, we naturally propose the following problem.

Problem 1. Which connected graphs S imply that every 2-connected claw-free (or claw-f-heavy or claw-o-heavy) and S-c-heavy graph is hamiltonian?

The solution to Problem 1 is one of the main results in this paper.

Theorem 6. *Let S be a connected graph of order at least 3 and let G be a 2-connected claw-o-heavy graph of order $n \geq 10$. Then G being S-c-heavy implies G is hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W.*

Note that the only subgraphs appearing in Theorem 2 but missed here are P_3 and C_3. Also note that every graph is P_3-c-heavy and C_3-c-heavy and there exist 2-connected claw-free graphs which are non-hamiltonian. By Theorem 2 and the fact that every claw-free (claw-f-heavy) graph is claw-o-heavy, we can see that Theorem 6 gives a complete solution to Problem 1.

We point out that a special case of our work results in a new proof of a stronger version of Theorem 5.

Theorem 7. *Let G be a 2-connected graph. If G is claw-o-heavy and N-c-heavy, then G is hamiltonian.*

Some previous theorems can also be obtained from this theorem as corollaries in a unified way.

Corollary 1 [17]**.** *Let G be a graph. If G is claw-f-heavy and N-c-heavy, then G is hamiltonian.*

Corollary 2 [20]**.** *Let G be a graph. If G is claw-o-heavy and N-f-heavy, then G is hamiltonian.*

Corollary 3 [19]**.** *Let G be a graph. If G is claw-f-heavy and B-f-heavy, then G is hamiltonian.*

Corollary 4 [2]**.** *Let G be a graph. If G is claw-f-heavy and Z_1-f-heavy, then G is hamiltonian.*

We remark that our methods used here are completely different from the ones in [17, 18, 20]. We mainly use the claw-o-heavy closure theory introduced by Čada [10], and many other results from the area of forbidden subgraphs. However, our technique here is new, and it is heavily dependent on some new concepts and tools developed by us recently. (See Lemma 7 in Sec.2 for example.) We point out that this is the first time to deal with Hamiltonicity of graphs under pairs of heavy subgraph conditions by using c-Closure theory systemically, compared with several previous works in [2, 19, 17, 9, 18, 20, 21].

The rest of this paper is organized as follows. In Section 2, we will present necessary and additional preliminaries (including the introduction to claw-free closure theory, claw-o-heavy closure theory and a useful theorem of Brousek). In Section 3, in the spirit of some previous works of Brousek et al. [8], we will study the stability of some subclasses of the class of claw-o-heavy graphs. In Section 4, by using the closure theory and a previous result of Brousek [7], we give the proof of Theorem 6. In Section 5, one useful remark is given to conclude this paper.

2. Preliminaries

The main tools in our paper are two kinds of closure theories introduced by Ryjáček [23] and Čada [10], respectively. These two closure theories are used to study hamiltonian properties of claw-free graphs and claw-o-heavy graphs, respectively. We will give some terminology and notation with a prefix or superscript r or c, respectively, to distinguish them.

r-Closure theory.

Let G be a claw-free graph and x be a vertex of G. Following [23], we call x an *r-eligible vertex* of G if $N(x)$ induces a connected graph in G but not a complete graph. The *completion of* G *at* x, denoted by G'_x, is the graph obtained from G by adding all missing edges uv with $u, v \in N(x)$.

Lemma 1 [23]**.** *Let G be a claw-free graph and x be an r-eligible vertex of G. Then*
(1) the graph G'_x is claw-free; and
(2) the circumferences of G'_x and G are equal.

The *r-closure* of a claw-free graph G, denoted by $\mathrm{cl}^{\mathrm{r}}(G)$, is defined by a sequence of graphs $G_1, G_2, \ldots, G_t$, and vertices $x_1, x_2 \ldots, x_{t-1}$ such that
(1) $G_1 = G$, $G_t = \mathrm{cl}^{\mathrm{r}}(G)$;
(2) x_i is an r-eligible vertex of G_i, $G_{i+1} = (G_i)'_{x_i}$, $1 \leq i \leq t-1$; and
(3) $\mathrm{cl}^{\mathrm{r}}(G)$ has no r-eligible vertices.

A claw-free graph G is *r-closed* if G has no r-eligible vertices, i.e., if $\mathrm{cl}^{\mathrm{r}}(G) = G$.

Theorem 8 [23]**.** *Let G be a claw-free graph. Then*
(1) the r-closure $\mathrm{cl}^{\mathrm{r}}(G)$ *is well defined;*
(2) there is a C_3-free graph H such that $\mathrm{cl}^{\mathrm{r}}(G)$ *is the line graph of H; and*
(3) the circumferences of $\mathrm{cl}^{\mathrm{r}}(G)$ *and G are equal.*

It is not difficult to get the following (see [8]).

Lemma 2 [8]**.** *Let G be a claw-free graph. Then* $\mathrm{cl}^{\mathrm{r}}(G)$ *is a $K_{1,1,2}$-free supergraph of G with the least number of edges.*

Following [8], we say a family $\mathcal{G}$ of graphs is *stable under the r-closure* (or shortly, r-stable) if for every graph in $\mathcal{G}$, its r-closure is also in $\mathcal{G}$. From Theorem 8, we can see that the class of all claw-free hamiltonian graphs and the class of all claw-free non-hamiltonian graphs are r-stable.

c-Closure theory.

Let G be a claw-o-heavy graph and let $x \in V(G)$. Let G' be the graph obtained from G by adding the missing edges uv with $u, v \in N(x)$ and $\{u, v\}$ is a heavy pair of G. We call x a *c-eligible* vertex of G if $N(x)$ is not a clique of G and one of the following is true:
(1) $G'[N(x)]$ is connected; or
(2) $G'[N(x)]$ consists of two disjoint cliques C_1 and C_2, and x is contained in a heavy pair $\{x, z\}$ of G such that $zy_1, zy_2 \in E(G)$ for some $y_1 \in C_1$ and $y_2 \in C_2$.
Note that if G is claw-free, then an r-eligible vertex is also c-eligible.

Lemma 3 [10]**.** *Let G be a claw-o-heavy graph and x be a c-eligible vertex of G. Then*
(1) for every vertex $y \in N(x)$, $d_{G'_x}(y) \geq d_{G'_x}(x)$;
(2) the graph G'_x is claw-o-heavy; and
(3) the circumferences of G'_x and G are equal.

The *c-closure* of a claw-o-heavy graph G, denoted by $\mathrm{cl}^{\mathrm{c}}(G)$, is defined by a sequence of graphs $G_1, G_2, \ldots, G_t$, and vertices $x_1, x_2 \ldots, x_{t-1}$ such that
(1) $G_1 = G$, $G_t = \mathrm{cl}^{\mathrm{c}}(G)$;
(2) x_i is a c-eligible vertex of G_i, $G_{i+1} = (G_i)'_{x_i}$, $1 \leq i \leq t-1$; and
(3) $\mathrm{cl}^{\mathrm{c}}(G)$ has no c-eligible vertices.

Theorem 9 [10]**.** *Let G be a claw-o-heavy graph. Then*
(1) the c-closure $\mathrm{cl}^{\mathrm{c}}(G)$ is well defined;
(2) there is a C_3-free graph H such that $\mathrm{cl}^{\mathrm{c}}(G)$ is the line graph of H; and
(3) the circumferences of $\mathrm{cl}^{\mathrm{c}}(G)$ and G are equal.

A claw-o-heavy graph G is *c-closed* if $\mathrm{cl}^{\mathrm{c}}(G) = G$. Note that every line graph is claw-free (see [3]). This implies that $\mathrm{cl}^{\mathrm{c}}(G)$ is a claw-free graph. Also note that for a claw-free graph, an r-eligible vertex is also c-eligible. This implies that every c-closed graph is also r-closed.

Similarly as the case of r-closure, we say a family $\mathcal{G}$ of graphs is stable under the c-closure (or shortly, c-stable) if for every graph in $\mathcal{G}$, its c-closure is also in $\mathcal{G}$.

The following lemma is an obvious but important fact, which can be deduced from Lemma 14 in [10] easily.

Lemma 4 [10]**.** *Let G be a claw-o-heavy graph. Then $\mathrm{cl}^{\mathrm{c}}(G)$ has no heavy pair.*

Here we list some new concepts introduced by us recently [21]. Let G be a claw-o-heavy graph and C be a maximal clique of $\mathrm{cl}^{\mathrm{c}}(G)$. We call $G[C]$ a *region* of G. For a vertex v of G, we call v an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions.

A graph G is nonseparable if it is connected and has no cut-vertex (i.e., either G is 2-connected, or $G = K_1$ or K_2). The following useful lemma originally appeared as Lemma 2 in [21], and it plays the crucial role of our proofs.

Lemma 5 [21]**.** *Let G be a claw-o-heavy graph and R be a region of G. Then*
(1) R is nonseparable;
(2) if v is a frontier vertex of R, then v has an interior neighbor in R or R is complete and has no interior vertices; and
(3) for any two vertices $u, v \in R$, there is an induced path of G from u to v such that every internal vertex of the path is an interior vertex of R.

Following [7], we define $\mathcal{P}$ to be the class of graphs obtained from two vertex-disjoint triangles $a_1a_2a_3a_1$ and $b_1b_2b_3b_1$ by joining every pair of vertices $\{a_i, b_i\}$ by a path P_{k_i}, where $k_i \geq 3$ or by a triangle. We use P_{x_1,x_2,x_3} to denote the graph in $\mathcal{P}$, where $x_i = k_i$ if a_i and b_i are joined by a path P_{k_i}, and $x_i = T$ if a_i and b_i are joined by a triangle. Note that $L_1 = P_{T,T,T}$ and $L_2 = P_{3,T,T}$.

We give the following useful result to finish this section.

Theorem 10 [7]**.** *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph $G' \in \mathcal{P}$.*

3. STABLE CLASSES UNDER CLOSURE OPERATION

Brousek et al. [8] studied the graphs S such that the class of claw-free and S-free graphs is r-stable. Before we present their result, we first remark that if S contains an induced claw or an induced $K_{1,1,2}$, then the class of claw-free and S-free graphs is trivially r-stable by Lemma 2. So in the following theorem we assume that S is claw-free and $K_{1,1,2}$-free.

Theorem 11 [8]**.** *Let S be a connected claw-free and $K_{1,1,2}$-free graph of order at least 3. Then the class of claw-free and S-free graphs is r-stable, if and only if*

$$S \in \{C_3, H\} \cup \{P_i : i \geq 3\} \cup \{Z_i : i \geq 1\} \cup \{N_{i,j,k} : i, j, k \geq 1\}.$$

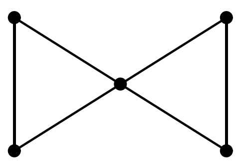


Figure 4. Graph H (hourglass).

In the spirit of previous works of Brousek et al. [8], we will consider the c-stability of the class of claw-o-heavy and S-c-heavy graphs. Before showing our results about this topic, we first remark the following trivial facts:

If S is the join of a complete graph and an empty graph (specially, if S is a complete graph or a star), then for every maximal clique C of S, $S-C$ has only trivial components. Thus by our definition, every graph will be S-c-heavy. Moreover, by our definition of c-stability, the class of claw-o-heavy and S-c-heavy graphs is c-stable. In the following, we will characterize all the other graphs S such that the class of claw-o-heavy and S-c-heavy graphs is c-stable.

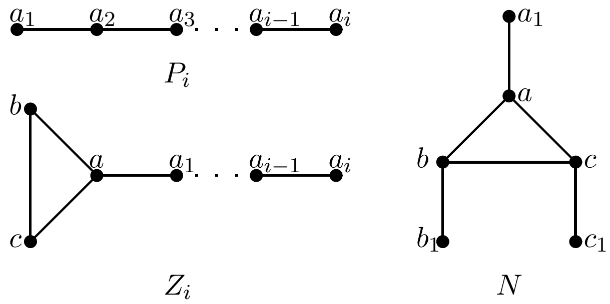


Figure 5. Graphs P_i, Z_i and N.

For a vertex x of a graph G, we set $B_G(x) = \{uv : u, v \in N(x) \text{ and } uv \notin E(G)\}$. For convenience, we say a vertex or a pair of nonadjacent vertices is *light* if it is not heavy.

Theorem 12. *Let G be a claw-o-heavy and P_i-c-heavy graph, $i \geq 4$, and x be a c-eligible vertex of G. Then G'_x is P_i-c-heavy.*

Proof. Let P be an induced P_i of G'_x. We denote the vertices of P as in Figure 5, and will prove that one vertex of $\{a_1, a_2\}$ is heavy in G'_x and one vertex of $\{a_{i-1}, a_i\}$ is heavy in G'_x. Note that $d_{G'_x}(v) \geq d(v)$ for every vertex $v \in V(G)$. If P is also an induced subgraph of G, then P is c-heavy in G, and then, is c-heavy in G'_x. So we assume that P is not an induced subgraph of G, which implies that $E(P) \cap B_G(x) \neq \emptyset$. Suppose that $a_j a_{j+1}$ is an edge in $E(P) \cap B_G(x)$, where $1 \leq j \leq i-1$.

Since $N(x)$ is a clique in G'_x, $N(x) \cap V(P) = \{a_j, a_{j+1}\}$ and there is only one edge in $E(P) \cap B_G(x)$. If $j \geq 2$, then $P' = a_1 a_2 \cdots a_j x a_{j+1} \cdots a_{i-1}$ is an induced P_i of G. Since G is P_i-c-heavy, one vertex of $\{a_1, a_2\}$ is heavy in G, and then, is heavy in G'_x. If $j = 1$, then $P' = a_1 x a_2 \cdots a_{i-1}$ is an induced P_i of G. Thus one vertex of $\{a_1, x\}$ is heavy in G. Note that $d_{G'_x}(a_1) \geq d_{G'_x}(x) = d(x)$ (see Lemma 3). Thus a_1 is heavy in G'_x. Hence in any case, we have shown that one vertex of $\{a_1, a_2\}$ is heavy in G'_x. By the symmetry, we can prove that one vertex of $\{a_{i-1}, a_i\}$ is heavy in G'_x. ■

Note that every c-closed graph has no heavy pairs, and note that every c-heavy P_i with $i \geq 5$ must have a heavy pair. By Theorem 12, we have

Corollary 5. *Let G be a claw-o-heavy and P_i-c-heavy graph with $i \geq 5$. Then* $\mathrm{cl}^{\mathrm{c}}(G)$ *is P_i-free.*

Corollary 6. *For $i \geq 3$, the class of claw-o-heavy and P_i-c-heavy graphs is c-stable.*

There are no counterpart results of Theorem 12 for the graph Z_i. In fact, there exist claw-free and Z_i-free graphs G with an r-eligible vertex x such that G'_x is not Z_i-free, see [8]. However, we can prove that the class of claw-o-heavy and Z_i-c-heavy graphs is also c-stable for $i \neq 2$.

Theorem 13. *Let G be a claw-o-heavy and Z_1-c-heavy graph. Then* $\mathrm{cl}^{\mathrm{c}}(G)$ *is also Z_1-c-heavy.*

Proof. Let Z be an induced Z_1 in $\mathrm{cl}^{\mathrm{c}}(G)$. We denote the vertices of Z as in Figure 5. We will prove that either b or c is heavy.

Claim 1. Let R be a region of G and $x \in V(R)$ be a frontier vertex. If y, y' are two neighbors of x in R, then one vertex in $\{y, y'\}$ is heavy in G.

Proof. Let z be a neighbor of x in $G-R$. Clearly $yz, y'z \notin E(G)$. If $yy' \in E(G)$, then the subgraph of G induced by $\{x, y, y', z\}$ is a Z_1. Since G is Z_1-c-heavy, either y or y' is heavy in G. Now we assume that $yy' \notin E(G)$. Then the subgraph of G induced by $\{x, y, y', z\}$ is a claw. Note that $\{y, z\}$ and $\{y', z\}$ are not heavy pairs in $\mathrm{cl}^{\mathrm{c}}(G)$, and then, are not heavy pairs in G. This implies that $\{y, y'\}$ is a heavy pair of G. Thus either y or y' is heavy in G. □

Suppose that both b and c are light. Let R be the region of G containing $\{a, b, c\}$. Note that R is a clique in $\mathrm{cl}^{\mathrm{c}}(G)$. If $|V(R)| \geq |V(G)|/2 + 1$, then b is heavy in $\mathrm{cl}^{\mathrm{c}}(G)$, a contradiction. So we assume that $|V(R)| \leq (|V(G)| + 1)/2$. This implies that every interior vertex of R is light in $\mathrm{cl}^{\mathrm{c}}(G)$, and also, light in G.

If R has no interior vertex, then by Lemma 5, R is a clique in G. By Claim 1, either b or c is heavy in G, a contradiction. So we assume that R has an interior vertex. By Lemma 5, R has an interior vertex adjacent to a. Since a has at least two neighbors in R, we may choose two neighbors x, y of a in R such that x is an interior vertex of R. Note that x is light in G. By Claim 1, y is heavy in G. Recall that b, c and every interior vertex of R are light. Hence $y \neq b, c$ and y is a frontier vertex of R.

If both by and cy are in $E(G)$, then by Claim 1, either b or c is heavy in G, a contradiction. So we conclude that $by \notin E(G)$ or $cy \notin E(G)$.

If $d_{G-R}(y)=1$, then $d(y)=d_R(y)+1\leq |V(R)|-2+1\leq (n-1)/2$. Hence y is light in G, a contradiction. So we conclude that $d_{G-R}(y)\geq 2$. Also note that $d_R(y)\geq 2$ by Lemma 5. Let x',x'' be two vertices in $N_R(y)$ and y',y'' be two vertices in $N_{G-R}(y)$. By Claim 1, one vertex of $\{x',x''\}$ is heavy in G, and one vertex of $\{y',y''\}$ is heavy in G. We assume without loss of generality that x',y' are heavy in G. Then $\{x',y'\}$ is a heavy pair in G, and also is a heavy pair of $\mathrm{cl^c}(G)$, a contradiction. ■

Theorem 14. *Let G be a claw-o-heavy and Z_i-c-heavy graph with $i\geq 3$. Then $\mathrm{cl^c}(G)$ is Z_i-free.*

Proof. The proof is almost the same as the proof of Lemma 3 in [21]. The only difference occurs when we find an induced Z_i in $\mathrm{cl^c}(G)$, instead of a Z_3 as done in the proof of Lemma 3 in [21], and when we use the c-heavy condition, instead of the f-heavy condition. But we still shall carry it in full, due to some specific details and the integrity of this paper. Now we give the proof along the outline in [21] step by step.

Suppose the contrary. Let Z be an induced Z_i in $\mathrm{cl^c}(G)$. We denote the vertices of Z as in Figure 5. Let R be the region of G containing $\{a,b,c\}$. Proofs of the first two claims are almost the same as Claims 1, 2 in the proof of Lemma 3 in [21].

Claim 1. [21, Claim 1 in the proof of Lemma 3]
$|N_R(a_2)\cup N_R(a_3)|\leq 1$.

Proof. Note that every vertex in $G-R$ has at most one neighbor in R. If $N_R(a_2)=\emptyset$, then the assertion is obviously true. Now we assume that $N_R(a_2)\neq\emptyset$. Let x be the vertex in $N_R(a_2)$. Clearly $x\neq a$ and $a_1x\notin E(\mathrm{cl^c}(G))$. If $a_3x\notin E(\mathrm{cl^c}(G))$, then $\{a_2,a_1,a_3,x\}$ induces a claw in $\mathrm{cl^c}(G)$, a contradiction. This implies that $a_3x\in E(\mathrm{cl^c}(G))$, and x is the unique vertex in $N_{\mathrm{cl^c}(G)}(a_3)\cap V(R)$. Thus $N_R(a_2)\cup N_R(a_3)=\{x\}$. □

We denote by I_R the set of interior vertices of R, and by F_R the set of frontier vertices of R.

Claim 2. [21, Claim 2 in the proof of Lemma 3]
Let x,y be two vertices in R.
(1) If $\{x,y\}$ is a heavy pair of G, then x,y have two common neighbors in I_R.
(2) If $x,y\in I_R\cup\{a\}$, $xy\in E(G)$ and $d(x)+d(y)\geq n$, then x,y have a common neighbor in I_R.

Proof. (1) Note that every vertex in F_R has at least one neighbor in $G-R$, and every vertex in $G-R$ has at most one neighbor in F_R. We have

$|N_{G-R}(F_R\backslash\{x,y\})| \geq |F_R\backslash\{x,y\}|$. Also note that $n = |I_R\backslash\{x,y\}|+|F_R\backslash\{x,y\}|+|V(G-R)|+2$. Thus

$$\begin{aligned}
n &\leq d(x)+d(y)\\
&= d_{I_R}(x)+d_{I_R}(y)+d_{F_R}(x)+d_{F_R}(y)+d_{G-R}(x)+d_{G-R}(y)\\
&\leq d_{I_R}(x)+d_{I_R}(y)+2|F_R\backslash\{x,y\}|+d_{G-R}(x)+d_{G-R}(y)\\
&\leq d_{I_R}(x)+d_{I_R}(y)+|F_R\backslash\{x,y\}|+|N_{G-R}(F_R\backslash\{x,y\})|+|N_{G-R}(x)|+|N_{G-R}(y)|\\
&= d_{I_R}(x)+d_{I_R}(y)+|F_R\backslash\{x,y\}|+|N_{G-R}(F_R)|\\
&\leq d_{I_R}(x)+d_{I_R}(y)+|F_R\backslash\{x,y\}|+|V(G-R)|,
\end{aligned}$$

and

$$d_{I_R}(x)+d_{I_R}(y) \geq n-|F_R\backslash\{x,y\}|-|V(G-R)| = |I_R\backslash\{x,y\}|+2.$$

This implies that x,y have two common neighbors in I_R.

(2) Note that if $a_2,a_3 \in N_{G-R}(R)$, then they have a common neighbor in $F_R\backslash\{a\}$. By Claim 1, we can see that

$$|V(G-R)| \geq |F_R|+1 \text{ and } |V(G-R)\backslash N_{G-R}(a)| \geq |F_R\backslash\{a\}|+1.$$

If $x,y \in I_R$, then

$$\begin{aligned}
n &\leq d(x)+d(y)\\
&= d_{I_R}(x)+d_{I_R}(y)+d_{F_R}(x)+d_{F_R}(y)\\
&\leq d_{I_R}(x)+d_{I_R}(y)+2|F_R|\\
&\leq d_{I_R}(x)+d_{I_R}(y)+|F_R|+|V(G-R)|-1,
\end{aligned}$$

and

$$d_{I_R}(x)+d_{I_R}(y) \geq n-|F_R|-|V(G-R)|+1 = |I_R|+1.$$

This implies that x,y have a common neighbor in I_R.

If one of x,y, say y, is equal to a, then

$$\begin{aligned}
n &\leq d(x)+d(a)\\
&= d_{I_R}(x)+d_{I_R}(a)+d_{F_R}(x)+d_{F_R}(a)+d_{G-R}(a)\\
&\leq d_{I_R}(x)+d_{I_R}(a)+|F_R|+|F_R\backslash\{a\}|+d_{G-R}(a)\\
&\leq d_{I_R}(x)+d_{I_R}(a)+|F_R|+|V(G-R)\backslash N_{G-R}(a)|-1+|N_{G-R}(a)|\\
&\leq d_{I_R}(x)+d_{I_R}(a)+|F_R|+|V(G-R)|-1,
\end{aligned}$$

and

$$d_{I_R}(x)+d_{I_R}(a) \geq n-|F_R|-|V(G-R)|+1 = |I_R|+1.$$

This implies that x,a have a common neighbor in I_R. □

From here, the main difference between the proof presented here and the proof of Lemma 3 in [21] will occur, considering that we would find an induced Z_i and use the Z_i-c-heavy condition.

By Lemma 5, G has an induced path P from a to a_i such that every vertex of P is either in $\{a_j : 0 \le j \le i\}$ or an interior vertex of some regions (we set $a_0 = a$). Let $a, a'_1, a'_2, \ldots, a'_i$ be the first $i+1$ vertices of P. Note that every vertex a'_i is nonadjacent to every vertex in $\{b, c\} \cup I_R$. If $abca$ is also a triangle in G, then $\{a, b, c, a'_1, \ldots, a'_i\}$ induces a Z_i in G. Thus one vertex of $\{b, c\}$ is heavy in G and one of $\{a'_{i-1}, a'_i\}$ is heavy in G. We assume without loss of generality that b, a'_{i-1} are heavy in G, and then, also are heavy in $\mathrm{cl}^{\mathrm{c}}(G)$. Then $\{b, a'_{i-1}\}$ is a heavy pair in $\mathrm{cl}^{\mathrm{c}}(G)$, a contradiction. So we only consider the case one edge of $\{ab, bc, ac\}$ does not exist in G.

If $I_R = \emptyset$, then R is a clique in G, and $ab, bc, ac \in E(G)$, a contradiction. Thus, $I_R \neq \emptyset$. By Lemma 5, a has a neighbor in I_R.

Claim 3. [21, Claim 3 in the proof of Lemma 3]
$d_{I_R}(a) = 1$.

Proof. If a is contained in a triangle $axya$ such that $x, y \in I_R$, then $\{a, x, y, a'_1, \ldots, a'_i\}$ induces a Z_i in G. Thus one vertex of $\{x, y\}$ is heavy in G and one vertex of $\{a'_{i-1}, a'_i\}$ is heavy in G, a contradiction. Hence, $N_{I_R}(a)$ is an independent set.

Suppose that $d_{I_R}(a) \ge 2$. Let x, y be two vertices in $N_{I_R}(a)$. Then $xy \notin E(G)$. Since $\{a, x, y, a'_1\}$ induces a claw in G, and $\{a'_1, x\}$, $\{a'_1, y\}$ are not heavy pairs of G, it follows $\{x, y\}$ is a heavy pair of G. Without loss of generality, suppose that x is heavy in G.

If a is also heavy in G, then by Claim 2, a, x have a common neighbor in I_R, contradicting the fact that $N_{I_R}(a)$ is independent. So we conclude that a is light in G.

Since $\{x, y\}$ is a heavy pair of G, by Claim 2, x, y have two common neighbors in I_R. Let x', y' be two vertices in $N_{I_R}(x) \cap N_{I_R}(y)$. Clearly $ax', ay' \notin E(G)$.

If $x'y' \in E(G)$, then $\{x, x', y', a, a'_1, \ldots, a'_{i-1}\}$ induces a Z_i in G. Thus one vertex of $\{a'_{i-2}, a'_{i-1}\}$ is heavy in G. This implies either $\{x, a'_{i-2}\}$ or $\{x, a'_{i-1}\}$ is a heavy pair of G, and also a heavy pair of $\mathrm{cl}^{\mathrm{c}}(G)$, a contradiction. So we conclude that $x'y' \notin E(G)$.

Note that $\{x, x', y', a\}$ induces a claw in G, and a is light in G. So one vertex of $\{x', y'\}$ is heavy in G. We assume without loss of generality that x' is heavy in G. By Claim 2, x, x' have a common neighbor x'' in I_R. Clearly $ax'' \notin E(G)$. Thus $\{x, x', x'', a, a'_1, \ldots, a'_{i-1}\}$ induces a Z_i, and hence one vertex of $\{a'_{i-2}, a'_{i-1}\}$ is heavy in G, a contradiction. □

Now let x be the vertex in $N_{I_R}(a)$. The left part is almost the same as in the proof of Lemma 3 in [21]. We rewrite it here.

Claim 4. [21, Claim 4 in the proof of Lemma 3]
$N_R(a) = V(R)\backslash\{a\}$.

Proof. Suppose that $V(R)\backslash\{a\}\backslash N_R(a) \neq \emptyset$. By Lemma 5, $R - x$ is connected. Let y be a vertex in $V(R)\backslash\{a\}\backslash N_R(a)$ such that a, y have a common neighbor z in $R-x$. Since $N_{I_R}(a) = \{x\}$ and $z \in N_R(a)\backslash\{x\}$, z is a frontier vertex of R. Let z' be a vertex in $N_{G-R}(z)$. Then $\{z, y, a, z'\}$ induces a claw in G. Since $\{a, z'\}$, $\{y, z'\}$ are not heavy pairs of G, $\{a, y\}$ is a heavy pair of G. By Claim 2, a, y have two common neighbors in I_R, contradicting Claim 3. □

By Claims 3 and 4, we can see that $|I_R| = 1$. Recall that one edge of $\{ab, bc, ac\}$ is not in $E(G)$. By Claim 4, $ab, ac \in E(G)$. This implies that $bc \notin E(G)$, and $\{a, b, c, a'_1\}$ induces a claw in G. Since $\{b, a'_1\}$, $\{c, a'_1\}$ are not heavy pairs of G, $\{b, c\}$ is a heavy pair of G. By Claim 2, b, c have two common neighbors in I_R, contradicting the fact that $|I_R| = 1$. ■

Corollary 7. *For $i = 1$ or $i \geq 3$, the class of claw-o-heavy and Z_i-c-heavy graphs is c-stable.*

Theorem 15. *Let S be a connected claw-free and $K_{1,1,2}$-free graph of order at least 3. Then the class of claw-o-heavy and S-c-heavy graphs is c-stable, if and only if*

$$S \in \{K_i : i \geq 3\} \cup \{P_i : i \geq 3\} \cup \{Z_i : i = 1 \text{ or } i \geq 3\}.$$

Proof. If $S = K_i$, $i \geq 3$, then every graph is S-c-heavy, and the class of claw-o-heavy and S-c-heavy graphs is c-stable. If $S = P_i$, $i \geq 3$ or $S = Z_i$, $i = 1$ or $i \geq 3$, then by Corollaries 6 and 7, the class of claw-o-heavy and S-c-heavy graphs is c-stable. This completes the 'if' part of the proof.

Now we consider the 'only if' part of the theorem. We first construct some claw-o-heavy graphs as in Figure 6.

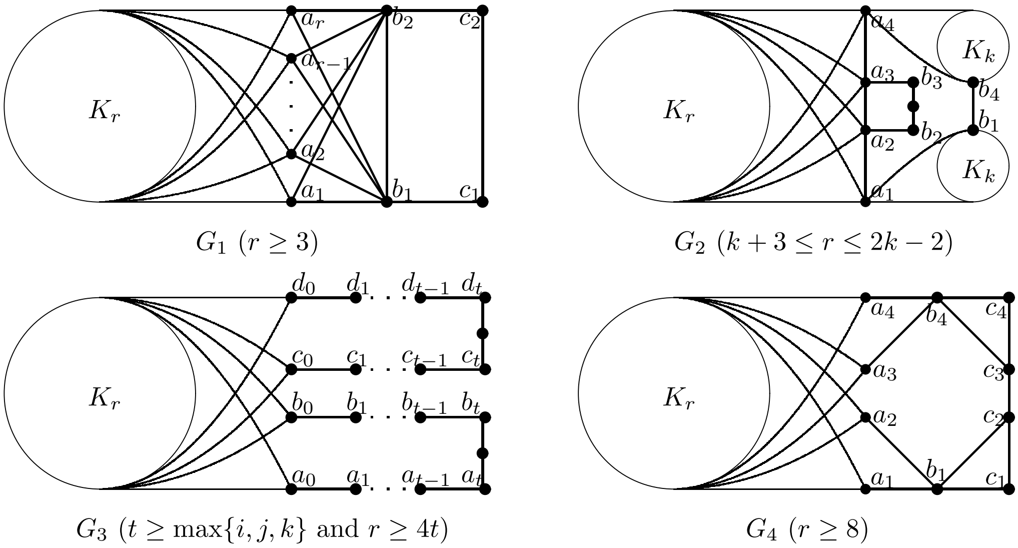


Figure 6. Some claw-o-heavy graphs.

Suppose S is a claw-free and $K_{1,1,2}$-free graph such that the class of claw-o-heavy and S-c-heavy graphs is c-stable. Consider the case where the class of claw-free and S-free graphs is r-stable. By Theorem 11, $S \in \{C_3, H\} \cup \{P_i : i \geq 1\} \cup \{Z_i : i \geq 1\} \cup \{N_{i,j,k} : i,j,k \geq 1\}$. Now we will explain why the graphs in Figure 6. are the required graphs.

- The graph G_1 is Z_2-c-heavy, and the closure $\mathrm{cl^c}(G_1)$ is obtained by adding all possible edges between vertices in the $V(K_r) \cup \{a_1, \ldots, a_r, b_1, b_2\}$. Notice that the subgraph of $\mathrm{cl^c}(G_1)$ induced by $\{a_1, a_2, b_1, c_1, c_2\}$ is a Z_2 which is not c-heavy in $\mathrm{cl^c}(G_1)$.

- The graph G_2 is N-c-heavy, and the closure $\mathrm{cl^c}(G_2)$ is obtained by adding all possible edges between vertices in the $V(K_r) \cup \{a_1, \ldots, a_4\}$. Notice that the subgraph of $\mathrm{cl^c}(G_2)$ induced by $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ is an N which is not c-heavy in $\mathrm{cl^c}(G_2)$ (noting that a_2, a_3 are not heavy in $\mathrm{cl^c}(G)$).

- The graph G_3 is $N_{i,j,k}$-c-heavy for $\max\{i,j,k\} \geq 2$ (in fact, it is $N_{i,j,k}$-free), and the closure $\mathrm{cl^c}(G_3)$ is obtained by adding all possible edges between vertices in the $V(K_r) \cup \{a_0, b_0, c_0, d_0\}$. Notice that the subgraph of $\mathrm{cl^c}(G_3)$ induced by $\{a_0, \ldots, a_i, b_0, \ldots, b_j, c_0, \ldots, c_k\}$ is an $N_{i,j,k}$ which is not c-heavy in $\mathrm{cl^c}(G_3)$.

- The graph G_4 is H-c-heavy ($\max\{i,j,k\} \geq 2$) (in fact, it is H-free), and the closure $\mathrm{cl^c}(G_4)$ is obtained by adding all possible edges between vertices in the $V(K_r) \cup \{a_1, \ldots, a_4\}$. Notice that the subgraph of $\mathrm{cl^c}(G_4)$ induced by $\{a_1, a_2, b_1, c_1, c_2\}$ is an H which is not c-heavy in $\mathrm{cl^c}(G_4)$.

Thus, we can see S is C_3, P_i, $i \geq 1$ or Z_i, $i = 1$ or $i \geq 3$.

Next we consider the case where the class of claw-free and S-free graphs is not r-stable. Let G' be a claw-free and S-free graph such that $\mathrm{cl}^{\mathrm{r}}(G)$ is not S-free. Let G be the disjoint union of G' and an empty graph of order $|V(G')|$. Clearly G is claw-free and S-free, and then, claw-o-heavy and S-c-heavy. Let G_i, $1 \leq i \leq r$, be the sequence of graphs in the definition of the c-closure of G, where $G = G_1$ and $\mathrm{cl}^{\mathrm{c}}(G) = G_r$. Note that for every i, every vertex of G_i has degree less than $|V(G)|/2$. This implies that the c-eligible vertices of G_i are exactly the r-eligible ones. Thus $\mathrm{cl}^{\mathrm{c}}(G) = \mathrm{cl}^{\mathrm{r}}(G)$ and $\mathrm{cl}^{\mathrm{c}}(G)$ contains an induced S. Note that $\mathrm{cl}^{\mathrm{c}}(G)$ has no heavy vertex. If S has a maximal clique C such that $S - C$ has a nontrivial component, then the induced S in $\mathrm{cl}^{\mathrm{c}}(G)$ is not c-heavy, a contradiction. So we conclude that for every maximal clique C of S, $S - C$ has only isolated vertex.

Let C be a maximal clique of S. If $V(S) \backslash V(C) = \emptyset$, then S is a complete graph K_k. Now we consider the case that $V(S) \backslash V(C) \neq \emptyset$. Note that every vertex of $S - C$ is an isolated vertex. Let x be a vertex in $S - C$. Since C is a maximal clique, $C \backslash N_S(x) \neq \emptyset$. If $|C \backslash N_S(x)| \geq 2$, then let C' be a maximal clique of S containing x. Then $S - C'$ will have a nontrivial component, a contradiction. So we conclude that $|C \backslash N_S(x)| = 1$. Let y be the vertex in $C \backslash N_S(x)$. By our assumption that S is connected, we obtain $|C| \geq 2$. If $|C| \geq 3$, letting z, z' be two vertices of $C \backslash \{y\}$, then $\{x, y, z, z'\}$ induces a $K_{1,1,2}$ of S, a contradiction. Thus we conclude that C has exactly two vertices. Let z be the vertex of C other than y. Note that $C' = C \cup \{x\} \backslash \{y\}$ is a maximal clique of S. Every vertex of $S - C'$ is nonadjacent to y. If $S - C$ has a vertex w other than x, then $\{z, x, y, w\}$ induces a claw in S, a contradiction. This implies that $S - C$ has only one vertex x, and $S = P_3$, a contradiction. ■

By Theorem 15, the class of claw-o-heavy and N-c-heavy graphs is not c-stable. However, we have a slightly larger class of graphs which is c-stable.

Let G be a graph and M be an induced N in G. We denote the vertices of M as in Figure 5. Note that M is c-heavy in G if and only if there are two vertices u, v of M which are heavy in G such that $\{u, v\} \notin \{\{a, a_1\}, \{b, b_1\}, \{c, c_1\}\}$. Now we say that M is p-heavy in G if there are two vertices u, v of M with $d(u) + d(v) \geq n$, such that $\{u, v\} \notin \{\{a, a_1\}, \{b, b_1\}, \{c, c_1\}\}$. Also, we say that G is N-p-heavy if every induced N in G is p-heavy. Note that an N-c-heavy graph is also N-p-heavy.

Now we prove that the class of claw-o-heavy and N-p-heavy graphs is c-stable.

Theorem 16. *Let G be a claw-o-heavy and N-p-heavy graph, and x be a c-eligible vertex of G. Then G'_x is N-p-heavy.*

Proof. Let M be an induced N in G'_x. We will prove that M is p-heavy. We denote the vertices of M as in Figure 5. Let $n = |V(G)|$. If M is also an induced subgraph of G, then M is p-heavy in G, and then, is p-heavy in G'_x.

Now we consider the case $E(M) \cap B_G(x) \neq \emptyset$. First suppose that $aa_1 \in B_G(x)$. Note that $N(x)$ is a clique in G'_x. This implies that $N(x) \cap V(M) = \{a, a_1\}$. Thus $\{a, x, b, b_1, c, c_1\}$ induces an N in G. Since G is N-p-heavy and $d_{G'_x}(a) \geq d_{G'_x}(x) \geq d(x)$, M is p-heavy in G'_x. Now we consider the case $aa_1 \notin B_G(G)$, and similarly, $bb_1, cc_1 \notin B_G(G)$. Thus at least one edge in $\{ab, ac, bc\}$ is in $B_G(x)$.

If $|B_G(x) \cap \{ab, ac, bc\}| = 1$, then without loss of generality, suppose that $ab \in B_G(x)$. Then $\{c, a, b, c_1\}$ induces a claw. Thus one of the three pairs $\{a, b\}, \{a, c_1\}, \{b, c_1\}$ is a heavy pair in G, and then has degree sum at least n in G'_x. Hence M is p-heavy in G'_x.

If $|B_G(x) \cap \{ab, ac, bc\}| = 2$, then without loss of generality, suppose that $ab, ac \in B_G(x)$. Then $\{x, a, b, b_1, c, c_1\}$ induces an N. Thus there are two vertices u, v in $\{x, a, b, b_1, c, c_1\}$ such that $\{u, v\} \notin \{\{x, a\}, \{b, b_1\}, \{c, c_1\}\}$, with degree sum at least n in G. Since $d_{G'_x}(a) \geq d(x)$, we can see that M is p-heavy.

If $|B_G(x) \cap \{ab, ac, bc\}| = 3$, then all the three edges $\{ab, ac, bc\}$ are in $B_G(x)$, which implies that $\{x, a, b, c\}$ induces a claw in G. So, one pair of $\{\{a, b\}, \{a, c\}, \{b, c\}\}$ is a heavy pair in G, and then has degree sum at least n in G'_x. Hence, M is p-heavy in G'_x. ■

Corollary 8. *The class of claw-o-heavy and N-p-heavy graphs is c-stable.*

4. Proof of Theorem 6

Note that every graph is P_3-c-heavy and C_3-c-heavy, and there indeed exist some 2-connected claw-o-heavy graphs which are not hamiltonian. The 'only if' part of the theorem can be deduced by Theorem 2 immediately. Now we prove the 'if' part of the theorem.

The cases $S = P_4$, P_5, P_6.

Note that every P_4-c-heavy graph is P_5-c-heavy and every P_5-c-heavy graph is P_6-c-heavy. We only need to prove the case $S = P_6$.

Let G be a claw-o-heavy and P_6-c-heavy graph. By Theorem 9 and Corollary 6, $\mathrm{cl}^{\mathrm{c}}(G)$ is claw-free and P_6-free. By Theorem 1, $\mathrm{cl}^{\mathrm{c}}(G)$ is hamiltonian, and by Theorem 9, so is G.

The cases $S = Z_1$, B, N.

Note that every Z_1-c-heavy graph is B-c-heavy and every B-c-heavy graph is N-c-heavy. We only need deal with the case $S = N$.

Let G be a claw-o-heavy and N-c-heavy graph. Note that every N-c-heavy graph is also N-p-heavy. By Theorem 9 and Corollary 8, $\mathrm{cl}^{\mathrm{c}}(G)$ is claw-free and

N-p-heavy. If $\mathrm{cl}^{\mathrm{c}}(G)$ is hamiltonian, then so is G. So we assume that $\mathrm{cl}^{\mathrm{c}}(G)$ is not hamiltonian. Since $\mathrm{cl}^{\mathrm{c}}(G)$ is 2-connected and claw-free, by Theorem 10, $\mathrm{cl}^{\mathrm{c}}(G)$ has an induced subgraph in $\mathcal{P}$. We denote the notation a_i, b_i $i = 1, 2, 3$ as in Section 2 and let $n = |V(G)|$.

Note that $\mathrm{cl}^{\mathrm{c}}(G)$ has no heavy pair. Since $\mathrm{cl}^{\mathrm{c}}(G)$ is N-p-heavy, every induced N of $\mathrm{cl}^{\mathrm{c}}(G)$ has two vertices in its triangle with degree sum at least n. Since both triangles $a_1a_2a_3a_1$ and $b_1b_2b_3b_1$ are contained in some induced N of $\mathrm{cl}^{\mathrm{c}}(G)$, two vertices of $\{a_1, a_2, a_3\}$ have degree sum at least n and two vertices of $\{b_1, b_2, b_3\}$ have degree sum at least n. We assume without loss of generality that a_1 has the maximum degree in $\mathrm{cl}^{\mathrm{c}}(G)$ among all the six vertices. Then two pairs of $\{\{a_1, b_1\}, \{a_1, b_2\}, \{a_1, b_3\}\}$ have degree sum at least n. Since a_1 is nonadjacent to b_2, b_3, $\mathrm{cl}^{\mathrm{c}}(G)$ has a heavy pair, a contradiction.

The cases $S = Z_2$, W.

Note that every Z_2-c-heavy graph is W-c-heavy. We only need to prove the case $S = W$. If G is W-c-heavy, then it is also W-o-heavy. By Theorem 3, G is hamiltonian.

The case $S = Z_3$.

Let G be a claw-o-heavy and Z_3-c-heavy graph. By Theorem 9 and Theorem 14, $\mathrm{cl}^{\mathrm{c}}(G)$ is claw-free and Z_3-free. By Theorem 1, $\mathrm{cl}^{\mathrm{c}}(G)$ is hamiltonian or $\mathrm{cl}^{\mathrm{c}}(G) = L_1$ or L_2 (see Figure 1). If $\mathrm{cl}^{\mathrm{c}}(G) = L_1$ or L_2, then G has no c-eligible vertices (any c-eligible vertex of G is an interior vertex and of degree at least 3 in $\mathrm{cl}^{\mathrm{c}}(G)$). Thus $G = \mathrm{cl}^{\mathrm{c}}(G) = L_1$ or L_2, contradicting the assumption $n \geq 10$.

5. One remark

In fact, in this paper we prove the following theorem, which is a common extension of the case $S = N$ in Theorems 3, 4 and 6.

Theorem 17. *Let G be a 2-connected graph. If G is claw-o-heavy and N-p-heavy, then G is hamiltonian.*

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