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Degree conditions restricted to induced paths for hamiltonicity of claw-heavy graphs

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Abstract Broersma and Veldman proved that every 2-connected claw-free and P_6-free graph is hamiltonian. Chen et al. extended this result by proving every 2-connected claw-heavy and P_6-free graph is hamiltonian. On the other hand, Li et al. constructed a class of 2-connected graphs which are claw-heavy and P_6-o-heavy but not hamiltonian. In this paper we further give some Ore-type degree conditions restricting to induced copies of P_6 of a 2-connected claw-heavy graph that can guarantee the graph to be hamiltonian. This improves some previous related results.

Keywords Hamiltonian graph, forbidden subgraph condition, degree condition, claw-heavy graph, closure theory

MR(2010) Subject Classification 05C07, 05C38, 05C45

1 Introduction

Throughout this paper, the graphs considered are undirected, finite and simple. For terminology and notation not defined here, we refer the reader to Bondy and Murty [3].

Let G be a graph. For a given graph H, we say that G is *H-free* if G contains no induced subgraph isomorphic to H. In this case, we call H a *forbidden subgraph* of G. Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

The bipartite graph $K_{1,3}$ is called a *claw*. Instead of $K_{1,3}$-free, we say that a graph is *claw-free* if it does not contain a copy of $K_{1,3}$ as an induced subgraph. As usual, we use P_i to denote the path of order i. Some other special graphs used in this paper are shown in Figure 1.

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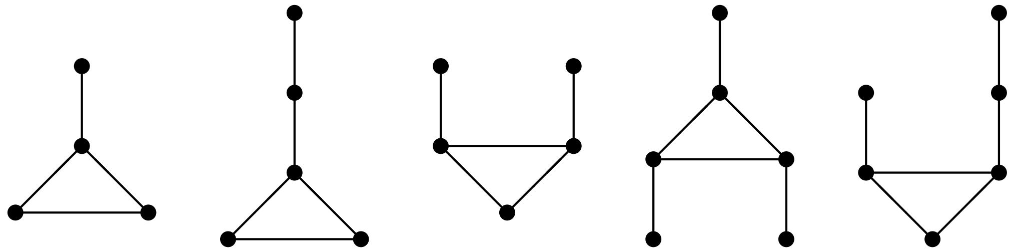


Figure 1 Graphs Z_1, Z_2, B, N and W

Forbidden subgraph conditions for hamiltonicity have been studied since the early 1980s, but till 1991, Bedrossian [1] firstly gave a characterization of all pairs of forbidden subgraphs for hamiltonian properties of graphs. First we note that a connected P_3-free graph is complete, and clearly is hamiltonian if it has at least three vertices. In fact, it is not difficult to see that P_3 is the only connected graph H such that every 2-connected H-free graph is hamiltonian. So the following result of Bedrossian deals with pairs of forbidden subgraphs, excluding P_3.

Theorem 1.1 (Bedrossian [1]) *Let R, S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being R-free and S-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

The above forbidden subgraph conditions for hamiltonicity are sometimes referred to as *structural conditions*. There is another type of conditions with respect to hamiltonian properties of graphs, so-called *numerical conditions*, of which degree conditions may be the most well-known.

Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H, respectively. We call $d_H(v)$ the *degree* of v in H. The *distance* between two vertices $x, y \in V(H)$ in H, denoted by $d_H(x, y)$, is the length of a shortest path between x and y in H. When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G(x, y)$ by $N(v)$, $d(v)$ and $d(x, y)$, respectively.

The followings are two well-known results concerning the degree conditions for hamiltonicity of graphs.

Theorem 1.2 (Dirac [9]) *Let G be a graph on $n \geq 3$ vertices. If $d(v) \geq n/2$ for every $v \in V(G)$, then G is hamiltonian.*

Theorem 1.3 (Ore [13]) *Let G be a graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u, v \in V(G)$, then G is hamiltonian.*

It is natural to relax the forbidden subgraph conditions to ones in which some of the forbidden subgraphs above are allowed, but some degree conditions are imposed on the subgraphs. Broersma et al. [4] introduced the class of 1-heavy (2-heavy) graphs by restricting Dirac's condition to induced claws of a graph. Instead of Broersma et al.'s restriction, Čada [8] put Ore's condition to induced claws of a graph, and call it an o-heavy graph (In this paper, we will call it a claw-o-heavy graph for convenience). Li et al. [10] extended Čada's concept of claw-o-heavy graphs to a more general one.

Let G be a graph on n vertices. Following [10], for a given graph H, G is called *H-o-heavy* (the authors used the notation 'H-heavy' in [10]), if every induced copy of H in G has two nonadjacent vertices with degree sum in G at least n. Note that an H-free graph is trivially H-o-heavy, and if H_1 is an induced subgraph of H_2, then an H_1-o-heavy graph is also H_2-o-heavy. Following [11], we say that a graph G is *H-f-heavy* if for every induced copy G' of H in G, and every two vertices $u, v \in V(G')$ with $d_{G'}(u, v) = 2$, there holds $\max\{d(u), d(v)\} \geq |V(G)|/2$. Note that every claw-$f$-heavy graph is also claw-o-heavy.

Li et al. [10] completely characterized pairs of Ore-type heavy subgraphs for a 2-connected graph to be hamiltonian, which extends Theorem 1.1. The main result in [10] is given as follows.

Theorem 1.4 (Li et al. [10]) *Let R, S be connected graphs of order at least 3 with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being R-o-heavy and S-o-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ or W.*

It is easy to see that P_6 is the only forbidden subgraph S appearing in Theorem 1.1 but missing here. Li et al. [10] constructed a class of 2-connected graphs which are claw-o-heavy and P_6-o-heavy but not hamiltonian.

In fact, earlier than Bedrossian [2], Broersma and Veldman [5] proved that every 2-connected claw-free and P_6-free graph is hamiltonian. Chen et al. [7] furthermore extended Broersma and Veldman's result as follows.

Theorem 1.5 (Chen et al. [7]) *Every 2-connected claw-o-heavy and P_6-free graph is hamiltonian.*

So one may ask the question: Which degree conditions can be used to restrict to all induced copies of P_6 in a 2-connected claw-o-heavy graph to make it hamiltonian?

A related result is as follows.

Theorem 1.6 (Ning and Zhang [11]) *Every 2-connected claw-o-heavy and P_6-f-heavy graph is hamiltonian.*

One may further ask: Can we still put Ore's condition (or Dirac's condition) to induced copies of P_6 in a graph but with some additional restrictions to guarantee that it is hamiltonian?

Our answers are the following two results. Note that the first theorem weakens the condition of Theorem 1.6.

Theorem 1.7 *Let G be a 2-connected claw-o-heavy graph of order at least n. If for every induced copy of $P_6 : v_1v_2 \cdots v_6$ in G, $d(v_i) + d(v_j) \geq n$ for some $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, then G is hamiltonian.*

Theorem 1.8 *Let G be a 2-connected claw-o-heavy graph of order at least n. If for every induced copy of $P_6 : v_1v_2 \cdots v_6$ in G, $\max\{d(v_1), d(v_6)\} \geq n/2$, then G is hamiltonian.*

Now we will go further on this direction. Before giving our main results, we will first introduce some necessary terminology and notation.

Let γ be a graph (possibly with loops) with vertex set $\mathfrak{I} = \{1, 2, 3, 4, 5, 6\}$. We say that a graph G is *P_6-γ-heavy* if, for every induced copy of $P_6 : v_1v_2v_3v_4v_5v_6$ in G, there exist $i, j \in \mathfrak{I}$ (possibly $i = j$) such that $ij \in E(\gamma)$ and $d(v_i) + d(v_j) \geq n$, where $n = |V(G)|$. Note that if γ' is a (spanning) subgraph of γ, then a P_6-γ'-heavy graph is also P_6-γ-heavy.

For two graphs γ and γ' on $\mathfrak{I}$ such that $ij \in E(\gamma)$ if and only if $(7-i)(7-j) \in E(\gamma')$, we say γ is *symmetrical* to γ'. Note that if γ and γ' are symmetrical to each other, then a graph G is P_6-γ-heavy if and only if G is P_6-γ'-heavy. If γ is symmetrical to itself, then we say γ is *symmetrical*.

Let ε be the empty graph on $\mathfrak{I}$. Then a graph G is P_6-free if and only if it is P_6-ε-heavy. Let σ be the graph on $\mathfrak{I}$ with edge set $E(\sigma) = \{ij : |j-i| \geq 2, i, j \in \mathfrak{I}\}$. Then a graph is P_6-o-heavy means it is P_6-σ-heavy. Let γ_1 be the graph on $\mathfrak{I}$ with edge set $\{ij : i = 1, 2, 3 \text{ and } j = 4, 5, 6\}$. Then Theorem 1.7 states that every 2-connected claw-o-heavy and P_6-γ_1-heavy graph is hamiltonian.

The goal of this paper is to find all symmetrical graphs γ on $\mathfrak{I}$ such that every 2-connected claw-o-heavy and P_6-γ-heavy graph is hamiltonian.

We describe the graphs $\gamma_1, \gamma_2, \gamma_3$ on $\mathfrak{I}$ by giving their edge sets (also see Figure 2):

$$E(\gamma_1) = \{14, 15, 16, 24, 25, 26, 34, 35, 36\};$$
$$E(\gamma_2) = \{11, 12, 14, 15, 16, 25, 26, 36, 56, 66\};$$
$$E(\gamma_3) = \{13, 14, 15, 25, 26, 36, 46\}.$$

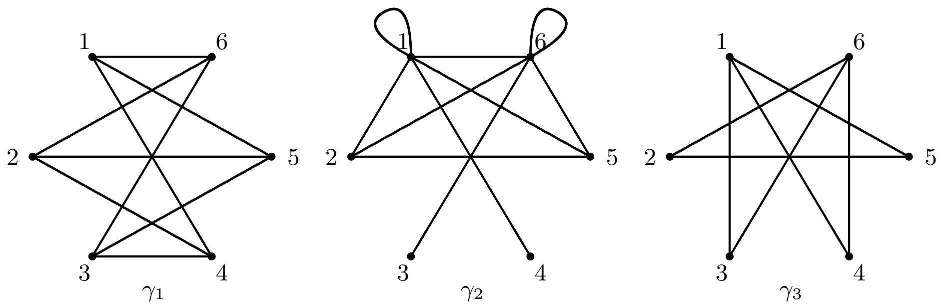


Figure 2 Graphs on $\mathfrak{I}$: γ_1, γ_2 and γ_3

The following is our main result. Note that both Theorems 7 and 8 are its corollaries.

Theorem 1.9 *Let γ be a symmetrical graph on $\mathfrak{I}$. Then every 2-connected claw-o-heavy and P_6-γ-heavy graph is hamiltonian if and only if γ is a subgraph of γ_1, γ_2 or γ_3.*

2 Preliminaries

In this section, we will introduce some preliminaries for the proof of the 'if' part of Theorem 1.9, which are similar to the ones in [12]. We first introduce Čada's closure theory of claw-o-heavy graphs [8], which is an extension of the closure theory of claw-free graphs invented by Ryjáček [14].

Let G be a graph of order n. We say that a vertex $x \in V(G)$ is *heavy* in G if $d(x) \geq n/2$; and a pair of vertices $\{x, y\}$ is *heavy* in G if $d(x) + d(y) \geq n$. We say that a vertex (a pair of vertices) is *light* if it is not heavy. Note that if $\{x, y\}$ is a heavy pair, then either x or y is a heavy vertex.

Let G be a graph and $x \in V(G)$. The *local completion of G at x*, denoted by G'_x, is the graph obtained from G by adding all missing edges in $G[N(x)]$. Define $B^o_x(G) = \{uv : \{u, v\} \subset$

$N(x)$ is a heavy pair of $G\}$. Let G_x^o be the graph with vertex set $V(G_x^o) = V(G)$ and edge set $E(G_x^o) = E(G) \cup B_x^o(G)$. If $G_x^o[N(x)]$ consists of two disjoint cliques C_1 and C_2, then we call a vertex $z \in V(G) \backslash (\{x\} \cup N(x))$ a *join vertex* of x in G if $\{x, z\}$ is a heavy pair in G, and there are two vertices $y_1 \in C_1$ and $y_2 \in C_2$ such that $zy_1, zy_2 \in E(G)$. The vertex x is an *o-eligible vertex* of G, if $N(x)$ is not a clique and, $G_x^o[N(x)]$ is connected or, $G_x^o[N(x)]$ consists of two disjoint cliques and there is some join vertex of x.

Let G be a claw-o-heavy graph. The *closure* of G, denoted by cl(G), is the graph such that there is a sequence of graphs $G_1, G_2, \ldots, G_t$ and a sequence of vertices $x_1, x_2, \ldots, x_{t-1}$ such that:
(1) $G = G_1$, $G_t = \mathrm{cl(G)}$;
(2) for $i = 1, 2, \ldots, t-1$, G_{i+1} is the local completion of G_i at some o-eligible vertex x_i of G_i; and
(3) there is no o-eligible vertex in G_t.

Theorem 2.1 (Čada [8]) *Let G be a claw-o-heavy graph. Then*
(1) the closure cl(G) *is uniquely determined;*
(2) there is a C_3-free graph H such that cl(G) *is the line graph of H; and*
(3) G is hamiltonian if and only if cl(G) *is.*

Note that every line graph is claw-free (see [2]). The above theorem implies that cl(G) is a claw-free graph.

Now we will give some terminology and notation firstly introduced in [12] by the authors. Let G be a claw-o-heavy graph and C be a maximal clique of cl(G). We call $G[C]$ a *region* of G. For a vertex v of G, we call v an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions. For two vertices $u, v \in V(G)$, we say u and v are *associated* if u, v are contained in a common region of G; otherwise u and v are *dissociated*. We denote by I_R the set of interior vertices of a region R, and by F_R the set of frontier vertices of R.

From [8], it is not difficult to get the following

Lemma 2.2 *Let G be a claw-o-heavy graph. Then*
(1) every vertex is either an interior vertex of a region or a frontier vertex of two regions;
(2) every two regions are either disjoint or have only one common vertex; and
(3) every pair of dissociated vertices have degree sum in cl(G) *(and in G) less than $|V(G)|$.*

We also need the following tools developed in [12].

Lemma 2.3 *Let G be a claw-o-heavy graph and R be a region of G. Then*
(1) for any two vertices $u, v \in V(R)$, there is an induced path of G from u to v such that every internal vertex of the path is in I_R; and
(2) for two vertices u, v in R, if $uv \notin E(G)$ and $\{u, v\}$ is a heavy pair of G, then u, v have two common neighbors in I_R.

For two associated vertices u, v, by Lemma 2.3 (1), we use $\Pi[u, v]$ to denote a shortest path between u and v of G such that every internal vertex of it is an interior vertex of the region containing u, v. Assume that u, v are two vertices in the region R and $\{x, y\}$ be a heavy pair of G contained in $\Pi[u, v]$. By Lemma 2.3 (2), x, y has two common neighbors in I_R, implying that

x, y have distance at most 2 in $\Pi[u,v]$. So we conclude that any pair of vertices with distance at least 3 in $\Pi[u,v]$ is not a heavy pair. Let $P = v_1v_2\cdots v_p$ be an induced path in cl(G) ($p \geq 3$). Then all the vertices v_i ($2 \leq i \leq p-1$) is a frontier vertices, common to the two regions containing $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$, respectively. Since P is an induced path, $\Pi[v_i, v_{i+1}]$ and $\Pi[v_j, v_{j+1}]$ are internally disjoint for $i \neq j$. So the path $P' = \Pi[v_1,v_2]\Pi[v_2,v_3]\cdots\Pi[v_{p-1},v_p]$ is an induced path of G.

Following [6], we define $\mathcal{P}$ to be the class of graphs obtained by taking two vertex-disjoint triangles $a_1a_2a_3a_1$, $b_1b_2b_3b_1$ and by joining every pair of vertices $\{a_i, b_i\}$ by a path P_{k_i} : $a_ic_i^1c_i^2\cdots c_i^{k_i-2}b_i$ for $k_i \geq 3$ or by a triangle $a_ib_ic_i$. We denote the graphs in $\mathcal{P}$ by P_{x_i,x_2,x_3}, where $x_i = k_i$ if a_i, b_i are joined by a path P_{k_i}, and $x_i = T$ if a_i, b_i are joined by a triangle.

The following theorem plays the central role in our proof.

Theorem 2.4 (Brousek [6]) *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph $H \in \mathcal{P}$.*

3 Proof of the 'if' part of Theorem 1.9

Let G be a claw-o-heavy non-hamiltonian graph of order n. For each γ_k, $k = 1,2,3$, we will show that there exists an induced P_6 : $v_1v_2\cdots v_6$ such that for every edge $ij \in E(\gamma_k)$, $d(v_i) + d(v_j) < n$. For convenience, we call such an induced P_6 a *bad* P_6 *to* γ_k in the following.

Let $G' = \text{cl(G)}$. By Theorem 2.1, G' is claw-free and non-hamiltonian. By Theorem 2.4, let $H \subseteq G'$ be an induced copy of some graph in $\mathcal{P}$. We denote the vertices of H as in Section 2. If $x_i = k_i$, then let a_i' be the neighbor of a_i on $\Pi[a_i, c_i^1]$, b_i' be the neighbor of b_i on $\Pi[b_i, c_i^{k_i-2}]$, and let $\Pi_i = \Pi[a_i,c_i^1]c_i^1\Pi[c_i^1,c_i^2]c_i^2\cdots c_i^{k_i-2}\Pi[c_i^{k_i-2},b_i]$. If $x_i = T$, then let a_i' be the neighbor of a_i on $\Pi[a_i, c_i]$, b_i' be the neighbor of b_i on $\Pi[b_i, c_i]$, and let $\Pi_i = \Pi[a_i, b_i]$. For $1 \leq i, j \leq 3$, let $\Pi_{ij}^a = \Pi[a_i, a_j]$ and $\Pi_{ij}^b = \Pi[b_i, b_j]$. Let a_{ij}' (b_{ij}') be the neighbor of a_i (b_i) on Π_{ij}^a (Π_{ij}^b). For convenient, we take Π_{ij}^a (Π_{ij}^b) and Π_{ji}^a (Π_{ji}^b) as the same path (with opposite direction). Set

$$S = \bigcup_{1\leq i\leq 3} (\{a_i', b_i'\} \cup V(\Pi_i)) \cup \bigcup_{1\leq i<j\leq 3} \left(V(\Pi_{ij}^a) \cup V(\Pi_{ij}^b)\right).$$

For a path P with the origin x, we use $P|_i^x$ (or briefly, $P|_i$) to denote the subpath of P consisting of the first i edges of P. If $P = v_1v_2\cdots v_p$, then we denote $\overleftarrow{P} = v_pv_{p-1}\ldots v_1$.

Claim 1 There is a heavy vertex of G in $S \backslash \{a_i, b_i : 1 \leq i \leq 3\}$, or there are two heavy vertices in $\{a_i, b_i : 1 \leq i \leq 3\}$.

Proof Up to symmetry, suppose that a_1 is the vertex with the largest degree among all vertices in $\{a_i, b_i : 1 \leq i \leq 3\}$. If G has no heavy vertex in S or has the only one heavy vertex a_1 in S, then $P = b_1'b_1\Pi_{12}^bb_2\overleftarrow{\Pi_2}a_2\Pi_{23}^aa_3a_3'$ is an induced path of order at least 6 and each vertex of P is not heavy in G. Thus $P|_5$ is a bad P_6 to every γ_k.

Note that any two heavy vertices are associated (see [8]). Up to symmetry, we have the following two cases:

Case 1 There is a heavy vertex in $\{a_1', b_1'\} \cup (V(\Pi_1) \backslash \{a_1, b_1\})$, or both a_1 and b_1 are heavy.

Suppose that there is a heavy vertex of G contained in $S \backslash (\{a_1', b_1'\} \cup V(\Pi_1))$, then two heavy vertices form a dissociated pair, a contradiction. Thus every heavy vertex of G contained in S is in $\{a_1', b_1'\} \cup V(\varPi_1)$. Also clearly either $\bigcup_{1\le i<j\le 3} V(\varPi^a_{ij})$ or $\bigcup_{1\le i<j\le 3} V(\varPi^b_{ij})$ contains no heavy pair of G. We suppose without loss of generality that $\bigcup_{1\le i<j\le 3} V(\varPi^a_{ij})$ contains no heavy pair of G. Let $Q_1 = a_1' a_1 \varPi^a_{12} a_2 \varPi_2 b_2 \varPi^b_{23} b_3 b_3'$. Then $\overleftarrow{Q_1}|_5$ is a bad P_6 to γ_1.

Suppose now that $\varPi_1 = a_1 x_1 x_2 \cdots x_{p-1} b_1$, where p is the length of $\varPi_1$.

Case 1.1 $p = 1$, i.e., $\varPi_1 = a_1 b_1$.

Let $Q_2 = (\varPi^a_{12} a_2 a_2')|_2$ and $Q_2' = (\varPi^b_{13} b_3 b_3')|_2$. Then $\overleftarrow{Q_2} a_1 b_1 Q_2'$ is a bad P_6 to γ_2, γ_3.

Case 1.2 $p = 2$, i.e., $\varPi_1 = a_1 x_1 b_1$.

Let $Q_2 = a_{13}' a_1 x_1 b_1 \varPi^b_{12} b_2 b_2'$. Then $Q_2|_5$ is a bad P_6 to γ_2, γ_3.

Case 1.3 $p = 3$, i.e., $\varPi_1 = a_1 x_1 x_2 b_1$.

Note that the pair $\{a_1, b_1\}$ is light by Lemma 2.3 (2). Suppose first that $\bigcup_{1\le i<j\le 3} V(\varPi^b_{ij})$ contains a heavy pair of G, say $\{b_1, x\}$. Then b_1 is heavy, a_1 is light, and $\{a_1, x_i\}$ is light for $i = 1, 2$ (for otherwise $\{a_1, b_1\}$ or $\{x_i, x\}$ is a heavy pair, a contradiction). Let $Q_2 = a_1 x_1 x_2 b_1 \varPi^b_{12} b_2 b_2'$. Then $Q_2|_5$ is a bad P_6 to γ_2. Let $Q_3 = x_1 a_1 \varPi^a_{12} a_2 \varPi_2 b_2\ \varPi^b_{23} b_3 b_3'$. Then $Q_3|_5$ is a bad P_6 to γ_3.

Now we suppose that $\bigcup_{1\le i<j\le 3} V(\varPi^b_{ij})$ contains no heavy pairs of G. Recalling that $\bigcup_{1\le i<j\le 3} V(\varPi^a_{ij})$ contains no heavy pairs of G, $Q_2 = a_{12}' a_1 x_1 x_2 b_1 b_{13}'$ is a bad P_6 to γ_2, γ_3.

Case 1.4 $p \ge 4$.

If both a_1 and x_1 are light, then $Q_2 = (x_1 a_1 \varPi^a_{12} a_2 \varPi_2 b_2 \varPi^b_{23} b_3 b_3')|_5$ contains no heavy vertices of G, and hence is bad to γ_2, γ_3. So we assume that either a_1 or x_1 is heavy, and similarly, either b_1 or x_{p-1} is heavy. By Lemma 2.3 (2), the only possible case is that $p = 4$, x_1, x_3 are heavy and a_1, b_1 are light.

Note that either $\{a_1, x_1\}$ is light or $\{b_1, x_3\}$ is light. we assume without loss of generality that $\{a_1, x_1\}$ is light. Thus $Q_2 = a_1 x_1 x_2 x_3 b_1 b_{12}'$ is bad to γ_2, γ_3.

Case 2 There is a heavy vertex in $\bigcup_{1\le i<j\le 3}(V(\varPi^a_{ij})\backslash\{a_i, a_j\})$, or two of $\{a_1, a_2, a_3\}$ are heavy.

Clearly every heavy vertex of G is in $\bigcup_{1\le i<j\le 3} V(\varPi^a_{ij})$, and at most one of $\{a_i', b_i'\} \cup V(\varPi_i)$ contains heavy pairs of G. We assume without loss of generality that both $\{a_1', b_1'\} \cup V(\varPi_1)$ and $\{a_2', b_2'\} \cup V(\varPi_2)$ contain no heavy pairs of G.

Let $Q_1 = b_2' b_2 \overleftarrow{\varPi^b_{12}} b_1 \overleftarrow{\varPi_1} a_1 \varPi^a_{13} a_3 a_3'$, then $Q_1|_5$ is a bad P_6 to γ_1.

Suppose now that $\varPi^a_{12} = a_1 x_1 x_2 \cdots x_{p-1} a_2$, where p is the length of $\varPi^a_{12}$.

Case 2.1 $p = 1$, i.e., $\varPi^a_{12} = a_1 a_2$.

Let $Q_2 = a_1' a_1 a_2 \varPi_2 b_2 \varPi^b_{23} b_3 b_3'$. Then $Q_2|_5$ is a bad P_6 to γ_2, γ_3.

Case 2.2 $p = 2$, i.e., $\varPi^a_{12} = a_1 x_1 a_2$.

Let $Q_2 = a_2' a_2 x_1 a_1 \varPi_1 b_1 b_{13}'$. Then $Q_2|_5$ is a bad P_6 to γ_2, γ_3.

Case 2.3 $p = 3$, i.e., $\varPi^a_{12} = a_1 x_1 x_2 a_2$.

Let $Q_2 = a_1' a_1 x_1 x_2 a_2 a_2'$. Then Q_2 is a bad P_6 to γ_2, γ_3.

Case 2.4 $p \ge 4$.

Let $Q_3 = x_1 a_1 \varPi_1 b_1 \varPi^b_{12} b_2 \overleftarrow{\varPi_2} a_2 x_{p-1}$. Then $Q_3|_5$ is a bad P_6 to γ_3.

If one of a_1, a_2 is heavy in G, say a_1 is heavy, then x_i $(i \ge 3)$ and a_2 are light. Thus $Q_2 = (a_1' a_1 \varPi^a_{12})|_5$ is a bad P_6 to γ_2. So we assume that a_1, a_2 are light.

Recall that for each two vertices with distance at least 3 in Π^a_{12}, at least one of them is light. This implies that there exists an integer i, $2 \le i \le p-2$, such that every vertex in $V(\Pi^a_{12})\backslash\{x_{i-1},x_i,x_{i+1}\}$ is light. Note that either $\{x_{i-2},x_{i-1}\}$ or $\{x_{i+1},x_{i+2}\}$ is light (we set $x_0=a_1$ and $x_p=a_2$). We assume without loss of generality that $\{x_{i-2},x_{i-1}\}$ is light. Then $Q_2=(x_{i-2}x_{i-1}\cdots x_{p-1}a_2a_2')|_5$ is a bad P_6 to γ_2.

The proof is complete.

4 Proof of the 'only if' part of Theorem 1.9

Let γ be a symmetrical graph on $\mathfrak{I}$ such that every 2-connected claw-o-heavy and P_6-γ-heavy graph is hamiltonian. We will prove that γ is a subgraph of γ_1, γ_2 or γ_3. Assume not. Then for every $k=1,2,3$, $E(\gamma)\backslash E(\gamma_k)\neq\emptyset$. Note that the graphs in Figure 3 are claw-o-heavy and non-hamiltonian. Hence they are not P_6-γ-heavy. Let $P=u_1u_2\cdots u_6$ and $Q=v_1v_2\cdots v_6$ be two induced copies of P_6 in a graph G of order n. We say P and Q are essentially same if for every $i,j\in[1,6]$, $d(u_i)+d(u_j)\ge n$ if and only if $d(v_i)+d(v_j)\ge n$.

Claim 2 None of $\{22,23,24,33,34,35,44,45,55\}$ is in $E(\gamma)$.

Proof Recall that $E(\gamma)\backslash E(\gamma_1)\neq\emptyset$, i.e., one of $\{11,12,13,22,23,33,44,45,46,55,56,66\}$ is in $E(\gamma)$. Since γ is symmetrical, one of $\{11,12,13,22,23,33\}$ is in $E(\gamma)$ and one of $\{44,45,46,55,56,66\}$ is in $E(\gamma)$.

Suppose that one of $\{22,23,24,33,34,35,44,45,55\}$ is in $E(\gamma)$. Since γ is symmetrical, one of $\{22,23,24,33,34,44\}$ is in $E(\gamma)$ and one of $\{33,34,35,44,45,55\}$ is in $E(\gamma)$. Consider the graph G_1. Let $P=v_1v_2\cdots v_6$ be an induced path of G_1, and let ij be an edge in $E(\gamma)$ such that

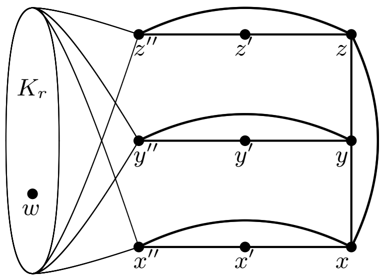
$$ij\in\begin{cases}\{11,12,13,22,23,33\}, & \text{if } P=x''wy''yzz';\\ \{22,23,24,33,34,44\}, & \text{if } P=x'x''wy''yz;\\ \{33,34,35,44,45,55\}, & \text{if } P=xyy''wz''z';\\ \{44,45,46,55,56,66\}, & \text{if } P=x'xyy''wz''.\end{cases}$$

Then $d(v_i)+d(v_j)\ge|V(G_1)|$. Note that G_1 has only the four essentially different induced copies of P_6. This implies that G_1 is P_6-γ-heavy, a contradiction.

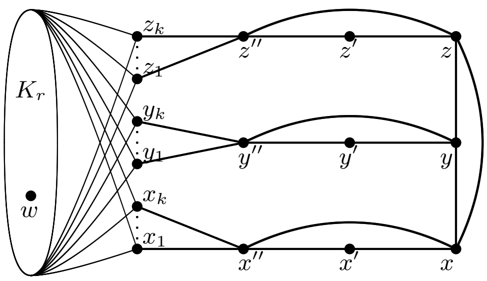
Let $\mathfrak{E}_1=\{22,23,24,33,34,35,44,45,55\}$. Then for $k=1,2,3$, $E(\gamma)\backslash(E(\gamma_k)\cup\mathfrak{E}_1)\neq\emptyset$. Note that $E(\overline{\gamma_2})\backslash\mathfrak{E}_1=\{13,46\}$. Since γ is symmetrical, we can see that $13,46\in E(\gamma)$.

Claim 3 None of $\{11,16,66\}$ is in $E(\gamma)$.

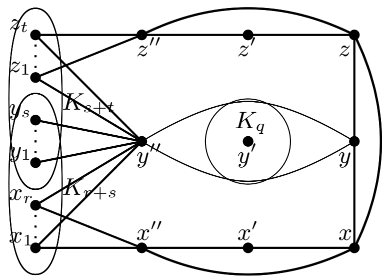
Proof Suppose not. Since γ is symmetrical, we can see that one of $\{11,16\}$ is in $E(\gamma)$ and one of $\{16,66\}$ is in $E(\gamma)$.



G_1 $(r \geq 7)$



G_2 $(k \geq 8$ and $r \geq 3k+7)$



G_3 $(q \geq 6,\, r, s \geq q+6,\, t \geq q+r+5)$

Figure 3 Three classes of claw-o-heavy non-hamiltonian graphs

Consider the graph G_2. Let $P = v_1v_2\cdots v_6$ be an induced path of G_3, and let ij be an edge in $E(\gamma)$ such that

$$ij = \begin{cases} 13, & \text{if } P = wx_1x''xyy'; \\ 11 \text{ or } 16, & \text{if } P = x_1x''xyy''y_1; \\ 46, & \text{if } P = x'xyy''y_1w; \\ 46, & \text{if } P = xyy''y_1wz_1; \\ 46, & \text{if } P = x'x''x_1wy_1y''; \\ 13, & \text{if } P = x''x_1wy_1y''y'; \\ 13, & \text{if } P = x_1wy_1y''yz. \end{cases}$$

Then $d(v_i) + d(v_j) \geq |V(G_2)|$. Note that G_2 has only the seven essentially different induced copies of P_6. This implies that G_2 is P_6-γ-heavy, a contradiction.

Let $\mathfrak{E}_2 = \mathfrak{E}_1 \cup \{11, 16, 66\}$. By Claims 2 and 3, $E(\gamma) \backslash (E(\gamma_3) \cup \mathfrak{E}_2) \neq \emptyset$. Note that $E(\overline{\gamma_3}) \backslash \mathfrak{E}_2 = \{12, 56\}$. Since γ is symmetrical, we can see that $12, 56 \in E(\gamma)$.

Let γ' be a graph on $\mathfrak{I}$ with edge set $E(\gamma') = \{12, 13, 46, 56\}$. Then γ' is a subgraph of γ. Similarly as in Claim 3, one can check that G_3 is P_6-γ'-heavy, and then is P_6-γ-heavy, a contradiction. This completes the proof of the 'only if' part of Theorem 1.9.

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