Standing Wave and Global Existence to a Nonlocal Nonlinear Schrödinger Equations: The Two-Dimensional Case

Zaihui Gan [1,2*]

[1] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

[2] College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610068, China

Abstract: In this paper, we consider the standing waves and the global existence for two-dimensional nonlocal nonlinear Schrödinger equations. It is a coupled system which describes the spontaneous generation of a magnetic field in a cold plasma under the static limit. The main difficulty in the proofs lies in exploring the inner structure of the equations due to the fact that the nonlocal terms violate the inner scaling invariance, which may cause the non-zero energy for the ground state. For this reason, we first make a proper use of the inner structure of the equations to establish the existence of standing waves, and then we apply an energy scaling to obtain the instability of standing waves. Finally we show a sharp threshold for the global existence of solutions to the nonlocal nonlinear Schrödinger equations by a variational method, which depends again on the inner structure of the equations under consideration.

Key words: Nonlocal nonlinear Schrödinger equations, Standing wave, Global existence, Instablity

AMS(2010): 35A15, 35Q55

1 Introduction

In 1971, Stamper et al. in [30] found that large quasi-steady magnetic fields were created in laser-produced plasmas. Later, Bezzerides et al. in [6] showed that the generation of magnetic fields is due to a solenoidal current $\mathbf{j}$ given by

$$\mathbf{j} = -i\frac{e\omega^{*2}}{16\pi m\omega_0^3}[\nabla \wedge (\mathbf{E} \wedge \overline{\mathbf{E}})], \qquad (A-1)$$

where ω^* is the plasma frequency, m the electron mass, $-e$ the electron charge, $\overline{\mathbf{E}}$ the complex conjugate of $\mathbf{E}$, and $\mathbf{E}$ the slowly varying complex amplitude of the high-frequency (ω_0) electric field $\tilde{\mathbf{E}}$:

$$\tilde{\mathbf{E}} = \frac{1}{2}\{\mathbf{E}(r,t)e^{-i\omega_0 t} + c.c.\}. \qquad (A-2)$$

*Zaihui Gan: ganzaihui2008cn@tju.edu.cn; ganzaihui2008cn@hotmail.com

Following the idea of introducing two time-scales which refer to the fast electron motion on a time-scale corresponding to the plasma frequency ω and to the ion motion, respectively [35], Thornhill and Ter Haar in [32] derived the following coupled system:

$$i\partial_t \mathbf{E} + \frac{3}{2}\omega^* \gamma_D^2 \nabla(\nabla \cdot \mathbf{E}) - \frac{1}{2}\frac{c^2}{\omega^*}[\nabla \wedge (\nabla \wedge \mathbf{E})] - \frac{1}{2}\omega^* \frac{n_s}{n_0}\mathbf{E} + \frac{ie}{2mc}[\mathbf{E} \wedge \mathbf{B}_s] = 0, \quad (A-3)$$

$$\partial_{tt} n_s - c_s^2 \Delta n_s = \frac{\nabla^2 |\mathbf{E}|^2}{16\pi M}, \quad (A-4)$$

$$\Delta \mathbf{B}_s - \frac{ie}{4mc\omega^*}[\nabla \wedge [\nabla \wedge [\mathbf{E} \wedge \overline{\mathbf{E}}]]] + \frac{\omega^{*2}}{c^2}\mathbf{B}_s = 0. \quad (A-5)$$

In the above system they used a two-fluid description of the plasma, which contains the equations of continuity, the equations of motion, and the Maxwell equations. Here M is the ion mass, c_s the ion-sound velocity with $c_s^2 \approx T_e/M$, T_e the electron temperature, γ_D the Debye redius, n_0 the equilibrium density of electron, n_s the fluctuation of electron density from its equilibrium and $\mathbf{B}_s$ the self-generation magnetic field in a cold plasma.

Normalizing equations (A-3)-(A-5), one gets

$$i\partial_t \mathbf{E} + \Delta \mathbf{E} - n\mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}) = 0, \quad (A-6)$$

$$\partial_{tt} n - \Delta n = \Delta |\mathbf{E}|^2, \quad (A-7)$$

$$\Delta \mathbf{B} - i\eta(\nabla \wedge (\nabla \wedge (\mathbf{E} \wedge \overline{\mathbf{E}}))) - \delta \mathbf{B} = 0, \quad (A-8)$$

where $n = n_s$, $\mathbf{B} = \mathbf{B}_s$, $\eta > 0$ and $\delta \geq 0$ are two dimensionless physical parameters. The system (A-6)-(A-8) describes the so-called spontaneous generation of a magnetic field in a cold plasma, in which $\mathbf{E}$ represents the slowly varying complex amplitude of the high-frequency electric field [8, 9, 10, 16, 35, 36]. We note that if one solves $\mathbf{B}$ from the equation (A-8) and substitutes it into the equation (A-6), one finds a differential -integral (nonlocal) equation for $\mathbf{E}$.

In order to illustrate the effects of the self-generation (induced) magnetic fields on dynamics of a cold plasma, we consider here a simplified version of the system (A-6)-(A-8). We restrict ourselves to the static limit(when the ion-sound speed c_s is large so that one can neglect the term $\partial_{tt} n$ in (A-7)), and obtain the generalized vector nonlinear Schrödinger equations for $\mathbf{E}$:

$$i\mathbf{E}_t + \Delta \mathbf{E} + |\mathbf{E}|^2 \mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}) = 0, \quad (A-9)$$

$$\Delta \mathbf{B} - i\eta \nabla \times \nabla \times (\mathbf{E} \wedge \overline{\mathbf{E}}) - \delta \mathbf{B} = 0. \quad (A-10)$$

In this paper we shall study the two-dimensional case of the above system. For $\mathbf{E} \in H^1(\mathbb{R}^2)$, after a Fourier transform of the equation (A-10), one obtains $\mathbf{B}(\mathbf{E}) \in L^2(\mathbb{R}^2)$ and

$$\mathbf{B}(\mathbf{E}) = \mathcal{F}^{-1}\left[\frac{i\eta}{|\xi|^2 + \delta}\left(\xi \wedge \left(\xi \wedge \mathcal{F}\left(\mathbf{E} \wedge \overline{\mathbf{E}}\right)\right)\right)\right], \quad (A-11)$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively (see [19] [21] [22] [23]). Due to the rotation invariance of equations (A-9)-(A-10), and in the two dimensional case, we may assume $\mathbf{E} = (E_1, E_2, 0)$ and $\xi = (\xi_1, \xi_2, 0)$ in previous discussions. Then equations (A-9)-(A-10) are equivalent to the following nonlinear Schrödinger system with nonlocal terms:

$$\begin{aligned} i\partial_t E_1 \quad &+\Delta E_1 + (|E_1|^2 + |E_2|^2)E_1 \\ &+E_2\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2)\right] = 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} i\partial_t E_2 \quad &+\Delta E_2 + (|E_1|^2 + |E_2|^2)E_2 \\ &+E_1\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{E_1}E_2 - E_1\overline{E_2})\right] = 0, \end{aligned} \tag{1.2}$$

where $(E_1, E_2) : (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C} \times \mathbb{C}$, η and δ are two constants with $\eta > 0$ and $\delta \geq 0$, $\overline{E_i}$ denotes the complex conjugate of E_i $(i = 1, 2)$. For equations (1.1)-(1.2), the initial data are taken to be:

$$E_1(0, x) = E_1^0(x), \quad E_2(0, x) = E_2^0(x),\ x \in \mathbb{R}^2. \tag{1.3}$$

Our main interest here is to understand the influence of the self-generation magnetic field on the solutions of (1.1)-(1.2). Let $E_1(t, x) = e^{iwt}u(x)$, $E_2(t, x) = e^{iwt}v(x)$ with $\omega > 0$. Here, $(u(x), v(x))$ is a pair of complex-valued functions, which satisfies the following nonlocal nonlinear elliptic equations:

$$\begin{aligned} -wu \quad &+\Delta u + (|u|^2 + |v|^2)u \\ &+v\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(u\overline{v})\right] - v\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{u}v)\right] = 0,\ x \in \mathbb{R}^2, \\ -wv \quad &+\Delta v + (|u|^2 + |v|^2)v \\ &+u\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{u}v)\right] - u\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(u\overline{v})\right] = 0,\ x \in \mathbb{R}^2. \end{aligned} \tag{1.4}$$

Here the nonlocal operator $\mathcal{F}^{-1}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}$ has the symbol $\sigma(\xi) = \frac{\eta|\xi|^2}{|\xi|^2+\delta}$ (see [13, 25, 26]). For related studies on these nonlocal nonlinear Schrödinger equations (1.1)-(1.2), we refer to our earlier works [11, 12]. In particular, we note the following conservation laws for the mass and energy:

$$\int_{\mathbb{R}^2}(|E_1|^2 + |E_2|^2)dx = \int_{\mathbb{R}^2}(|E_1^0|^2 + |E_2^0|^2)dx, \tag{1.5}$$

$$\begin{aligned} \mathcal{H}(E_1, E_2) \quad &= \int_{\mathbb{R}^2}(|\nabla E_1|^2 + |\nabla E_2|^2)dx \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^2}(|E_1|^4 + |E_2|^4)dx - \int_{\mathbb{R}^2}|E_1|^2|E_2|^2dx \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2\right)d\xi \\ &\quad +Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)}d\xi \\ &= \mathcal{H}(E_1^0, E_2^0). \end{aligned} \tag{1.6}$$

For the classical nonlinear Schrödinger equations without nonlocal terms, there have been many works on the local and global existence, finite time blowup and instability of standing waves (see [1, 15, 14, 27, 28, 20, 33, 37, 38]). For equations (1.1)-(1.2), however, the existence and instability of standing waves do not seem to be a trivial issue, and the sharp sufficient conditions of global existence in $\mathbb{R}^2$ are far from well-understood.

First of all, the presence of the nonlocal terms eliminates the invariance of inner scaling with respect to the space variables which is often a useful fact in known arguments. Secondly, the nonlocality may yield the non-zero energy of the ground states, which leads to new difficulties in establishing the instability of the standing waves for the nonlocal nonlinear Schrödinger equations (1.1)-(1.2). Based on some a priori estimates on these nonlocal terms and several new techniques, we obtain the existence and instability of the standing waves (Theorem 2.1, Theorem 3.1) as well as the sharp sufficient condition of global existence for the equations (1.1)-(1.2) in $\mathbb{R}^2$ (Theorem 4.1).

In contrast to the case of nonlinear Schrödinger equations without any external field effect, the nonlocal term due to the self-generation of magnetic field complicates, in particular, the applications of the virial identities (see the proof of Theorem 4.1). Moreover, from a physical point of view, self-generation magnetic field may alter the energy of the ground state and it may, in particular, become positive (Proposition 3.2). Hence it could lead to an additional force to act on the position (the orbit center) of this particle like solution. The latter may be a reason for the orbital instability of standing waves to occur (Theorem 3.1).

Our analysis may provide some preliminary understandings of the effect of the self-generation magnetic field in a cold plasma through this very specific static limiting case.

The following remark on the parameter δ is elementary but it is helpful to understand various arguments.

Remark 1.1. For $\delta > 0$, let $E_i(t,x) = \delta\psi_i(\delta^2 t, \delta x)$, $(i = 1,2)$, we can scale the parameter δ in equations (1.1)-(1.2) to unity, that is, $\psi_i(t,x)$ $(i = 1,2)$ solves the Cauchy problem below:

$$\begin{aligned} i\partial_t\psi_1 \quad &+\Delta\psi_1 + (|\psi_1|^2 + |\psi_2|^2)\psi_1 \\ &+\psi_2\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+1}\mathcal{F}(\psi_1\overline{\psi_2} - \overline{\psi_1}\psi_2)\right] = 0, \end{aligned} \tag{1.7}$$

$$\begin{aligned} i\partial_t\psi_2 \quad &+\Delta\psi_2 + (|\psi_1|^2 + |\psi_2|^2)\psi_2 \\ &+\psi_1\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+1}\mathcal{F}(\overline{\psi_1}\psi_2 - \psi_1\overline{\psi_2})\right] = 0, \end{aligned} \tag{1.8}$$

$$\psi_1(0,x) = \tfrac{1}{\delta}E_1^0(\tfrac{x}{\delta}), \quad \psi_2(0,x) = \tfrac{1}{\delta}E_2^0(\tfrac{x}{\delta}), \tag{1.9}$$

where $(\psi_1,\psi_2) : (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}\times\mathbb{C}$. In addition, we keep in this paper the parameter δ still in the equations (1.1)-(1.2) so that the perturbation or regularization nature would be clear. □

This paper is organized as follows. The existence of standing wave for (1.1)-(1.2) is shown in Section 2. The orbital instability of the standing wave is established in Section 3. At the last section, we derive a sharp sufficient condition of global existence to the Cauchy problem (1.1)-(1.3). Throughout this paper, we denote various positive constants by C, and employ the standard notations:

$$H_r^1(\mathbb{R}^2) = \{u, \quad \text{radially symmetric functions on } \mathbb{R}^2, \\ \|u\|_{H_r^1(\mathbb{R}^2)} = (\textstyle\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |u|^2 dx)^{\frac{1}{2}} < \infty\},$$

$$L_r^2(\mathbb{R}^2) = \{u, \quad \text{radially symmetric functions on } \mathbb{R}^2, \\ \|u\|_{L_r^2(\mathbb{R}^2)} = (\textstyle\int_{\mathbb{R}^2} |u|^2 dx)^{\frac{1}{2}} < \infty\}.$$

2 Existence of Standing Waves

For $\eta > 0$, $\delta \geq 0$ and $(u, v) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$, we define two functionals $J(u, v)$ and $I(u, v)$ as follows:

$$J(u,v) = \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2) dx + \tfrac{\omega}{2} \int_{\mathbb{R}^2} (|u|^2 + |v|^2) dx, \tag{2.1}$$

$$\begin{aligned} I(u,v) \quad &= \tfrac{1}{4} \textstyle\int_{\mathbb{R}^2} (|u|^4 + |v|^4) dx + \tfrac{1}{2} \int_{\mathbb{R}^2} |u|^2 |v|^2 dx \\ &+ \tfrac{1}{4} \textstyle\int_{\mathbb{R}^2} \frac{\eta |\xi|^2}{|\xi|^2 + \delta} (|\mathcal{F}(u\overline{v})|^2 + |\mathcal{F}(\overline{u}v)|^2) d\xi \\ &- \tfrac{1}{2} Re \textstyle\int_{\mathbb{R}^2} \frac{\eta |\xi|^2}{|\xi|^2 + \delta} \mathcal{F}(u\overline{v}) \overline{\mathcal{F}(\overline{u}v)} d\xi. \end{aligned} \tag{2.2}$$

We further define a set B by

$$B = \left\{ (u,v) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2), I(u,v) = 1 \right\}, \tag{2.3}$$

and a constrained variational problem by

$$d := \inf_{(u,v) \in B} J(u,v). \tag{2.4}$$

From $(u, v) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$, $\eta > 0$, $\delta \geq 0$, the Sobolev's embedding theorem and the related properties of Fourier transform, it follows that functionals $J(u, v)$ and $I(u, v)$ are both well defined. The result on the existence of standing wave for (1.1)-(1.2) is the following.

Theorem 2.1. For $\eta > 0$ and $\delta \geq 0$, there exists $(Q, P) \in B$ such that (Q, P) is a solution of the elliptic equations (1.4) and $(E_1(t, x), E_2(t, x)) = \left(e^{i\omega t} Q(x), e^{i\omega t} P(x)\right)$ is a standing wave solution to (1.1)-(1.2). In addition, $(Q(x), P(x))$ are functions of $|x|$ alone with exponential decay at infinity.

The main tools that we will need in order to show Theorem 2.1 are Hölder's inequality and the following lemma.

Lemma 2.1 [31, 34]. For $1 < \sigma < \infty$, the embedding $H_r^1(\mathbb{R}^2) \mapsto L_r^\sigma(\mathbb{R}^2)$ is compact. □

Proof of Theorem 2.1. Let $\left\{(u_n,v_n)\in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2), n\in\mathbb{N}\right\}$ be a minimizing sequence for (2.4), that is, $I(u_n,v_n)=1$ and

$$J(u_n,v_n)\to \inf_{(u,v)\in B} J(u,v)=d \text{ as } n\to\infty. \tag{2.5}$$

This implies that $\|u_n\|_{H_r^1(\mathbb{R}^2)}$ and $\|v_n\|_{H_r^1(\mathbb{R}^2)}$ are both bounded for all $n\in\mathbb{N}$. Thus, there exists a subsequence $\{u_{nk},k\in\mathbb{N}\}\subset\{u_n,n\in\mathbb{N}\}$ such that as $k\to\infty$,

$$u_{nk}\rightharpoonup u_\infty \text{ weakly in } H_r^1(\mathbb{R}^2),\quad u_{nk}\to u_\infty \ a.e. \text{ in } \mathbb{R}^2.$$

On the other hand, for a subsequence $\{v_{nk},k\in\mathbb{N}\}\subset\{v_n,n\in\mathbb{N}\}$, there also exists a subsequence $\{v_{nkm},m\in\mathbb{N}\}\subset\{v_{nk},k\in\mathbb{N}\}$ such that as $m\to\infty$,

$$v_{nkm}\rightharpoonup v_\infty \text{ weakly in } H_r^1(\mathbb{R}^2),\quad v_{nkm}\to v_\infty \text{ a.e. in } \mathbb{R}^2. \tag{2.6}$$

Meanwhile as $n\to\infty$,

$$u_{nkm}\rightharpoonup u_\infty \text{ weakly in } H_r^1(\mathbb{R}^2),\quad u_{nkm}\to u_\infty \text{ a.e. in } \mathbb{R}^2. \tag{2.7}$$

Then we can extract a subsequence $\{(u_{nkm},v_{nkm}),m\in\mathbb{N}\}$ from $\{(u_n,v_n),n\in\mathbb{N}\}$ such that (2.6) and (2.7) hold. We represent $\{(u_{nkm},v_{nkm}),m\in\mathbb{N}\}$ by $\{(u_n,v_n),n\in\mathbb{N}\}$ for simplicity. Hence as $n\to\infty$,

$$\begin{cases} u_n\rightharpoonup u_\infty, & v_n\rightharpoonup v_\infty \text{ weakly in } H_r^1(\mathbb{R}^2),\\ u_n\to u_\infty, & v_n\to v_\infty \text{ a.e. in } \mathbb{R}^2.\end{cases} \tag{2.8}$$

In view of Lemma 2.1 and (2.8), there then holds that as $n\to\infty$,

$$u_n\to u_\infty,\quad v_n\to v_\infty \text{ strongly in } L_r^4(\mathbb{R}^2). \tag{2.9}$$

Furthermore, $\eta>0$, $\delta\ge 0$, (2.6), (2.7) and Hölder's inequality yield the following estimates:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}(|u_n|^2|v_n|^2-|u_\infty|^2|v_\infty|^2)dx\right|\\
&\quad\le \int_{\mathbb{R}^2}\left||u_n|^2|v_n|^2-|u_\infty|^2|v_\infty|^2\right|dx\\
&\quad= \int_{\mathbb{R}^2}\left||u_n|^2(|v_n|^2-|v_\infty|^2)+|v_\infty|^2(|u_n|^2-|u_\infty|^2)\right|dx\\
&\quad\le \int_{\mathbb{R}^2}|u_n|^2\left||v_n|^2-|v_\infty|^2\right|dx+\int_{\mathbb{R}^2}|v_\infty|^2\left||u_n|^2-|u_\infty|^2\right|dx\\
&\quad\le \left(\int_{\mathbb{R}^2}|u_n|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}\left||v_n|^2-|v_\infty|^2\right|^2dx\right)^{\frac12}\\
&\qquad+\left(\int_{\mathbb{R}^2}|v_\infty|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}\left||u_n|^2-|u_\infty|^2\right|^2dx\right)^{\frac12}\\
&\quad\le \left(\int_{\mathbb{R}^2}|u_n|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}||v_n|+|v_\infty||^2\,||v_n|-|v_\infty||^2dx\right)^{\frac12}\\
&\qquad+\left(\int_{\mathbb{R}^2}|v_\infty|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}||u_n|+|u_\infty||^2\,||u_n|-|u_\infty||^2dx\right)^{\frac12}\\
&\quad\le \left(\int_{\mathbb{R}^2}|u_n|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}||v_n|+|v_\infty||^4dx\right)^{\frac14}\left(\int_{\mathbb{R}^2}||v_n|-|v_\infty||^4dx\right)^{\frac14}\\
&\qquad+\left(\int_{\mathbb{R}^2}|v_\infty|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}||u_n|+|u_\infty||^4dx\right)^{\frac14}\left(\int_{\mathbb{R}^2}||u_n|-|u_\infty||^4dx\right)^{\frac14}\\
&\quad\le \left(\int_{\mathbb{R}^2}|u_n|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}||v_n|+|v_\infty||^4dx\right)^{\frac14}\left(\int_{\mathbb{R}^2}|v_n-v_\infty|^4dx\right)^{\frac14}\\
&\qquad+\left(\int_{\mathbb{R}^2}|v_\infty|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}||u_n|+|u_\infty||^4dx\right)^{\frac14}\left(\int_{\mathbb{R}^2}|u_n-u_\infty|^4dx\right)^{\frac14},
\end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}|u_n|^4dx-\int_{\mathbb{R}^2}|u_\infty|^4dx\right| \\
&\quad=\left|\int_{\mathbb{R}^2}\left(|u_n|^2+|u_\infty|^2\right)\left(|u_n|^2-|u_\infty|^2\right)dx\right| \\
&\quad\le\int_{\mathbb{R}^2}\left||u_n|^2+|u_\infty|^2\right|\left||u_n|^2-|u_\infty|^2\right|dx \\
&\quad\le\left(\int_{\mathbb{R}^2}\left||u_n|^2+|u_\infty|^2\right|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\left||u_n|^2-|u_\infty|^2\right|^2dx\right)^{\frac{1}{2}} \\
&\quad\le\left(\int_{\mathbb{R}^2}\left||u_n|^2+|u_\infty|^2\right|^2dx\right)^{\frac{1}{2}} \\
&\qquad\cdot\left(\int_{\mathbb{R}^2}||u_n|+|u_\infty||^2\,||u_n|-|u_\infty||^2dx\right)^{\frac{1}{2}} \\
&\quad\le\left(\int_{\mathbb{R}^2}\left||u_n|^2+|u_\infty|^2\right|^2dx\right)^{\frac{1}{2}} \\
&\qquad\cdot\left(\int_{\mathbb{R}^2}||u_n|+|u_\infty||^4dx\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}^2}|u_n-u_\infty|^4dx\right)^{\frac{1}{4}},
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\left(|\mathcal{F}(u_n\overline{v_n})|^2-|\mathcal{F}(u_\infty\overline{v_\infty})|^2\right)d\xi\right| \\
&\quad\le\eta\int_{\mathbb{R}^2}\left||\mathcal{F}(u_n\overline{v_n})|^2-|\mathcal{F}(u_\infty\overline{v_\infty})|^2\right|d\xi \\
&\quad=\eta\int_{\mathbb{R}^2}||\mathcal{F}(u_n\overline{v_n})|+|\mathcal{F}(u_\infty\overline{v_\infty})||\,||\mathcal{F}(u_n\overline{v_n})|-|\mathcal{F}(u_\infty\overline{v_\infty})||\,d\xi \\
&\quad\le\eta\left(\int_{\mathbb{R}^2}||\mathcal{F}(u_n\overline{v_n})|+|\mathcal{F}(u_\infty\overline{v_\infty})||^2d\xi\right)^{\frac{1}{2}} \\
&\qquad\cdot\left(\int_{\mathbb{R}^2}|\mathcal{F}(u_n\overline{v_n})-\mathcal{F}(u_\infty\overline{v_\infty})|^2d\xi\right)^{\frac{1}{2}} \\
&\quad=\eta\left(\int_{\mathbb{R}^2}||\mathcal{F}(u_n\overline{v_n})|+|\mathcal{F}(u_\infty\overline{v_\infty})||^2d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|u_n\overline{v_n}-u_\infty\overline{v_\infty}|^2dx\right)^{\frac{1}{2}} \\
&\quad\le C\left(\int_{\mathbb{R}^2}||\mathcal{F}(u_n\overline{v_n})|+|\mathcal{F}(u_\infty\overline{v_\infty})||^2d\xi\right)^{\frac{1}{2}} \\
&\qquad.\left(\int_{\mathbb{R}^2}|u_n|^2|\overline{v_n}-\overline{v_\infty}|^2dx+\int_{\mathbb{R}^2}|\overline{v_\infty}|^2|u_n-u_\infty|^2dx\right)^{\frac{1}{2}} \\
&\quad\le C\left(\int_{\mathbb{R}^2}||\mathcal{F}(u_n\overline{v_n})|+|\mathcal{F}(u_\infty\overline{v_\infty})||^2d\xi\right)^{\frac{1}{2}}\left\{\left(\int_{\mathbb{R}^2}|u_n|^4|dx\right)^{\frac{1}{2}}\right. \\
&\qquad\left..\left(\int_{\mathbb{R}^2}|\overline{v_n}-\overline{v_\infty}|^4dx\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^2}|\overline{v_\infty}|^4dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|u_n-u_\infty|^4dx\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\left(|\mathcal{F}(\overline{u_n}v_n)|^2-|\mathcal{F}(\overline{u_\infty}v_\infty)|^2\right)d\xi\right| \\
&\quad\le C\left(\int_{\mathbb{R}^2}||\mathcal{F}(\overline{u_n}v_n)|+|\mathcal{F}(\overline{u_\infty}v_\infty)||^2d\xi\right)^{\frac{1}{2}}\left\{\left(\int_{\mathbb{R}^2}|\overline{u_n}|^4dx\right)^{\frac{1}{2}}\right. \\
&\qquad\left..\left(\int_{\mathbb{R}^2}|v_n-v_\infty|^4dx\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^2}|v_\infty|^4dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|\overline{u_n}-\overline{u_\infty}|^4dx\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&\left|Re\int_{\mathbb{R}^2}\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\left(\mathcal{F}(u_n\overline{v_n})\overline{\mathcal{F}(\overline{u_n}v_n)}-\mathcal{F}(u_\infty\overline{v_\infty})\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\right)d\xi\right|\\
&\quad\le\eta\int_{\mathbb{R}^2}\left|\mathcal{F}(u_n\overline{v_n})\overline{\mathcal{F}(\overline{u_n}v_n)}-\mathcal{F}(u_\infty\overline{v_\infty})\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\right|d\xi\\
&\quad=\eta\int_{\mathbb{R}^2}\Big|\mathcal{F}(u_n\overline{v_n})\left(\overline{\mathcal{F}(\overline{u_n}v_n)}-\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\right)\\
&\qquad\quad+\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\left(\mathcal{F}(u_n\overline{v_n})-\mathcal{F}(u_\infty\overline{v_\infty})\right)\Big|\,d\xi\\
&\quad\le\eta\int_{\mathbb{R}^2}\left|\mathcal{F}(u_n\overline{v_n})\left(\overline{\mathcal{F}(\overline{u_n}v_n)}-\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\right)\right|d\xi\\
&\qquad+\eta\int_{\mathbb{R}^2}\left|\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\left(\mathcal{F}(u_n\overline{v_n})-\mathcal{F}(u_\infty\overline{v_\infty})\right)\right|d\xi\\
&\quad\le\eta\left(\int_{\mathbb{R}^2}|\mathcal{F}(u_n\overline{v_n})|^2d\xi\right)^{\frac12}\left(\int_{\mathbb{R}^2}\left|\overline{\mathcal{F}(\overline{u_n}v_n)}-\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\right|^2d\xi\right)^{\frac12}\\
&\qquad+\eta\left(\int_{\mathbb{R}^2}\left|\overline{\mathcal{F}(\overline{u_\infty}v_\infty)}\right|^2d\xi\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\mathcal{F}(u_n\overline{v_n})-\mathcal{F}(u_\infty\overline{v_\infty})|^2\,d\xi\right)^{\frac12}\\
&\quad=\eta\left(\int_{\mathbb{R}^2}|u_n\overline{v_n}|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\overline{u_n}v_n-\overline{u_\infty}v_\infty|^2\,dx\right)^{\frac12}\\
&\qquad+\eta\left(\int_{\mathbb{R}^2}|\overline{u_\infty}v_\infty|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|u_n\overline{v_n}-u_\infty\overline{v_\infty}|^2\,dx\right)^{\frac12}\\
&\quad=\eta\left(\int_{\mathbb{R}^2}|u_n\overline{v_n}|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\overline{u_n}(v_n-v_\infty)+v_\infty(\overline{u_n}-\overline{u_\infty})|^2\,dx\right)^{\frac12}\\
&\qquad+\eta\left(\int_{\mathbb{R}^2}|\overline{u_\infty}v_\infty|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|u_n(\overline{v_n}-\overline{v_\infty})+\overline{v_\infty}(u_n-u_\infty)|^2\,dx\right)^{\frac12}\\
&\quad\le C\left(\int_{\mathbb{R}^2}|u_n\overline{v_n}|^2\,dx\right)^{\frac12}\\
&\qquad\quad\cdot\left[\int_{\mathbb{R}^2}|\overline{u_n}(v_n-v_\infty)|^2\,dx+\int_{\mathbb{R}^2}|v_\infty(\overline{u_n}-\overline{u_\infty})|^2\,dx\right]^{\frac12}\\
&\qquad+C\left(\int_{\mathbb{R}^2}|\overline{u_\infty}v_\infty|^2\,dx\right)^{\frac12}\\
&\qquad\quad\cdot\left[\int_{\mathbb{R}^2}|u_n(\overline{v_n}-\overline{v_\infty})|^2\,dx+\int_{\mathbb{R}^2}|\overline{v_\infty}(u_n-u_\infty)|^2\,dx\right]^{\frac12}\\
&\quad\le C\left(\int_{\mathbb{R}^2}|u_n\overline{v_n}|^2\,dx\right)^{\frac12}\Bigg[\left(\int_{\mathbb{R}^2}|\overline{u_n}|^4\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|v_n-v_\infty|^4\,dx\right)^{\frac12}\\
&\qquad\qquad\qquad+\left(\int_{\mathbb{R}^2}|v_\infty|^4\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\overline{u_n}-\overline{u_\infty}|^4\,dx\right)^{\frac12}\Bigg]^{\frac12}\\
&\qquad+C\left(\int_{\mathbb{R}^2}|\overline{u_\infty}v_\infty|^2\,dx\right)^{\frac12}\Bigg[\left(\int_{\mathbb{R}^2}|u_n|^4\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\overline{v_n}-\overline{v_\infty}|^4\,dx\right)^{\frac12}\\
&\qquad\qquad\qquad+\left(\int_{\mathbb{R}^2}|\overline{v_\infty}|^4\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|u_n-u_\infty|^4\,dx\right)^{\frac12}\Bigg]^{\frac12}.
\end{aligned}
\tag{2.14}
$$

Estimates (2.9)-(2.14) yield that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^2}|u_n|^2|v_n|^2dx=\int_{\mathbb{R}^2}|u_\infty|^2|v_\infty|^2dx, \tag{2.15}
$$

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(u_n\overline{v_n})|^2+|\mathcal{F}(\overline{u_n}v_n)|^2)d\xi\\
&\qquad=\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(u_\infty\overline{v_\infty})|^2+|\mathcal{F}(\overline{u_\infty}v_\infty)|^2)d\xi,
\end{aligned}
\tag{2.16}
$$

$$
\lim_{n\to\infty}Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(u_n\overline{v_n})\overline{\mathcal{F}(\overline{u_n}v_n)}d\xi
$$

$$= Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \overline{\mathcal{F}(\overline{u_\infty} v_\infty)} d\xi. \tag{2.17}$$

It is easy to check

$$I(u_\infty, v_\infty) = \lim_{n\to\infty} I(u_n, v_n) = 1,$$

$$J(u_\infty, v_\infty) \le \lim_{n\to\infty} J(u_n, v_n) = d.$$

We then achieve that

$$J(u_\infty, v_\infty) = d = \min_{(u,v)\in B} J(u, v). \tag{2.18}$$

Therefore, there exists a positive Lagrange multiplier Λ such that

$$\delta'_{\overline{u_\infty}} J(u_\infty, v_\infty) = \Lambda \delta'_{\overline{u_\infty}} I(u_\infty, v_\infty), \quad \delta'_{\overline{v_\infty}} J(u_\infty, v_\infty) = \Lambda \delta'_{\overline{v_\infty}} I(u_\infty, v_\infty), \tag{2.19}$$

where $\delta'_u T(u,v) = \frac{\partial}{\partial \tau} T(u + \tau\delta' u, v)|_{\tau=0}$. Furthermore, the following identities hold:

$$\begin{aligned} & Re \int_{\mathbb{R}^2} \tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \overline{\mathcal{F}(\overline{u_\infty} v_\infty)} d\xi \\ &= \tfrac{1}{2} \int_{\mathbb{R}^2} \tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \overline{\mathcal{F}(\overline{u_\infty} v_\infty)} d\xi \\ &\quad + \tfrac{1}{2} \int_{\mathbb{R}^2} \tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \overline{\mathcal{F}(u_\infty \overline{v_\infty})} \mathcal{F}(\overline{u_\infty} v_\infty) d\xi \\ &= \tfrac{1}{2} \int_{\mathbb{R}^2} \mathcal{F}^{-1} \left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \right] u_\infty \overline{v_\infty} d\xi \\ &\quad + \tfrac{1}{2} \int_{\mathbb{R}^2} \mathcal{F}^{-1} \left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(\overline{u_\infty} v_\infty) \right] \overline{u_\infty} v_\infty d\xi, \end{aligned} \tag{2.20}$$

$$\begin{aligned} & \tfrac{1}{2} \int_{\mathbb{R}^2} \tfrac{\eta|\xi|^2}{|\xi|^2+\delta} |\mathcal{F}(u_\infty \overline{v_\infty})|^2 d\xi + \tfrac{1}{2} \int_{\mathbb{R}^2} \tfrac{\eta|\xi|^2}{|\xi|^2+\delta} |\mathcal{F}(\overline{u_\infty} v_\infty)|^2 d\xi \\ &= \tfrac{1}{2} \int_{\mathbb{R}^2} \mathcal{F}^{-1} \left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \right] \overline{u_\infty} v_\infty d\xi \\ &\quad + \tfrac{1}{2} \int_{\mathbb{R}^2} \mathcal{F}^{-1} \left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(\overline{u_\infty} v_\infty) \right] u_\infty \overline{v_\infty} d\xi \\ &= \int_{\mathbb{R}^2} \mathcal{F}^{-1} \left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \right] \overline{u_\infty} v_\infty d\xi \\ &= \int_{\mathbb{R}^2} \mathcal{F}^{-1} \left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(\overline{u_\infty} v_\infty) \right] u_\infty \overline{v_\infty} d\xi. \end{aligned} \tag{2.21}$$

Taking $\delta' \overline{u_\infty} = \overline{u_\infty}$, $\delta' \overline{v_\infty} = \overline{v_\infty}$, we get

$$\begin{aligned} -\Delta u_\infty + \omega u_\infty = \Lambda \Big\{ & |u_\infty|^2 u_\infty + u_\infty |v_\infty|^2 \\ & + v_\infty \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \right] - v_\infty \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(\overline{u_\infty} v_\infty) \right] \Big\}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} -\Delta v_\infty + \omega v_\infty = \Lambda \Big\{ & |v_\infty|^2 u_\infty + v_\infty |u_\infty|^2 \\ & + u_\infty \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(\overline{u_\infty} v_\infty) \right] - u_\infty \mathcal{F}^{-1} \left[\frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u_\infty \overline{v_\infty}) \right] \Big\}. \end{aligned} \tag{2.23}$$

Direct calculation, (2.22) and (2.23) yield that

$$\begin{aligned} & \tfrac{1}{2} \int_{\mathbb{R}^2} (|\nabla u_\infty|^2 + |\nabla v_\infty|^2) dx + \tfrac{\omega}{2} \int_{\mathbb{R}^2} (|u_\infty|^2 + |v_\infty|^2) dx \\ &= \Lambda \Big\{ \int_{\mathbb{R}^2} \left[\tfrac{1}{2}(|u_\infty|^4 + |v_\infty|^4) + |u_\infty|^2 |v_\infty|^2 \right] dx \\ &\quad + \tfrac{1}{2} \int_{\mathbb{R}^2} \tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(u_\infty \overline{v_\infty})|^2 + |\mathcal{F}(\overline{u_\infty} v_\infty)|^2 \right) d\xi \\ &\quad - Re \int_{\mathbb{R}^2} \tfrac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(\overline{u_\infty} v_\infty) \overline{\mathcal{F}(u_\infty \overline{v_\infty})} d\xi \Big\}. \end{aligned} \tag{2.24}$$

Then $I(u_\infty, v_\infty) = 1$ implies that $\Lambda > 0$. Putting $Q = \Lambda^{\frac{1}{2}} u_\infty$, $P = \Lambda^{\frac{1}{2}} v_\infty$, thanks to (2.22) and (2.23), one obtains that (Q, P) solves

$$\begin{aligned} &-\omega Q + \Delta Q + (|Q|^2 + |P|^2)Q \\ &\quad + P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}(\mathcal{F}(Q\overline{P})\right] - P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}(\mathcal{F}(\overline{Q}P)\right] = 0, \qquad (2.25) \\ &-\omega P + \Delta P + (|Q|^2 + |P|^2)P \\ &\quad + Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}(\mathcal{F}(\overline{Q}P)\right] - Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}(\mathcal{F}(Q\overline{P})\right] = 0. \qquad (2.26) \end{aligned}$$

In addition, it is evident that $(Q(x), P(x))$ are functions of $|x|$ alone. By [4, 5], we obtain that $(Q(x), P(x))$ has exponential decay at infinity.

The proof of Theorem 2.1 is fulfilled. □

3 Orbital Instability of Standing Waves

We tackle here the orbital instability of the standing wave with ground state $(Q(x), P(x))$ for (1.1)-(1.2) in $\mathbb{R}^2$. We first establish two key propositions.

Proposition 3.1. For (Q, P) in Theorem 2.1, $\eta > 0$ and $\delta \geq 0$, there holds that $\mathcal{H}(Q, P) \geq 0$, where the energy functional $\mathcal{H}$ is defined by (1.6).

Proof. Since (Q, P) is a solution of (1.4), it solves

$$\begin{aligned} &-\omega Q + \Delta Q + (|Q|^2 + |P|^2)Q \\ &\quad + P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\right] - P\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{Q}P)\right] = 0, \qquad (3.1) \\ &-\omega P + \Delta P + (|Q|^2 + |P|^2)P \\ &\quad + Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{Q}P)\right] - Q\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\right] = 0. \qquad (3.2) \end{aligned}$$

Multiplying (3.1) by $x\nabla\overline{Q}$ and (3.2) by $x\nabla\overline{P}$, then integrating with respect to x on $\mathbb{R}^2$ and taking real part for the resulting equations, we obtain

$$\begin{aligned} &Re\int_{\mathbb{R}^2}(-\omega Q x\nabla\overline{Q} - \omega P x\nabla\overline{P})dx \\ &\quad + Re\int_{\mathbb{R}^2}(\Delta Q x\nabla\overline{Q} + \Delta P x\nabla\overline{P})dx \\ &\quad + Re\int_{\mathbb{R}^2}(|Q|^2 + |P|^2)(Q x\nabla\overline{Q} + P x\nabla\overline{P})dx \\ &\quad + Re\int_{\mathbb{R}^2} P x\nabla\overline{Q}\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\right]dx \qquad (3.3) \\ &\quad + Re\int_{\mathbb{R}^2} Q x\nabla\overline{P}\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{Q}P)\right]dx \\ &\quad - Re\int_{\mathbb{R}^2} P x\nabla\overline{Q}\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{Q}P)\right]dx \\ &\quad - Re\int_{\mathbb{R}^2} Q x\nabla\overline{P}\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\right]dx = 0. \end{aligned}$$

Direct calculation yields the following identities:

$$Re\int_{\mathbb{R}^2}(-\omega Qx\nabla\overline{Q}-\omega Px\nabla\overline{P})dx=\omega\int_{\mathbb{R}^2}(|Q|^2+|P|^2)dx, \qquad (E-1)$$

$$Re\int_{\mathbb{R}^2}(\Delta Qx\nabla\overline{Q}+\Delta Px\nabla\overline{P})dx=0, \qquad (E-2)$$

$$\begin{aligned}&Re\int_{\mathbb{R}^2}(|Q|^2+|P|^2)(Qx\nabla\overline{Q}+Px\nabla\overline{P})dx\\&=-\frac{1}{2}\int_{\mathbb{R}^2}(|Q|^4+|P|^4)dx-\int_{\mathbb{R}^2}|Q|^2|P|^2dx,\end{aligned} \qquad (E-3)$$

$$\begin{aligned}&Re\int_{\mathbb{R}^2}Px\nabla\overline{Q}\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\right]dx\\&+Re\int_{\mathbb{R}^2}Qx\nabla\overline{P}\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{Q}P)\right]dx\\&=-\frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi\\&+\frac{\delta}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi,\end{aligned} \qquad (E-4)$$

$$\begin{aligned}&Re\int_{\mathbb{R}^2}Px\nabla\overline{Q}\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{Q}P)\right]dx\\&+Re\int_{\mathbb{R}^2}Qx\nabla\overline{P}\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\right]dx\\&=-Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi\\&+\delta Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi.\end{aligned} \qquad (E-5)$$

Putting (E-1)-(E-5) into (3.3), one gets

$$\begin{aligned}&\omega\textstyle\int_{\mathbb{R}^2}(|Q|^2+|P|^2)dx\\&-\tfrac{1}{2}\textstyle\int_{\mathbb{R}^2}(|Q|^4+|P|^4)dx-\int_{\mathbb{R}^2}|Q|^2|P|^2dx\\&-\tfrac{1}{2}\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi\\&+\tfrac{\delta}{2}\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi\\&+Re\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi\\&-\delta Re\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi=0.\end{aligned} \qquad (3.4)$$

On the other hand, multiplying (3.1) by $\overline{Q}$ and (3.2) by $\overline{P}$, then integrating with respect to x on $\mathbb{R}^2$, we get

$$\begin{aligned}&-\omega\textstyle\int_{\mathbb{R}^2}(|Q|^2+|P|^2)dx-\int_{\mathbb{R}^2}(|\nabla Q|^2+|\nabla P|^2)dx\\&+\textstyle\int_{\mathbb{R}^2}(|Q|^4+|P|^4)dx+2\int_{\mathbb{R}^2}|Q|^2|P|^2dx\\&+\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi\\&-2Re\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi=0.\end{aligned} \qquad (3.5)$$

(1.6), (3.4) and (3.5) then yield that

$$-\mathcal{H}(Q,P)+\frac{\delta}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi$$
$$-\delta Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi=0. \tag{3.6}$$

It is easy to check

$$\left|Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi\right|$$
$$\leq\frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi, \tag{3.7}$$

which together with $\eta>0$, $\delta\geq 0$, (3.6) and (3.7) implies that $\mathcal{H}(Q,P)\geq 0$.

This completes the proof of Proposition 3.1. □

Proposition 3.2. For (Q,P) in Theorem 2.1, there exists $\lambda_0=\lambda_0(Q,P)\geq 1$ such that $\mathcal{H}(\lambda_0 Q,\lambda_0 P)=0$. Moreover,

$$\mathcal{H}(\lambda Q,\lambda P)<0\quad for\quad \lambda>\lambda_0;\quad \mathcal{H}(\lambda Q,\lambda P)>0\quad for\quad \lambda<\lambda_0.$$

Here, $\mathcal{H}(Q,P)$ is defined by (1.6).

Proof. By (1.6), it is easy to write the expression of $\mathcal{H}(\lambda Q,\lambda P)$ as follows:

$$\begin{aligned}&\mathcal{H}(\lambda Q,\lambda P)\\&\quad=\lambda^2\textstyle\int_{\mathbb{R}^2}(|\nabla Q|^2+|\nabla P|^2)dx\\&\qquad-\lambda^4\Big(\tfrac{1}{2}\textstyle\int_{\mathbb{R}^2}(|Q|^4+|P|^4)dx+\int_{\mathbb{R}^2}|Q|^2|P|^2dx\\&\qquad\qquad+\tfrac{1}{2}\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi\\&\qquad\qquad-Re\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi\Big)\\&\quad=\lambda^2A(Q,P)-\lambda^4D(Q,P)\\&\quad=\lambda^2[A(Q,P)-\lambda^2D(Q,P)],\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}A(Q,P)&=\textstyle\int_{\mathbb{R}^2}(|\nabla Q|^2+|\nabla P|^2)dx\\D(Q,P)&=\Big(\tfrac{1}{2}\textstyle\int_{\mathbb{R}^2}(|Q|^4+|P|^4)dx+\int_{\mathbb{R}^2}|Q|^2|P|^2dx\\&\qquad+\tfrac{1}{2}\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi\\&\qquad-Re\textstyle\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi\Big).\end{aligned}$$

Since

$$\left|Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q\overline{P})\overline{\mathcal{F}(\overline{Q}P)}d\xi\right|$$
$$\leq\frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q\overline{P})|^2+|\mathcal{F}(\overline{Q}P)|^2)d\xi,$$

one has $D(Q,P)>0$ by $(Q,P)\neq(0,0)$. On the other hand, one has

$$A(Q,P)-\lambda^2D(Q,P)\to A(Q,P)-D(Q,P)=\mathcal{H}(Q,P)\geq 0 \text{ as } \lambda\to 1,$$

and

$$A(Q,P) - \lambda^2 D(Q,P) < 0 \text{ as } \lambda \to \infty.$$

Hence, there always exists $\lambda_0 = \lambda_0(Q,P) \geq 1$ such that

$$A(Q,P) - \lambda_0^2 D(Q,P) = 0.$$

One can also check that

$$A(Q,P) - \lambda^2 D(Q,P) < 0 \text{ for } \lambda > \lambda_0,$$

$$A(Q,P) - \lambda^2 D(Q,P) > 0 \text{ for } \lambda < \lambda_0.$$

By (3.8), we finish the proof of Proposition 3.2. □

According to Proposition 3.1 and Proposition 3.2, we claim the following instability result of the standing wave for (1.1)-(1.2).

Theorem 3.1. Let $\eta > 0$ and $\delta \geq 0$. For λ_0 in Proposition 3.2, and for arbitrary $\varepsilon > 0$, there exists $(E_1^0, E_2^0) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ with

$$\|E_1^0 - \lambda_0 Q\|_{H_r^1(\mathbb{R}^2)} < \varepsilon, \qquad \|E_2^0 - \lambda_0 P\|_{H_r^1(\mathbb{R}^2)} < \varepsilon, \tag{3.9}$$

such that the solution $(E_1(t,x), E_2(t,x))$ to the Cauchy problem (1.1)-(1.3) exists on $[0,T)$, $(E_1(t,x), E_2(t,x)) \in C([0,T); H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2))$ and

$$\lim_{t\to T} \left(\|E_1\|_{H_r^1(\mathbb{R}^2)} + \|E_2\|_{H_r^1(\mathbb{R}^2)} \right) = +\infty. \tag{3.10}$$

Here, $(Q,P) \in B$ is given by Theorem 2.1.

Proof. Let $(E_1, E_2) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ be a solution to the equations (1.1)-(1.2) with initial data (E_1^0, E_2^0) on $[0,T)$. Put

$$J(t) = \int_{\mathbb{R}^2} |x|^2 (|E_1|^2 + |E_2|^2) dx. \tag{3.11}$$

A direct calculation gives that

$$\begin{aligned}
\frac{d^2}{dt^2} J(t) &= 8 \int_{\mathbb{R}^2} (|\nabla E_1|^2 + |\nabla E_2|^2) dx \\
&\quad -4 \int_{\mathbb{R}^2} (|E_1|^4 + |E_2|^4) dx - 8 \int_{\mathbb{R}^2} |E_1|^2 |E_2|^2) dx \\
&\quad -4 \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2 \right) d\xi \\
&\quad -4\delta \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2 \right) d\xi \\
&\quad +8Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi \\
&\quad +8\delta Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi \\
&= 8\mathcal{H}(E_1^0, E_2^0) \\
&\quad -4\delta \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2 \right) d\xi \\
&\quad +8\delta Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi.
\end{aligned} \tag{3.12}$$

(3.7) together with $\eta > 0$ and $\delta \geq 0$ then implies

$$\frac{d^2}{dt^2} J(t) \leq 8\mathcal{H}(E_{10}, E_{20}). \tag{3.13}$$

Let

$$E_1^0(x) = \lambda(\lambda_0 Q(x)), \quad E_2^0(x) = \lambda(\lambda_0 P(x)), \quad \lambda > 1. \tag{3.14}$$

Then

$$\begin{aligned}
\mathcal{H}(E_1^0, E_2^0) &= \textstyle\int_{\mathbb{R}^2} (|\nabla E_1^0|^2 + |\nabla E_2^0|^2) dx \\
&\quad - \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2} (|E_1^0|^4 + |E_2^0|^4) dx - \int_{\mathbb{R}^2} |E_1^0|^2 |E_2^0|^2) dx \\
&\quad - \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(E_1^0 \overline{E_2^0})|^2 + |\mathcal{F}(\overline{E_1^0} E_2^0)|^2 \right) d\xi \\
&\quad + Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(E_1^0 \overline{E_2^0}) \overline{\mathcal{F}(\overline{E_1^0} E_2^0)} d\xi \\
&= \lambda^2 \left\{ \textstyle\int_{\mathbb{R}^2} (|\nabla(\lambda_0 Q)|^2 + |\nabla(\lambda_0 P)|^2) dx \right. \\
&\quad - \tfrac{1}{2}\lambda^2 \textstyle\int_{\mathbb{R}^2} (|\lambda_0 Q|^4 + |\lambda_0 P|^4) dx - \lambda^2 \int_{\mathbb{R}^2} |\lambda_0 Q|^2 |\lambda_0 P|^2 dx \\
&\quad - \tfrac{1}{2}\lambda^2 \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(\lambda_0^2 Q\overline{P})|^2 + |\mathcal{F}(\lambda_0^2 \overline{Q} P)|^2 \right) d\xi \\
&\quad \left. + \lambda^2 Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(\lambda_0^2 Q\overline{P}) \overline{\mathcal{F}(\lambda_0^2 \overline{Q} P)} d\xi \right\}.
\end{aligned} \tag{3.15}$$

Combining (3.15), Proposition 3.2 with (3.7) yields that

$$\mathcal{H}(E_1^0, E_2^0) < \lambda^2 \mathcal{H}(\lambda_0 Q, \lambda_0 P) = 0. \tag{3.16}$$

In addition, Theorem 2.1 implies

$$(|x|E_1^0, |x|E_2^0) \in L_r^2(\mathbb{R}^2) \times L_r^2(\mathbb{R}^2). \tag{3.17}$$

Thus (3.10) follows from (3.13).

On the other hand, for arbitrary $\varepsilon > 0$, we can always take $\lambda > 1$ such that

$$\|E_1^0 - \lambda_0 Q\|_{H_r^1(\mathbb{R}^2)} = (\lambda - 1)\|\lambda_0 Q\|_{H_r^1(\mathbb{R}^2)} < \varepsilon,$$

$$\|E_2^0 - \lambda_0 P\|_{H_r^1(\mathbb{R}^2)} = (\lambda - 1)\|\lambda_0 P\|_{H_r^1(\mathbb{R}^2)} < \varepsilon.$$

The proof of Theorem 3.1 is accomplished. □

Theorem 3.2. For $\eta > 0$ and $\delta \geq 0$, let $(E_1^0, E_2^0) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ with

$$E_1^0(x) = \mu Q(x), \qquad E_2^0(x) = \mu P(x), \tag{3.18}$$

where $(Q, P) \in B$ is given by Theorem 2.1,

$$0 < \mu < \left\{ \frac{\|\phi\|^2_{L^2(\mathbb{R}^2)}}{2(1+\eta)\max\{\|Q\|^2_{L^2(\mathbb{R}^2)}, \|P\|^2_{L^2(\mathbb{R}^2)}\}} \right\}^{\frac{1}{2}}, \tag{3.19}$$

and ϕ is the ground state solution of the equation

$$\Delta u - u + u^3 = 0, \qquad u \in H^1(\mathbb{R}^2). \tag{3.20}$$

Then the Cauchy problem (1.1)-(1.3) admits a global solution.

Proof. We first recall an interpolation estimate in [34]:

$$\|f\|_{L^4(\mathbb{R}^2)}^4 \le C_{1,2}^4 \|\nabla f\|_{L^2(\mathbb{R}^2)}^2 \|f\|_{L^2(\mathbb{R}^2)}^2, \quad C_{1,2} = \left(\frac{2}{\|\phi\|_{L^2(\mathbb{R}^2)}^2}\right)^{\frac{1}{4}}, \tag{3.21}$$

where ϕ is the ground state solution of (3.20). (1.5),(1.6) and (3.21) yield that

$$\begin{aligned}
&\int_{\mathbb{R}^2}(|\nabla E_1|^2 + |\nabla E_2|^2)dx \le \mathcal{H}(E_1^0, E_2^0) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}(|E_1|^4 + |E_2|^4)dx + \int_{\mathbb{R}^2}|E_1|^2|E_2|^2dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2\right)d\xi \\
&\quad - Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)}d\xi \\
&\le \mathcal{H}(E_1^0, E_2^0) + (1+\eta)\int_{\mathbb{R}^2}(|E_1|^4 + |E_2|^4)dx \\
&\le \mathcal{H}(E_1^0, E_2^0) + (1+\eta)C_{1,2}^4\left(\|E_1^0\|_{L^2(\mathbb{R}^2)}^2\|\nabla E_1\|_{L^2(\mathbb{R}^2)}^2\right. \\
&\qquad\qquad \left. + \|E_2^0\|_{L^2(\mathbb{R}^2)}^2\|\nabla E_2\|_{L^2(\mathbb{R}^2)}^2\right) \\
&\le \mathcal{H}(E_1^0, E_2^0) + (1+\eta)C_{1,2}^4\max\{\|E_1^0\|_{L^2(\mathbb{R}^2)}^2, \|E_2^0\|_{L^2(\mathbb{R}^2)}^2\}\cdot \\
&\qquad\qquad \left(\|\nabla E_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_2\|_{L^2(\mathbb{R}^2)}^2\right) \\
&\le \mathcal{H}(E_1^0, E_2^0) + (1+\eta)C_{1,2}^4\mu^2\max\{\|Q\|_{L^2(\mathbb{R}^2)}^2, \|P\|_{L^2(\mathbb{R}^2)}^2\} \\
&\qquad\qquad \left(\|\nabla E_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_2\|_{L^2(\mathbb{R}^2)}^2\right).
\end{aligned} \tag{3.22}$$

Then one has

$$\begin{aligned}
&\left[1 - \frac{2(1+\eta)\mu^2}{\|\phi\|_{L^2(\mathbb{R}^2)}^2}\max\{\|Q\|_{L^2(\mathbb{R}^2)}^2, \|P\|_{L^2(\mathbb{R}^2)}^2\}\right] \\
&\qquad \left(\|\nabla E_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_2\|_{L^2(\mathbb{R}^2)}^2\right) \le \mathcal{H}(E_1^0, E_2^0).
\end{aligned}$$

This together with (3.19) manifests that

$$\left(\|\nabla E_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_2\|_{L^2(\mathbb{R}^2)}^2\right) \le C\mathcal{H}(E_1^0, E_2^0) \le C, \tag{3.23}$$

where C is a time-independent constant. Combining (1.5) with (3.23) implies that the Cauchy problem (1.1)-(1.3) admits a global solution $(E_1(t,x), E_2(t,x))$ in $H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$. □

Remark 3.1. For λ_0 in Theorem 3.1, let

$$\left\{\frac{\|\phi\|_{L^2(\mathbb{R}^2)}^2}{2(1+\eta)\mu^2\max\{\|Q\|_{L^2(\mathbb{R}^2)}^2, \|P\|_{L^2(\mathbb{R}^2)}^2\}}\right\}^{\frac{1}{2}} = \lambda_0.$$

Then the instability of the standing waves for (1.1)-(1.2) is sharp. But for

$$\left\{\frac{\|\phi\|_{L^2(\mathbb{R}^2)}^2}{2(1+\eta)\mu^2\max\{\|Q\|_{L^2(\mathbb{R}^2)}^2, \|P\|_{L^2(\mathbb{R}^2)}^2\}}\right\}^{\frac{1}{2}} < \lambda_0,$$

on the interval $\left[\left\{\frac{\|\phi\|_{L^2(\mathbb{R}^2)}^2}{2(1+\eta)\mu^2\max\{\|Q\|_{L^2(\mathbb{R}^2)}^2, \|P\|_{L^2(\mathbb{R}^2)}^2\}}\right\}^{\frac{1}{2}}, \lambda_0\right)$, whether the standing wave $(e^{i\omega t}Q(x), e^{i\omega t}P(x))$ for (1.1)-(1.2) is stable or not is unclear to our best knowledge. □

4 A Sharp Threshold for Global Existence in $\mathbb{R}^2$

In this section, we first establish the local well-posedness of the Cauchy problem (1.1)-(1.3), and then derive the variational structure for (1.4), through which we set up a sharp threshold for global existence to the Cauchy problem (1.1)-(1.3) motivated by the works [2, 3].

4.1 Local Well-posedness

The Cauchy problem (1.1)-(1.3) is equivalent to the integral equations:

$$\mathbf{E}(t) = U(t)\mathbf{E}^0(x) + i\int_0^t U(t-\tau)\big[(|E_1|^2+|E_2|^2)\mathbf{E}+\mathbf{K}(\mathbf{E})\big](\tau)\,d\tau, \tag{4.1}$$

where $U(t) = \exp\{it\Delta\}$ is the unitary semigroup generated by the free Schrödinger equation $i\mathbf{E}_t + \Delta\mathbf{E} = 0$ in the Hilbert space $H^s(\mathbb{R}^2)$ (with $s \in \mathbb{R}$), $\mathbf{E} = (E_1, E_2)$, $\mathbf{E}^0 = (E_1^0, E_2^0)$,

$$\begin{aligned}\mathbf{K}(\mathbf{E}) &= (K_1(\mathbf{E}), K_2(\mathbf{E}))\\ &= \left(E_2\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right], E_1\mathcal{F}^{-1}\left[\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(\overline{E_1}E_2-E_1\overline{E_2})\right]\right).\end{aligned}$$

Let $\mathbf{E}^j = (E_1^j, E_2^j)$ $(j = 1, 2)$ be two solutions of the integral equation (4.1), $\mathbf{K}^j(\mathbf{E}) = (K_1^j(\mathbf{E}), K_2^j(\mathbf{E})) = (K_1(\mathbf{E}^j), K_2(\mathbf{E}^j))$. By a direct calculation, we have

$$\begin{aligned}K_1^1(\mathbf{E}) - K_1^2(\mathbf{E}) &= K_1(\mathbf{E}^1) - K_1(\mathbf{E}^2)\\ &= E_2^1\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1^1\overline{E_2^1}-\overline{E_1^1}E_2^1)\right] - E_2^2\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1^2\overline{E_2^2}-\overline{E_1^2}E_2^2)\right]\\ &= (E_2^1-E_2^2)\mathcal{F}^{-1}\left[\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1^1\overline{E_2^1}-\overline{E_1^1}E_2^1)\right]\\ &\quad +E_2^2\mathcal{F}^{-1}\left\{\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}\left[(E_1^1-E_1^2)\overline{E_2^2}+E_2^1(\overline{E_1^2}-\overline{E_2^2})\right.\right.\\ &\qquad\left.\left.-\overline{E_1^1}(E_1^2-E_2^2)+E_2^2(\overline{E_2^1}-\overline{E_1^1})\right]\right\}.\end{aligned} \tag{4.2}$$

The expression for $K_2^1(\mathbf{E}) - K_2^2(\mathbf{E}) = K_2(\mathbf{E}^1) - K_2(\mathbf{E}^2)$ can be obtained similarly. Since $\frac{\eta|\xi|^2}{|\xi|^2+\delta} \le \eta$ by $\eta > 0$ and $\delta \ge 0$, one can establish the local well-posedness of the Cauchy problem (1.1)-(1.3) in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ by employing contraction mapping principle for the integral equation (4.1) (see [15, 17, 18, 21, 22, 23, 24, 29]). More precisely, we have

Proposition 4.1. For $\eta > 0$ and $\delta \ge 0$, let $(E_1^0, E_2^0) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Then the Cauchy problem (1.1)-(1.3) admits a solution $(E_1, E_2) \in X^1_{4,loc}([0,T)) \times X^1_{4,loc}([0,T))$ for some positive time $T = T(E_1^0, E_2^0)$, and for any $0 \le T_1 < T_2 < T$, the mapping $(E_1^0, E_2^0)(\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)) \mapsto (E_1, E_2)(t) \left(\in X_4^1([T_1,T_2]) \times X_4^1([T_1,T_2])\right)$ is continuous. Moreover, there holds that either $T = +\infty$, or $T < +\infty$ with

$$\lim_{t\to T}\left(\|E_1\|_{H^1(\mathbb{R}^2)} + \|E_2\|_{H^1(\mathbb{R}^2)}\right) = +\infty.$$

Here, for any interval $I \subset \mathbb{R}$, $0 \leq \frac{2}{q} = 1 - \frac{2}{\theta} < 1$, $s \in \mathbb{R}$,

$$X_\theta^s(I) = (C \cap L^\infty)(I; H^s) \cap L^q(I; H_\theta^s),$$
$$X_{\theta,loc}^s(I) = \{u;\, u \in X_\theta^s(J), \quad \forall\, J \subset\subset I\},$$
$$H_\theta^s = J_s(L^\theta), \quad J_s = (I - \Delta)^{-\frac{s}{2}}.$$

Proof. We omit the details since the essential arguments will be repeated for the analysis of the corresponding classical nonlinear Schrödinger equations. One may verify them according to the proof of Theorem 1.4 in Chapter 12 in [21]. □

In view of Proposition 4.1, using some ideas in [7, 15, 17, 18, 19, 24, 29], we can establish the local well-posedness for the Cauchy problem (1.1)-(1.3): For $\eta > 0$, $\delta \geq 0$ and $(E_1^0(x), E_2^0(x)) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, the Cauchy problem (1.1)-(1.3) admits a solution $(E_1, E_2) \in C\left([0,T); H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\right)$ for some $T \in (0, +\infty)$ with $T = +\infty$ or $T < +\infty$ with $\lim\limits_{t\to T}\left(\|E_1\|_{H^1(\mathbb{R}^2)} + \|E_2\|_{H^1(\mathbb{R}^2)}\right) = +\infty$. □

4.2 Variational Structures

For $(u,v) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$, we define two functionals $S(u,v)$ and $R(u,v)$ as

$$S(u,v) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + |\nabla v|^2)dx, \tag{4.3}$$

$$\begin{aligned} R(u,v) = {} & \frac{\omega}{2}\int_{\mathbb{R}^2}(|u|^2 + |v|^2)dx \\ & -\frac{1}{4}\int_{\mathbb{R}^2}(|u|^4 + |v|^4)dx - \frac{1}{2}\int_{\mathbb{R}^2}|u|^2|v|^2dx \\ & -\frac{1}{4}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(u\overline{v})|^2 + |\mathcal{F}(\overline{u}v)|^2)d\xi \\ & -\frac{\delta}{4}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(u\overline{v})|^2 + |\mathcal{F}(\overline{u}v)|^2)d\xi \\ & +\frac{1}{2}Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(u\overline{v})\overline{\mathcal{F}(\overline{u}v)}d\xi \\ & +\frac{1}{2}\delta Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(u\overline{v})\overline{\mathcal{F}(\overline{u}v)}d\xi. \end{aligned} \tag{4.4}$$

We further define a set M by

$$M = \left\{(u,v) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2) \setminus \{(0,0)\}, R(u,v) = 0\right\}. \tag{4.5}$$

According to (4.3),(4.4) and (4.5), we construct a constrained variational problem

$$d_1 := \inf_{(u,v)\in M} S(u,v). \tag{4.6}$$

Note that $(u,v) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$, $\eta > 0$, $\delta \geq 0$, and the Sobolev's embedding theorem, we know that the functionals $S(u,v)$ and $R(u,v)$ are both well defined. In (4.3) and (4.4), (u,v) is a pair of complex-valued functions for the purpose of understanding the relation between the solution (u,v) to (1.4) and the solution (E_1, E_2) to (1.1)-(1.2). We then achieve the following propositions and lemmas.

Proposition 4.2 For $\eta > 0$ and $\delta \geq 0$, there holds that $d_1 > 0$.

Proof. By (4.3)-(4.7), $\eta > 0$, $\delta \geq 0$ and $(u,v) \in M$, one has

$$\begin{aligned}
&\tfrac{\omega}{2}\int_{\mathbb{R}^2}(|u|^2+|v|^2)dx \\
&\quad \leq \tfrac{1}{4}\int_{\mathbb{R}^2}(|u|^4+|v|^4)dx + \tfrac{1}{2}\int_{\mathbb{R}^2}|u|^2|v|^2dx \\
&\quad\quad + \tfrac{1}{4}\int_{\mathbb{R}^2}\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(u\overline{v})|^2+|\mathcal{F}(\overline{u}v)|^2)d\xi \\
&\quad\quad + \tfrac{\delta}{4}\int_{\mathbb{R}^2}\tfrac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(u\overline{v})|^2+|\mathcal{F}(\overline{u}v)|^2)d\xi \\
&\quad\quad - \tfrac{1}{2}Re\int_{\mathbb{R}^2}\tfrac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(u\overline{v})\overline{\mathcal{F}(\overline{u}v)}d\xi \\
&\quad\quad - \tfrac{1}{2}\delta Re\int_{\mathbb{R}^2}\tfrac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(u\overline{v})\overline{\mathcal{F}(\overline{u}v)}d\xi \\
&\quad \leq C\int_{\mathbb{R}^2}(|u|^4+|v|^4)dx \\
&\quad \leq C\int_{\mathbb{R}^2}(|u|^2+|v|^2)dx\cdot\int_{\mathbb{R}^2}(|\nabla u|^2+|\nabla v|^2)dx.
\end{aligned}$$

This concludes that

$$\int_{\mathbb{R}^2}(|\nabla u|^2+|\nabla v|^2)dx \geq \frac{1}{C} > 0.$$

□

Proposition 4.3. For $\eta > 0$ and $\delta \geq 0$, there exists $(Q,P) \in M$ such that

$$S(Q,P) = d_1 = \min_{(u,v)\in M} S(u,v). \tag{4.7}$$

The following lemma is key to the proof of Proposition 4.3.

Lemma 4.1 For $\eta > 0$ and $\delta \geq 0$, the minimization problem

$$\begin{aligned}
d_1 &= \inf_{(u,v)\in M} S(u,v) \\
&= \inf\left\{S(u,v) : R(u,v) = 0, (u,v) \in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)\backslash\{(0,0)\}\right\},
\end{aligned} \tag{4.8}$$

is equivalent to

$$d_1 = \inf\left\{S(u,v) : R(u,v) \leq 0, (u,v) \in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)\backslash\{(0,0)\}\right\}. \tag{4.9}$$

Proof of Lemma 4.1. We choose $(u_0,v_0) \in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)$ such that $R(u_0,v_0) < 0$. Let $u_\lambda(x) = \lambda^{-1}u_0(\lambda x)$, $v_\lambda(x) = \lambda^{-1}v_0(\lambda x)$ with $\lambda \geq 1$. We then have

$$\begin{aligned}
&R(u_\lambda(x),v_\lambda(x)) \\
&\quad = \lambda^{-4}\Big\{\tfrac{\omega}{2}\int_{\mathbb{R}^2}(|u_0|^2+|v_0|^2)dx \\
&\quad\quad - \tfrac{1}{4}\lambda^{-2}\int_{\mathbb{R}^2}(|u_0|^4+|v_0|^4)dx - \tfrac{1}{2}\lambda^{-2}\int_{\mathbb{R}^2}|u_0|^2|v_0|^2dx \\
&\quad\quad - \tfrac{1}{4}\lambda^{-2}\int_{\mathbb{R}^2}\tfrac{\eta\lambda^2|\xi|^2}{\lambda^2|\xi|^2+\delta}(|\mathcal{F}(u_0\overline{v}_0)|^2+|\mathcal{F}(\overline{u}_0v_0)|^2)d\xi \\
&\quad\quad - \tfrac{\delta}{4}\lambda^{-2}\int_{\mathbb{R}^2}\tfrac{\eta\lambda^2|\xi|^2}{(\lambda^2|\xi|^2+\delta)^2}(|\mathcal{F}(u_0\overline{v}_0)|^2+|\mathcal{F}(\overline{u}_0v_0)|^2)d\xi \\
&\quad\quad + \tfrac{1}{2}\lambda^{-2}Re\int_{\mathbb{R}^2}\tfrac{\eta\lambda^2|\xi|^2}{\lambda^2|\xi|^2+\delta}\mathcal{F}(u_0\overline{v}_0)\overline{\mathcal{F}(\overline{u}_0v_0)}d\xi \\
&\quad\quad + \tfrac{\delta}{2}\lambda^{-2}Re\int_{\mathbb{R}^2}\tfrac{\eta\lambda^2|\xi|^2}{(\lambda^2|\xi|^2+\delta)^2}\mathcal{F}(u_0\overline{v}_0)\overline{\mathcal{F}(\overline{u}_0v_0)}d\xi\Big\} \\
&\quad = \lambda^{-4}R_\lambda^*(u_0,v_0).
\end{aligned} \tag{4.10}$$

Note that $R_\lambda^*(u_0,v_0) = R(u_0,v_0) < 0$ for $\lambda = 1$, and $R_\lambda^*(u_0,v_0) = \frac{\omega}{2}\int_{\mathbb{R}^2}(|u_0|^2 + |v_0|^2)dx > 0$ for λ tends to ∞, by continuity, there exists $\mu \in (1,\infty)$ such that

$R^*_\mu(u_0, v_0) = 0$. (4.10) then yields $R(u_\mu, v_\mu) = 0$ (i.e., $(u_\mu, v_\mu) \in M_1$). In addition, for $\mu \in (1, \infty)$ one achieves

$$\begin{aligned} S(u_\mu, v_\mu) &= \tfrac{1}{2}\mu^{-2} \textstyle\int_{\mathbb{R}^2}(|\nabla u_0|^2 + |\nabla v_0|^2)dx \\ &< \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2}(|\nabla u_0|^2 + |\nabla v_0|^2)dx \\ &= S(u_0, v_0). \end{aligned} \tag{4.11}$$

Therefore, (4.8) and (4.9) are equivalent. □

We are now in the position to prove Proposition 4.3.

Proof of Proposition 4.3. Combining (4.3) with (4.4) and (4.5) yields that $S(u, v) > 0$ on M. Let $\{(u_n, v_n), n \in \mathbb{N}\} \subset M$ be a minimizing sequence of (4.8), that is,

$$S(u_n, v_n) \to \inf_{(u,v)\in M} S(u, v) = d_1 \text{ as } n \to \infty. \tag{4.12}$$

Put

$$\lambda_n = \left[\frac{\omega}{2}\left(\|u_n\|^2_{L^2_r(\mathbb{R}^2)} + \|v_n\|^2_{L^2_r(\mathbb{R}^2)}\right)\right]^{\frac{1}{2}}, \tag{4.13}$$

$$Q_n(x) = u_n(\lambda_n x),\ P_n(x) = v_n(\lambda_n x),\ \text{for } \lambda_n \in [1, \infty), \tag{4.14}$$

and

$$Q_n(x) = u_n(x),\ P_n(x) = v_n(x),\ \text{for } \lambda_n \in (0, 1). \tag{4.15}$$

We now claim the following conclusion (Lemma 4.2) according to (4.13)-(4.15).

Lemma 4.2 Let $\eta > 0$ and $\delta \geq 0$. Then

(1) $(Q_n(x), P_n(x))$ is still a minimizing sequence for (4.8);

(2) $\|Q_n\|_{H^1_r(\mathbb{R}^2)}$ and $\|P_n\|_{H^1_r(\mathbb{R}^2)}$ are both bounded for any $\lambda_n \in (0, \infty)$.

Here, $(Q_n(x), P_n(x))$ is defined by (4.14) and (4.15).

Proof of Lemma 4.2. We prove this lemma by dividing the proof into two cases:

Case 1: $\lambda_n \in [1, \infty)$; **Case 2:** $\lambda_n \in (0, 1)$.

We first consider **Case 1:** $\lambda_n \in [1, \infty)$. In this case, let

$$Q_n(x) = u_n(\lambda_n x), \quad P_n(x) = v_n(\lambda_n x). \tag{4.16}$$

We then have

$$\|Q_n\|^2_{L^2_r(\mathbb{R}^2)} = \lambda_n^{-2}\|u_n\|^2_{L^2_r(\mathbb{R}^2)} = \frac{\|u_n\|^2_{L^2_r(\mathbb{R}^2)}}{\frac{\omega}{2}\left(\|u_n\|^2_{L^2_r(\mathbb{R}^2)} + \|v_n\|^2_{L^2_r(\mathbb{R}^2)}\right)} \leq \frac{2}{\omega}, \tag{4.17}$$

$$\|P_n\|^2_{L^2_r(\mathbb{R}^2)} = \lambda_n^{-2}\|v_n\|^2_{L^2_r(\mathbb{R}^2)} = \frac{\|v_n\|^2_{L^2_r(\mathbb{R}^2)}}{\frac{\omega}{2}\left(\|u_n\|^2_{L^2_r(\mathbb{R}^2)} + \|v_n\|^2_{L^2_r(\mathbb{R}^2)}\right)} \leq \frac{2}{\omega}. \tag{4.18}$$

It is easy to check that

$$\begin{aligned} &S(Q_n(x), P_n(x)) \\ &\quad = S(u_n(\lambda_n x), v_n(\lambda_n x)) \\ &\quad = \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2}(|\nabla u_n|^2 + |\nabla v_n|^2)dx \\ &\quad = S(u_n, v_n), \end{aligned} \tag{4.19}$$

$$\begin{aligned}
&R(Q_n(x), P_n(x)) \\
&\quad = R(u_n(\lambda_n x), v_n(\lambda_n x)) \\
&\quad = \tfrac{\omega}{2}\lambda_n^{-2} \int_{\mathbb{R}^2} (|u_n|^2 + |v_n|^2)dx \\
&\quad\quad - \tfrac{1}{4}\lambda_n^{-2} \int_{\mathbb{R}^2} (|u_n|^4 + |v_n|^4)dx - \tfrac{1}{2}\lambda_n^{-2} \int_{\mathbb{R}^2} |u_n|^2|v_n|^2 dx \\
&\quad\quad - \tfrac{1}{4}\lambda_n^{-2} \int_{\mathbb{R}^2} \tfrac{\eta\lambda_n^2|\xi|^2}{\lambda_n^2|\xi|^2+\delta} (|\mathcal{F}(u_n\overline{v}_n)|^2 + |\mathcal{F}(\overline{u}_n v_n)|^2)d\xi \\
&\quad\quad - \tfrac{\delta}{4}\lambda_n^{-2} \int_{\mathbb{R}^2} \tfrac{\eta\lambda_n^2|\xi|^2}{(\lambda_n^2|\xi|^2+\delta)^2} (|\mathcal{F}(u_n\overline{v}_n)|^2 + |\mathcal{F}(\overline{u}_n v_n)|^2)d\xi \\
&\quad\quad + \tfrac{1}{2}\lambda_n^{-2} Re \int_{\mathbb{R}^2} \tfrac{\eta\lambda_n^2|\xi|^2}{\lambda_n^2|\xi|^2+\delta} \mathcal{F}(u_n\overline{v}_n)\overline{\mathcal{F}(\overline{u}_n v_n)}d\xi \\
&\quad\quad + \tfrac{\delta}{2}\lambda_n^{-2} Re \int_{\mathbb{R}^2} \tfrac{\eta\lambda_n^2|\xi|^2}{(\lambda_n^2|\xi|^2+\delta)^2} \mathcal{F}(u_n\overline{v}_n)\overline{\mathcal{F}(\overline{u}_n v_n)}d\xi.
\end{aligned} \tag{4.20}$$

Let $G(\lambda_n) = \frac{\lambda_n^2|\xi|^2}{\lambda_n^2|\xi|^2+\delta} + \frac{\delta\lambda_n^2|\xi|^2}{(\lambda_n^2|\xi|^2+\delta)^2} = \frac{\lambda_n^4|\xi|^4+2\delta\lambda_n^2|\xi|^2}{(\lambda_n^2|\xi|^2+\delta)^2}$. We have $G'(\lambda_n) = \frac{4\delta^2\lambda_n|\xi|^2}{(\lambda_n^2|\xi|^2+\delta)^3} \geq 0$, and thus $G(\lambda_n)$ is an increasing function with respect to $\lambda_n \in [1, \infty)$. Note that

$$\begin{aligned}
&\left| Re \int_{\mathbb{R}^2} \left(\tfrac{\lambda_n^2|\xi|^2}{\lambda_n^2|\xi|^2+\delta} + \tfrac{\delta\lambda_n^2|\xi|^2}{(\lambda_n^2|\xi|^2+\delta)^2} \right) \mathcal{F}(u_n\overline{v}_n)\overline{\mathcal{F}(\overline{u}_n v_n)}d\xi \right| \\
&\quad \leq \tfrac{1}{2} \int_{\mathbb{R}^2} \left(\tfrac{\lambda_n^2|\xi|^2}{\lambda_n^2|\xi|^2+\delta} + \tfrac{\delta\lambda_n^2|\xi|^2}{(\lambda_n^2|\xi|^2+\delta)^2} \right) (|\mathcal{F}(u_n\overline{v}_n)|^2 + |\mathcal{F}(\overline{u}_n v_n)|^2)d\xi.
\end{aligned} \tag{4.21}$$

Combining (4.21) with $\eta > 0$, $\delta \geq 0$ and (4.20) yields that

$$R(Q_n(x), P_n(x)) \leq \lambda_n^{-2} R(u_n, v_n) = 0. \tag{4.22}$$

Hence, $\{(Q_n, P_n)\}$ is still a minimizing sequence for (4.9) by Lemma 4.1. From (4.14), (4.17) and (4.18) it follows that $\|Q_n\|_{H_r^1(\mathbb{R}^2)}$ and $\|P_n\|_{H_r^1(\mathbb{R}^2)}$ are both bounded for all $n \in \mathbb{N}$.

We next consider **Case 2** : $\lambda_n \in (0, 1)$. Let $Q_n(x) = u_n(x)$, $P_n(x) = v_n(x)$. (4.15) implies that

$$\|Q_n\|_{L_r^2(\mathbb{R}^2)} \leq \frac{2}{\omega}, \quad \|P_n\|_{L_r^2(\mathbb{R}^2)} \leq \frac{2}{\omega}. \tag{4.23}$$

This together with (4.14) completes the proof of Lemma 4.2. □

We now continue to prove Proposition 4.3.

By Lemma 4.2, it follows that there exists a subsequence $\{Q_{nk}, k \in \mathbb{N}\} \subset \{Q_n, n \in \mathbb{N}\}$ such that as $k \to \infty$,

$$Q_{nk} \rightharpoonup Q_\infty \text{ weakly in } H_r^1(\mathbb{R}^2), \quad Q_{nk} \to Q_\infty \text{ a.e. in } \mathbb{R}^2. \tag{4.24}$$

Now for $\{P_{nk}, k \in \mathbb{N}\} \subset \{P_n, n \in \mathbb{N}\}$, there also exists a subsequence $\{P_{nkm}, m \in \mathbb{N}\} \subset \{P_{nk}, k \in \mathbb{N}\}$ such that as $m \to \infty$,

$$P_{nkm} \rightharpoonup P_\infty \text{ weakly in } H_r^1(\mathbb{R}^2), \quad P_{nkm} \to P_\infty \text{ a.e. in } \mathbb{R}^2. \tag{4.25}$$

Naturally, we can also verify that as $m \to \infty$,

$$Q_{nkm} \rightharpoonup Q_\infty \text{ weakly in } H_r^1(\mathbb{R}^2), \quad Q_{nkm} \to Q_\infty \text{ a.e. in } \mathbb{R}^2. \tag{4.26}$$

Therefore, we can extract a subsequence $\{(Q_{nkm}, P_{nkm}), m \in \mathbb{N}\}$ from $\{(Q_n, P_n), n \in \mathbb{N}\}$ such that (4.25) and (4.26) hold. We still denote $\{(Q_{nkm}, P_{nkm}), m \in \mathbb{N}\}$ by

$\{(Q_n, P_n), n \in \mathbb{N}\}$ for simplicity. That is, as $n \to \infty$,

$$\begin{aligned} &Q_n \rightharpoonup Q_\infty,\ P_n \rightharpoonup P_\infty \text{ weakly in } H_r^1(\mathbb{R}^2),\\ &Q_n \to Q_\infty,\ P_n \to P_\infty \text{ a.e. in } \mathbb{R}^2. \end{aligned} \tag{4.27}$$

Combining Lemma 2.1 with (4.27) yields that as $n \to \infty$,

$$Q_n \to Q_\infty,\ P_n \to P_\infty \text{ strongly in } L_r^4(\mathbb{R}^2). \tag{4.28}$$

By $\left|\frac{\delta\eta|\xi|^2}{(|\xi|^2+\delta)^2}\right| \le \eta \left|\frac{(|\xi|^2+\delta)(|\xi|^2+\delta)}{(|\xi|^2+\delta)^2}\right| \le \eta$ with $\eta > 0$ and $\delta \ge 0$, one gets following estimates:

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (|Q_n|^4 + |P_n|^4)dx = \int_{\mathbb{R}^2} (|Q_\infty|^4 + |P_\infty|^4)dx, \tag{4.29}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |Q_n|^2|P_n|^2 dx = \int_{\mathbb{R}^2} |Q_\infty|^2|P_\infty|^2 dx, \tag{4.30}$$

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q_n\overline{P_n})|^2 + |\mathcal{F}(\overline{Q_n}P_n)|^2)d\xi\\ &\quad = \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q_\infty\overline{P_\infty})|^2 + |\mathcal{F}(\overline{Q_\infty}P_\infty)|^2)d\xi, \end{aligned} \tag{4.31}$$

$$\begin{aligned} &\lim_{n\to\infty} -\delta \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(Q_n\overline{P_n})|^2 + |\mathcal{F}(\overline{Q_n}P_n)|^2)d\xi\\ &\quad = -\delta \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(Q_\infty\overline{P_\infty})|^2 + |\mathcal{F}(\overline{Q_\infty}P_\infty)|^2)d\xi, \end{aligned} \tag{4.32}$$

$$\begin{aligned} &\lim_{n\to\infty} Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q_n\overline{P_n})\overline{\mathcal{F}(\overline{Q_n}P_n)}d\xi\\ &\quad = Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q_\infty\overline{P_\infty})\overline{\mathcal{F}(\overline{Q_\infty}P_\infty)}d\xi. \end{aligned} \tag{4.33}$$

$$\begin{aligned} &\lim_{n\to\infty} -\delta Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(Q_n\overline{P_n})\overline{\mathcal{F}(\overline{Q_n}P_n)}d\xi\\ &\quad = -\delta Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(Q_\infty\overline{P_\infty})\overline{\mathcal{F}(\overline{Q_\infty}P_\infty)}d\xi. \end{aligned} \tag{4.34}$$

In light of (4.17), (4.18) and (4.23), one has

$$0 < \frac{\omega}{2} \int_{\mathbb{R}^2} (|Q_n|^2 + |P_n|^2)dx \le 2,$$

which implies that there exists $0 < c \le 2$ such that

$$\frac{\omega}{2} \int_{\mathbb{R}^2} (|Q_n|^2 + |P_n|^2)dx = c. \tag{4.35}$$

Combining (4.22) and (4.23) with (4.29)-(4.35) then yields that

$$\begin{aligned} &\tfrac{1}{4} \textstyle\int_{\mathbb{R}^2} (|Q_\infty|^4 + |P_\infty|^4)dx + \tfrac{1}{2} \int_{\mathbb{R}^2} |Q_\infty|^2|P_\infty|^2 dx\\ &\quad + \tfrac{1}{4} \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta}(|\mathcal{F}(Q_\infty\overline{P_\infty})|^2 + |\mathcal{F}(\overline{Q_\infty}P_\infty)|^2)d\xi\\ &\quad + \tfrac{\delta}{4} \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}(|\mathcal{F}(Q_\infty\overline{P_\infty})|^2 + |\mathcal{F}(\overline{Q_\infty}P_\infty)|^2)d\xi\\ &\quad - \tfrac{1}{2} Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(Q_\infty\overline{P_\infty})\overline{\mathcal{F}(\overline{Q_\infty}P_\infty)}d\xi\\ &\quad - \tfrac{1}{2}\delta Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(Q_\infty\overline{P_\infty})\overline{\mathcal{F}(\overline{Q_\infty}P_\infty)}d\xi \ge c. \end{aligned} \tag{4.36}$$

Note that for $\theta \in \{1, 2\}$, $\eta > 0$ and $\delta \geq 0$,

$$\left| Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^\theta} \mathcal{F}(Q_\infty \overline{P_\infty}) \overline{\mathcal{F}(\overline{Q_\infty} P_\infty)} d\xi \right|$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^\theta} (|\mathcal{F}(Q_\infty \overline{P_\infty})|^2 + |\mathcal{F}(\overline{Q_\infty} P_\infty)|^2) d\xi,$$

which together with (4.36) yields that $(Q_\infty, P_\infty) \neq (0, 0)$. Let

$$Q(x) = Q_\infty(\mu x), \quad P(x) = P_\infty(\mu x), \quad \text{with } \mu \geq 1. \tag{4.37}$$

According to (4.22) and (4.23), we have $R(Q, P) \leq 0$ and

$$S(Q,P) = S(Q_\infty(\mu x), P_\infty(\mu x)) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla Q_\infty|^2 + |\nabla P_\infty|^2) dx$$

$$= S(Q_\infty, P_\infty) \leq \lim_{n\to\infty} S(Q_n, P_n) = d_1. \tag{4.38}$$

Collecting Lemma 4.1 and (4.38) together yields that

$$S(Q,P) = d_1 = \min_{(u,v)\in M} S(u,v), \tag{4.39}$$

which accomplishes the proof of Proposition 4.3. □

4.3 Sharp threshold of Global Existence

For $(u, v) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ we define an auxiliary functional $\mathcal{H}_1(u, v)$ as

$$\begin{aligned} \mathcal{H}_1(u,v) = & \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2) dx + \tfrac{\omega}{2} \textstyle\int_{\mathbb{R}^2} (|u|^2 + |v|^2) dx \\ & - \tfrac{1}{4} \textstyle\int_{\mathbb{R}^2} (|u|^4 + |v|^4) dx - \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2} |u|^2 |v|^2 dx \\ & - \tfrac{1}{4} \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(u\overline{v})|^2 + |\mathcal{F}(\overline{u}v)|^2 \right) d\xi \\ & + \tfrac{1}{2} Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(u\overline{v}) \overline{\mathcal{F}(\overline{u}v)} d\xi. \end{aligned} \tag{4.40}$$

We then have

Theorem 4.1. For $\eta > 0$ and $\delta \geq 0$, let $(E_1^0(x), E_2^0(x)) \in H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ satisfy

$$\mathcal{H}_1(E_1^0, E_2^0) < d_1. \tag{4.41}$$

Then we have

1. If $(|x|E_1^0, |x|E_2^0) \in L_r^2(\mathbb{R}^2) \times L_r^2(\mathbb{R}^2)$ and

$$R(E_1^0, E_2^0) < 0, \tag{4.42}$$

then the solution $(E_1(t,x), E_2(t,x))$ to the Cauchy problem (1.1)-(1.3) blows up in finite time.

2. If

$$R(E_1^0, E_2^0) > 0, \tag{4.43}$$

then the solution $(E_1(t,x), E_2(t,x))$ to the Cauchy problem (1.1)-(1.3) exists globally on $t \in [0, +\infty)$. Moreover, $(E_1(t,x), E_2(t,x))$ satisfies

$$\|\nabla E_1\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2\|^2_{L^2(\mathbb{R}^2)} < 2d_1. \tag{4.44}$$

Here, $R(u,v)$ and d_1 are defined by (4.4) and (4.5), respectively.

The following proposition plays an important role for the proof of Theorem 4.1.

Proposition 4.4 For $\eta > 0$ and $\delta \geq 0$, let

$$K_1 = \left\{(u,v) \in H^1_r(\mathbb{R}^2) \times H^1_r(\mathbb{R}^2), R(u,v) > 0, \mathcal{H}_1(u,v) < d_1\right\}$$

$$K_2 = \left\{(u,v) \in H^1_r(\mathbb{R}^2) \times H^1_r(\mathbb{R}^2), R(u,v) < 0, \mathcal{H}_1(u,v) < d_1\right\}.$$

Then K_1 and K_2 are two invariant manifolds generated by the Cauchy problem (1.1)-(1.3).

Proof. Let $(E_1^0(x), E_2^0(x)) \in K_1$ and $(E_1(t), E_2(t))$ be the solution to the Cauchy problem (1.1)-(1.3). By (1.5) and (1.6), one has

$$\mathcal{H}_1(E_1(t), E_2(t)) = \mathcal{H}_1(E_1^0, E_2^0), \quad t \in [0,T). \tag{4.45}$$

$\mathcal{H}_1(E_1^0, E_2^0) < d_1$ then implies that

$$\mathcal{H}_1(E_1(t), E_2(t)) < d_1, \quad t \in [0,T). \tag{4.46}$$

We now need to prove that for $t \in [0,T)$,

$$R(E_1(t), E_2(t)) > 0. \tag{4.47}$$

Otherwise, note that $\eta > 0$, $\delta \geq 0$ and for $\theta = 1, 2$

$$\left| Re \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^\theta} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi \right|$$
$$\leq \frac{1}{2} \int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^\theta} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2\right) d\xi, \tag{4.48}$$

(4.3), (4.4) and (4.41) yield that

$$S(E_1, E_2) + R(E_1, E_2) \leq \mathcal{H}_1(E_1, E_2) < d_1. \tag{4.49}$$

On the other hand, in view of $R(E_1^0, E_2^0) > 0$, by continuity, there would exist a $t^* > 0$ such that

$$R(E_1(t^*), E_2(t^*)) = 0, \tag{4.50}$$

which implies that $(E_1(t^*), E_2(t^*)) \in M$. From (4.46), (4.49) and (4.50) it follows that

$$S(E_1(t^*), E_2(t^*)) < d_1 \quad \text{with} \quad (E_1(t^*), E_2(t^*)) \in M. \tag{4.51}$$

This is contradictory to (4.6), Proposition 4.2 and Proposition 4.3. Thus, K_1 is an invariant manifold generated by the Cauchy problem (1.1)-(1.3).

Similarly, we can prove that K_2 is also an invariant manifold generated by the Cauchy problem (1.1)-(1.3). □

We are now in the position to prove Theorem 4.1.

Proof of Theorem 4.1.

From (4.40) and (4.41) it follows that $(E_1^0, E_2^0) \in K_2$. Then $(E_1(t), E_2(t)) \in K_2$ for $t \in [0, T)$ by Proposition 4.4, that is,

$$R(E_1(t), E_2(t)) < 0, \quad \mathcal{H}_1(E_1(t), E_2(t)) < d_1, \quad t \in [0, T), \tag{4.52}$$

where $(E_1(t), E_2(t))$ is the solution to the Cauchy problem (1.1)-(1.3). Since $(|x|E_1^0, |x|E_2^0) \in L_r^2(\mathbb{R}^2) \times L_r^2(\mathbb{R}^2)$, by Ginibre-Velo [15], we get $(|x|E_1, |x|E_2) \in L_r^2(\mathbb{R}^2) \times L_r^2(\mathbb{R}^2)$. A direct computation then yields that

$$\begin{aligned}
&\tfrac{d^2}{dt^2} \textstyle\int_{\mathbb{R}^2} (|x|^2|E_1|^2 + |x|^2|E_2|^2)dx \\
&\quad = 8 \textstyle\int_{\mathbb{R}^2} (|\nabla E_1|^2 + |\nabla E_2|^2)dx \\
&\qquad -4 \textstyle\int_{\mathbb{R}^2} (|E_1|^4 + |E_2|^4)dx - 8 \int_{\mathbb{R}^2} |E_1|^2|E_2|^2 dx \\
&\qquad -4 \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2\right) d\xi \\
&\qquad -4\delta \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2\right) d\xi \\
&\qquad +8Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi \\
&\qquad +8\delta Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi.
\end{aligned} \tag{4.53}$$

Combining (1.6) with (4.53) gives that

$$\begin{aligned}
&\tfrac{d^2}{dt^2} \textstyle\int_{\mathbb{R}^2} (|x|^2|E_1|^2 + |x|^2|E_2|^2)dx \\
&\quad = 8\mathcal{H}(E_1^0, E_2^0) \\
&\qquad -4\delta \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2\right) d\xi \\
&\qquad +8\delta Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi.
\end{aligned} \tag{4.54}$$

On the other hand, (4.4), $R(E_1^0, E_2^0) < 0$, $\eta > 0$, $\delta \geq 0$ and

$$\begin{aligned}
&\left| Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(E_1\overline{E_2})\overline{\mathcal{F}(\overline{E_1}E_2)} d\xi \right| \\
&\qquad \leq \tfrac{1}{2} \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(E_1\overline{E_2})|^2 + |\mathcal{F}(\overline{E_1}E_2)|^2\right) d\xi,
\end{aligned} \tag{4.55}$$

yield that there exists $0 < \lambda < 1$ such that $R(\lambda E_1^0, \lambda E_2^0) = 0$, that is,

$$\begin{aligned}
&\tfrac{\omega}{2} \textstyle\int_{\mathbb{R}^2} (|E_1^0|^2 + |E_2^0|^2)dx \\
&\qquad -\tfrac{1}{4}\lambda^2 \textstyle\int_{\mathbb{R}^2} (|E_1^0|^4 + |E_2^0|^4)dx - \tfrac{1}{2}\lambda^2 \int_{\mathbb{R}^2} |E_1^0|^2|E_2^0|^2 dx \\
&\qquad -\tfrac{1}{4}\lambda^2 \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \left(|\mathcal{F}(E_1^0\overline{E_2^0})|^2 + |\mathcal{F}(\overline{E_1^0}E_2^0)|^2\right) d\xi \\
&\qquad -\tfrac{\delta}{4}\lambda^2 \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \left(|\mathcal{F}(E_1^0\overline{E_2^0})|^2 + |\mathcal{F}(\overline{E_1^0}E_2^0)|^2\right) d\xi \\
&\qquad +\tfrac{1}{2}\lambda^2 Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{|\xi|^2+\delta} \mathcal{F}(E_1^0\overline{E_2^0})\overline{\mathcal{F}(\overline{E_1^0}E_2^0)} d\xi \\
&\qquad +\tfrac{\delta}{2}\lambda^2 Re \textstyle\int_{\mathbb{R}^2} \frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2} \mathcal{F}(E_1^0\overline{E_2^0})\overline{\mathcal{F}(\overline{E_1^0}E_2^0)} d\xi = 0.
\end{aligned} \tag{4.56}$$

From (4.3), (4.4), (4.6), Proposition 4.2 and Proposition 4.3 it follows that

$$S(\lambda E_1^0, \lambda E_2^0) \geq d_1. \tag{4.57}$$

On the other hand, $\mathcal{H}_1(E_1^0, E_2^0) < d_1$ and (4.57) imply $S(\lambda E_1^0, \lambda E_2^0) > \mathcal{H}_1(E_1^0, E_2^0)$, which is equivalent to

$$\begin{aligned}
&\lambda^2 \int_{\mathbb{R}^2}(|\nabla E_1^0|^2 + |\nabla E_2^0|^2)dx \\
&\quad > \int_{\mathbb{R}^2}(|\nabla E_1^0|^2 + |\nabla E_2^0|^2)dx + \omega \int_{\mathbb{R}^2}(|E_1^0|^2 dx + |E_2^0|^2)dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}(|E_1^0|^4 + |E_2^0|^4)dx - \int_{\mathbb{R}^2}|E_1^0|^2|E_2^0|^2 dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\left(|\mathcal{F}(E_1^0\overline{E_2^0})|^2 + |\mathcal{F}(\overline{E_1^0}E_2^0)|^2\right)d\xi \\
&\quad + Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1^0\overline{E_2^0})\overline{\mathcal{F}(\overline{E_1^0}E_2^0)}d\xi.
\end{aligned} \tag{4.58}$$

That is,

$$\begin{aligned}
&(\lambda^2 - 1)\int_{\mathbb{R}^2}(|\nabla E_1^0|^2 + |\nabla E_2^0|^2)dx \\
&\quad > \int_{\mathbb{R}^2}\omega(|E_1^0|^2 + |E_2^0|^2)dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}(|E_1^0|^4 + |E_2^0|^4)dx - \int_{\mathbb{R}^2}|E_1^0|^2|E_2^0|^2 dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\left(|\mathcal{F}(E_1^0\overline{E_2^0})|^2 + |\mathcal{F}(\overline{E_1^0}E_2^0)|^2\right)d\xi \\
&\quad + Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1^0\overline{E_2^0})\overline{\mathcal{F}(\overline{E_1^0}E_2^0)}d\xi.
\end{aligned} \tag{4.59}$$

(4.56) and (4.59) then yield that

$$\begin{aligned}
&(\lambda^2 - 1)\Big[\int_{\mathbb{R}^2}(|\nabla E_1^0|^2 + |\nabla E_2^0|^2)dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}(|E_1^0|^4 + |E_2^0|^4)dx - \int_{\mathbb{R}^2}|E_1^0|^2|E_2^0|^2 dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\left(|\mathcal{F}(E_1^0\overline{E_2^0})|^2 + |\mathcal{F}(\overline{E_1^0}E_2^0)|^2\right)d\xi \\
&\quad + Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2+\delta}\mathcal{F}(E_1^0\overline{E_2^0})\overline{\mathcal{F}(\overline{E_1^0}E_2^0)}d\xi\Big] \\
&\quad > \frac{\delta}{2}\lambda^2\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\left(|\mathcal{F}(E_1^0\overline{E_2^0})|^2 + |\mathcal{F}(\overline{E_1^0}E_2^0)|^2\right)d\xi \\
&\quad - \lambda^2\delta Re\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{(|\xi|^2+\delta)^2}\mathcal{F}(E_1^0\overline{E_2^0})\overline{\mathcal{F}(\overline{E_1^0}E_2^0)}d\xi.
\end{aligned} \tag{4.60}$$

From $\delta \geq 0$, $\eta > 0$, (1.6), (4.48) and (4.60) it follows that $(\lambda^2 - 1)\mathcal{H}(E_1^0, E_2^0) > 0$. Thus, $\mathcal{H}(E_1^0, E_2^0) < 0$ for $\lambda \in (0, 1)$. In view of (4.54), (4.55), $\delta \geq 0$ and $\eta > 0$, one gets that $\int_{\mathbb{R}^2}(|x|^2|E_1|^2 + |x|^2|E_2|^2)dx$ being a positive function of t can't verify (4.54) with $\mathcal{H}(E_1^0, E_2^0) < 0$ for all time. Hence, it must be the case that $T < +\infty$ and

$$\lim_{t \to T}\left(\|E_1\|_{H_r^1(\mathbb{R}^2)} + \|E_2\|_{H_r^1(\mathbb{R}^2)}\right) = +\infty.$$

This accomplishes the proof of 1 of Theorem 4.1.

Next, we verify 2 of Theorem 4.1.

(4.40) and (4.42) conclude that $(E_1^0, E_2^0) \in K_1$. By Proposition 4.4, one sees that for $t \in [0, T)$, $(E_1, E_2) \in K_1$ and

$$R(E_1, E_2) > 0, \quad \mathcal{H}_1(E_1, E_2) < d_1, \tag{4.61}$$

where (E_1, E_2) is the solution to the Cauchy problem (1.1)-(1.3). By (4.3), (4.4) and (4.49) we have

$$\int_{\mathbb{R}^2} (|\nabla E_1|^2 + |\nabla E_2|^2) dx < 2d_1, \tag{4.62}$$

which together with (1.5) yields the boundedness of (E_1, E_2) in $H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ for $t \in [0, T)$. So it must be $T = +\infty$, i.e.,(E_1, E_2) exists globally on $t \in [0, +\infty)$. Furthermore, (4.62) yields the estimate (4.43).

This completes the proof of Theorem 4.1. □

Acknowledgements

The author would like to thank the anonymous referee for many valuable comments. The author is supported by the National Natural Science Foundation of China (Grant No. 11171241, No. 11571254) and The Program for New Century Excellent Talents in University (No. NCET-12-1058).

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