Ordered Partitions Avoiding a Permutation Pattern of Length 3

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Abstract

An ordered partition of $[n] = \{1, 2, \ldots, n\}$ is a partition whose blocks are endowed with a linear order. Let $\mathcal{OP}_{n,k}$ be the set of ordered partitions of $[n]$ with k blocks and $\mathcal{OP}_{n,k}(\sigma)$ be the set of ordered partitions in $\mathcal{OP}_{n,k}$ that avoid a pattern σ. For any permutation pattern σ of length three, Godbole, Goyt, Herdan and Pudwell obtained formulas for the number of ordered partitions of $[n]$ with 3 blocks avoiding σ as well as the number of ordered partitions of $[n]$ with $n-1$ blocks avoiding σ. They also showed that $|\mathcal{OP}_{n,k}(\sigma)| = |\mathcal{OP}_{n,k}(123)|$ for any permutation σ of length 3. Moreover, they raised a question concerning the enumeration of $\mathcal{OP}_{n,k}(123)$, and conjectured that the number of ordered partitions of $[2n]$ with blocks of size 2 avoiding σ satisfied a second order linear recurrence relation. In answer to the question of Godbole, et al., we establish a connection between $|\mathcal{OP}_{n,k}(123)|$ and the number $e_{n,d}$ of 123-avoiding permutations of $[n]$ with d descents. Using the bivariate generating function of $e_{n,d}$ given by Barnabei, Bonetti and Silimbani, we obtain the bivariate generating function of $|\mathcal{OP}_{n,k}(123)|$. Meanwhile, we confirm the conjecture of Godbole, et al. by deriving the generating function for the number of 123-avoiding ordered partitions of $[2n]$ with n blocks of size 2.

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1 Introduction

The notion of pattern avoiding permutations was introduced by Knuth [10], and it has been extensively studied. Klazar [7] initiated the study of pattern avoiding set partitions. Further studies of pattern avoiding set partitions can be found in [4, 5, 8, 9, 11].

Recently, Godbole, Goyt, Herdan and Pudwell [3] considered pattern avoiding ordered set partitions. Let $[n] = \{1, 2, \ldots, n\}$. For a permutation σ of length 3, Godbole, et al. obtained a formula for the number of σ-avoiding ordered partitions of $[n]$ with 3 blocks and a formula for the number of σ-avoiding ordered partitions of $[n]$ with $n-1$ blocks. Moreover, they raised a question of finding the number of σ-avoiding ordered partitions of $[n]$ with k blocks.

In answer to the above question, we establish a connection between the number of 123-avoiding ordered partitions of $[n]$ with k blocks and the number of 123-avoiding permutations of $[n]$ with d descents. This enables us to derive a bivariate generating function for the number of 123-avoiding ordered partitions of $[n]$ with k blocks. Meanwhile, we confirm the conjecture of Godbole, Goyt, Herdan and Pudwell [3] on a recurrence relation concerning the number of 123-avoiding ordered partitions of $[2n]$ with blocks of size 2.

Let us give an overview of notation and terminology. Let S_n be the set of permutations of $[n]$. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in S_k$, where $1 \le k \le n$, we say that π contains a pattern σ if there exists a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ $(1 \le i_1 < i_2 < \cdots < i_k \le n)$ of π that is order-isomorphic to σ, in other words, for all $l, m \in [k]$, we have $\pi_{i_l} < \pi_{i_m}$ if and only if $\sigma_l < \sigma_m$. Otherwise, we say that π avoids a pattern σ, or π is σ-avoiding. Let $S_n(\sigma)$ denote the set of permutations of S_n that avoid a pattern σ. For example, 41532 is 123-avoiding, while it contains a pattern 312 corresponding to the subsequence 412.

A partition π of a set $[n]$, written $\pi \vdash [n]$, is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \ldots, B_k$ of $[n]$ such that $\cup_{i=1}^{k} B_i = [n]$, where each B_i $(1 \le i \le k)$ is called a block. We write $\pi = B_1/B_2/\cdots/B_k$ and define the length of π, denoted $b(\pi)$, to be the number of blocks. An ordered partition of $[n]$ is a partition of $[n]$ whose blocks are endowed with a linear order. Let $\mathcal{OP}_{n,k}$ denote the set of ordered partitions of $[n]$ with k blocks, let $\mathcal{OP}_n$ denote the set of ordered partitions of $[n]$, and let $\mathcal{OP}_{[b_1,b_2,\ldots,b_k]}$ denote the set of ordered partitions of $[b_1 + b_2 + \cdots + b_k]$ such that the i-th block contains b_i elements. If $b_1 = \cdots = b_k = s$, we write $\mathcal{OP}_{[s^k]}$ for $\mathcal{OP}_{[b_1,b_2,\ldots,b_k]}$. Let $\mathrm{op}_{n,k} = |\mathcal{OP}_{n,k}|$, $\mathrm{op}_n = |\mathcal{OP}_n|$, $\mathrm{op}_{[b_1,b_2,\ldots,b_k]} = |\mathcal{OP}_{[b_1,b_2,\ldots,b_k]}|$ and $\mathrm{op}_{[s^k]} = |\mathcal{OP}_{[s^k]}|$.

Given an ordered partition $\pi = B_1/B_2/\cdots/B_k \in \mathcal{OP}_{n,k}$ and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in S_m$, we say that π contains a pattern σ if there exist blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_m}$ with $1 \le i_1 < i_2 < \cdots < i_m \le k$ and elements $b_1 \in B_{i_1}, b_2 \in B_{i_2}, \ldots, b_m \in B_{i_m}$ such that $b_1b_2\cdots b_m$ is order-isomorphic to σ. Otherwise, we say that π avoids a pattern σ. For example, the ordered partition $14/35/2 \in \mathcal{OP}_{5,3}$ is 123-avoiding, while it contains a pattern 132. Similarly, let $\mathcal{OP}_{n,k}(\sigma)$ denote the set of ordered partitions of $\mathcal{OP}_{n,k}$ that are σ-avoiding. Let $\mathrm{op}_{n,k}(\sigma) = |\mathcal{OP}_{n,k}(\sigma)|$, $\mathrm{op}_n(\sigma) = |\mathcal{OP}_n(\sigma)|$, $\mathrm{op}_{[b_1,b_2,\ldots,b_k]}(\sigma) = |\mathcal{OP}_{[b_1,b_2,\ldots,b_k]}(\sigma)|$ and $\mathrm{op}_{[s^k]}(\sigma) = |\mathcal{OP}_{[s^k]}(\sigma)|$.

Godbole, et al. [3] obtained the following formulas for $\mathrm{op}_{n,3}(\sigma)$ and $\mathrm{op}_{n,n-1}(\sigma)$ for any $\sigma \in S_3$.

Theorem 1.1 *For $n \geq 1$, $1 \leq k \leq n$, and for any permutation σ of length 3, we have*

$$\begin{aligned} \mathrm{op}_{n,3}(\sigma) &= \left(\frac{n^2}{8} + \frac{3n}{8} - 2\right) 2^n + 3, \\ \mathrm{op}_{n,n-1}(\sigma) &= \frac{3(n-1)^2}{n(n+1)} \binom{2n-2}{n-1}. \end{aligned} \tag{1.1}$$

Godbole, et al. [3] also showed that

$$\mathrm{op}_{n,k}(\sigma) = \mathrm{op}_{n,k}(123), \tag{1.2}$$

$$\mathrm{op}_{[b_1,b_2,\ldots,b_k]}(\sigma) = \mathrm{op}_{[b_1,b_2,\ldots,b_k]}(123) \tag{1.3}$$

for any $\sigma \in S_3$. They raised a question concerning the enumeration of $\mathcal{OP}_{n,k}(123)$. Using Zeilberger's Maple package *FindRec* [12], they conjectured that $\mathrm{op}_{[2^k]}(123)$ satisfied the following second order linear recurrence relation.

Conjecture 1.1 *For $k \geq 0$, we have*

$$\begin{aligned} \mathrm{op}_{[2^{k+2}]}(123) = & \frac{329k^3 + 1215k^2 + 1426k + 528}{2(k+2)(2k+5)(7k+5)} \mathrm{op}_{[2^{k+1}]}(123) \\ & + \frac{3(k+1)(2k+1)(7k+12)}{(k+2)(2k+5)(7k+5)} \mathrm{op}_{[2^k]}(123). \end{aligned} \tag{1.4}$$

In this paper, we provide an answer to the above question by deriving a bivariate generating function for $\mathrm{op}_{n,k}(123)$ and w confirm the conjectured recurrence relation by computing the generating function of $\mathrm{op}_{[2^k]}(123)$.

2 The generating function of $\mathrm{op}_{n,k}(123)$

In this section, we obtain the bivariate generating function of $\mathrm{op}_{n,k}(123)$. Let $F(x,y)$ be the generating function of $\mathrm{op}_{n,k}(123)$, that is,

$$F(x,y) = \sum_{n\geq 0} \sum_{k\geq 0} \mathrm{op}_{n,k}(123) x^n y^k. \tag{2.1}$$

We show that $F(x,y)$ can be expressed in terms of the bivariate generating function $E(x,y)$ of 123-avoiding permutations of $[n]$ with respect to the number of descents. More precisely, for a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, the descent set of σ is defined by

$$D(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$$

and the number of descents of σ is denoted by $des(\sigma) = |D(\sigma)|$. Barnabei, Bonetti and Silimbani [2] defined the generating function

$$E(x,y) = \sum_{n\geq 0} \sum_{\sigma\in S_n(123)} x^n y^{des(\sigma)} = \sum_{n\geq 0}\sum_{d\geq 0} e_{n,d} x^n y^d, \tag{2.2}$$

where

$$e_{n,d} = |\{\sigma \mid \sigma \in S_n(123), des(\sigma) = d\}|.$$

Furthermore, they obtained the following formula:

$$E(x,y) = \frac{-1 + 2xy + 2x^2y - 2xy^2 - 4x^2y^2 + 2x^2y^3 + \sqrt{1 - 4xy - 4x^2y + 4x^2y^2}}{2xy^2(xy - 1 - x)}. \tag{2.3}$$

The following theorem gives the generating function $F(x,y)$ in terms of $E(x,y)$.

Theorem 2.1 *We have*

$$F(x,y) = E(xy, 1 + y^{-1}),$$

which implies that

$$F(x,y) = \frac{-y - 2xy - 2x + 2x^2y + 2x^2 + y\sqrt{1 - 4xy - 4x + 4x^2y + 4x^2}}{2x(y+1)^2(x-1)}. \tag{2.4}$$

To prove the above theorem, we establish a connection between $\mathrm{op}_{n,k}(123)$ and $e_{n,d}$.

Theorem 2.2 *For $n \geq 1$ and $1 \leq k \leq n$, we have*

$$\mathrm{op}_{n,k}(123) = \sum_{d=n-k}^{n-1} \binom{d}{n-k} e_{n,d}. \tag{2.5}$$

Proof. Define a map $\varphi\colon \mathcal{OP}_{n,k}(123) \to S_n(123)$ as a canonical representation of an ordered partition. Given an ordered partition $\pi = B_1/B_2/\cdots/B_k \in \mathcal{OP}_{n,k}(123)$. If we list the elements of each block in decreasing order and ignore the symbol '/' between two adjacent blocks, we get a permutation $\varphi(\pi) = \sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. It can be shown that φ is well-defined, that is, $\sigma = \varphi(\pi)$ is a 123-avoiding permutation of S_n. Assume to the contrary that σ contains a 123-pattern, that is, there exist $i < j < l$ such that $\sigma_i\sigma_j\sigma_l$ is a 123-pattern in σ. By the construction of σ, we see that the elements σ_i, σ_j and σ_l are in different blocks in π. This implies that $\sigma_i\sigma_j\sigma_l$ is a 123-pattern of π, a contradiction. Thus $\sigma \in S_n(123)$. Moreover, according to the construction of σ, we find that

$$des(\sigma) \geq \sum_{s=1}^{k}(|B_s| - 1) = n - k. \tag{2.6}$$

Conversely, given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ in $S_n(123)$ with d descents, we aim to count the preimages π in $\mathcal{OP}_{n,k}(123)$ such that $\varphi(\pi) = \sigma$. If $d < n-k$, by inequality (2.6), it is impossible for any π in $\mathcal{OP}_{n,k}(123)$ to be a preimage of σ. So we may assume that $d \geq n-k$. Let $\pi' = \sigma_1/\sigma_2/\cdots/\sigma_n$. Clearly, $\varphi(\pi') = \sigma$. If $i \in D(\sigma)$, we may merge σ_i and σ_{i+1} of π' into a block to form a new ordered partition π''. It is easily verified that $\varphi(\pi'') = \sigma$ and $b(\pi'') = n-1$. Moreover, we may iterate this process if $des(\pi'') > 0$. Note that at each step we get a preimage of σ with one less block. To obtain the preimages π with k blocks, we need to repeat this process $n-k$ times. Observe that the resulting ordered partition depends only on the positions we choose in $D(\sigma)$. Hence we conclude that there are $\binom{d}{n-k}$ ordered partitions π in $\mathcal{OP}_{n,k}(123)$ such that $\varphi(\pi) = \sigma$. Hence the theorem follows from summing over d. ∎

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Theorem 2.2, we have

$$\begin{aligned}
\sum_{k=0}^{n} \mathrm{op}_{n,k}(123)x^n y^k &= \sum_{k=0}^{n}\sum_{d=n-k}^{n-1}\binom{d}{n-k}e_{n,d}x^n y^k \\
&= \sum_{d=0}^{n-1}\sum_{k=n-d}^{n}\binom{d}{n-k}e_{n,d}x^n y^k \\
&= \sum_{d=0}^{n-1}\sum_{j=0}^{d}\binom{d}{j}e_{n,d}x^n y^{n-j} \\
&= \sum_{d=0}^{n-1} e_{n,d}(xy)^n(1+y^{-1})^d.
\end{aligned}$$

Summing over n, we obtain that $F(x,y) = E(xy,(1+y^{-1}))$. ∎

An alternative proof of the formula (2.4) for $F(x,y)$ was given by Kasraoui [6]. Setting $y = 1$ in the generating function $F(x,y)$, we are led to the generating function of $\mathrm{op}_n(123)$.

Corollary 2.3 *Let $H(x)$ be the generating function of $\mathrm{op}_n(123)$, that is*

$$H(x) = \sum_{n\geq 0}\mathrm{op}_n(123)x^n.$$

Then we have

$$H(x) = \frac{1}{2} + \frac{1}{1+\sqrt{1-8x+8x^2}}.$$

The connection between $\mathrm{op}_{n,k}(123)$ and $e_{n,d}$ can be used to derive the following generating function of $\mathrm{op}_{n,n-1}(123)$.

Corollary 2.4 *Let $G(x)$ be the generating function of* $\mathrm{op}_{n,n-1}(123)$*, that is,*

$$G(x) = \sum_{n\geq 1} \mathrm{op}_{n,n-1}(123)x^n.$$

Then we have

$$G(x) = \frac{2\,x^2 - 7\,x + 2 + 3\,x\sqrt{1-4\,x} - 2\,\sqrt{1-4\,x}}{2x\sqrt{1-4\,x}}. \tag{2.7}$$

Proof. By Theorem 2.2, we have

$$\mathrm{op}_{n,n-1}(123) = \sum_{d=1}^{n-1} d\,e_{n,d}. \tag{2.8}$$

It follows that

$$\begin{aligned} G(x) &= \sum_{n\geq 1}\sum_{d=1}^{n-1} d\,e_{n,d}x^n \\ &= \frac{\partial E(x,y)}{\partial y}\bigg|_{y=1}. \end{aligned}$$

By expression (2.3) for $E(x,y)$, we obtain (2.7). ∎

Notice that formula (1.1) for $\mathrm{op}_{n,n-1}$ can be deduced from (2.7).

3 The generating function of $\mathrm{op}_{[2^k]}(123)$

In this section, we compute the generating function of $\mathrm{op}_{[2^k]}(123)$ which leads to the recurrence relation of $\mathrm{op}_{[2^k]}(123)$ as in Conjecture 1.1

Theorem 3.1 *Let $Q(x)$ be the generating function of* $\mathrm{op}_{[2^k]}(123)$*, that is,*

$$Q(x) = \sum_{k\geq 0} \mathrm{op}_{[2^k]}(123)x^{2k}.$$

Then we have

$$Q(x) = \sqrt{\frac{2}{1+2\,x^2+\sqrt{1-12\,x^2}}}. \tag{3.1}$$

Let $Q'(x)$, $Q''(x)$ and $Q'''(x)$ denote the first derivative, second derivative and third derivative of $Q(x)$, respectively. The following theorem shows that $Q(x)$ satisfies a third order differential equation.

Theorem 3.2 *We have*

$$\left(\frac{21}{2}x^7+\frac{329}{8}x^5-\frac{7}{2}x^3\right)Q^{'''}(x)+\left(99x^6+\frac{1443}{8}x^4-5x^2\right)Q^{''}(x)$$
$$+\left(207x^5+\frac{717}{8}x^3+11x\right)Q^{'}(x)+(72x^4-12x^2)Q(x)=0. \tag{3.2}$$

Equating coefficients of x^{2n+4} in (3.2), we obtain the recurrence relation (1.4) for $\mathrm{op}_{[2^k]}(123)$.

To prove Theorem 3.1, we construct a bijection between ordered partitions and permutations on multisets. Given an ordered partition $\pi = B_1/B_2/\cdots/B_k \in \mathcal{OP}_{n,k}$, its canonical sequence, denoted $\psi(\pi)$, is defined to be a sequence $\rho=\rho_1\rho_2\cdots\rho_n$ with $\rho_i = j$ if $i \in B_j$. Let $\mathcal{W}_{[1^{b_1}2^{b_2}\cdots k^{b_k}]}$ denote the set of permutations on a multiset $\{1^{b_1}, 2^{b_2}, \ldots, k^{b_k}\}$, where i^r means r occurrences of i. It is easily verified that ψ is a bijection between $\mathcal{OP}_{[b_1,b_2,\ldots,b_k]}$ and $\mathcal{W}_{[1^{b_1}2^{b_2}\cdots k^{b_k}]}$.

Any permutation $\sigma \in S_m$ corresponds naturally to a unique ordered partition of $[m]$ with each element in its own block. Define the canonical sequence of σ to be the canonical sequence of the corresponding ordered partition. It is not hard to see that the canonical sequence of σ is its inverse σ^{-1}. For example, the canonical sequence of 43512 is 45213.

By the definition of pattern avoiding ordered partitions, we see that an ordered partition π contains a pattern σ if and only if its canonical sequence $\psi(\pi)$ contains a pattern σ^{-1}. This implies that ψ is a bijection between $\mathcal{OP}_{[b_1,b_2,\ldots,b_k]}(\sigma)$ and $\mathcal{W}_{[1^{b_1}2^{b_2}\cdots k^{b_k}]}(\sigma^{-1})$, where $\mathcal{W}_{[1^{b_1}2^{b_2}\cdots k^{b_k}]}(\tau)$ is the set of τ-avoiding permutations in $\mathcal{W}_{[1^{b_1}2^{b_2}\cdots k^{b_k}]}$. Hence we have

$$\mathrm{op}_{[b_1,b_2,\ldots,b_k]}(\sigma) = |\mathcal{W}_{[1^{b_1}2^{b_2}\cdots k^{b_k}]}(\sigma^{-1})|. \tag{3.3}$$

In order to establish the recurrence relation for $\mathrm{op}_{[2^k]}(123)$, we need to use $\mathrm{op}_{[2^k,1]}(123)$ and $\mathrm{op}_{[2^k,1,1]}(123)$. Combining (3.3) and (1.3), we obtain

$$\begin{aligned}
\mathrm{op}_{[2^n]}(123) =& |\mathcal{W}_{[1^22^2\cdots n^2]}(132)|,\\
\mathrm{op}_{[2^n,1]}(123) =& |\mathcal{W}_{[1^22^2\cdots n^2(n+1)]}(132)|,\\
\mathrm{op}_{[2^n,1,1]}(123) =& |\mathcal{W}_{[1^22^2\cdots n^2(n+1)(n+2)]}(132)|.
\end{aligned}$$

Let

$$\begin{aligned}
u_{2n} &= |\mathcal{W}_{[1^22^2\cdots n^2]}(132)|,\\
u_{2n+1} &= |\mathcal{W}_{[1^22^2\cdots n^2(n+1)]}(132)|,\\
v_{2n} &= |\mathcal{W}_{[1^22^2\cdots (n-1)^2n(n+1)]}(132)|,
\end{aligned}$$

where we set $u_0 = v_0 = 1$ and set $u_n = v_n = 0$ for $n < 0$.

We proceed to derive recurrence relations for u_{2n}, u_{2n+1} and v_{2n} that can be used to obtain a system of equations on the generating functions. In particular, we get the generating function of u_{2n}, that is, the generating function of $\mathrm{op}_{[2^n]}(123)$.

Let $U_e(x)$, $U_o(x)$ and $V(x)$ denote the generating functions of u_{2n}, u_{2n+1} and v_{2n}, namely,

$$U_e(x) = \sum_{n\geq 0} u_{2n}x^{2n},$$
$$U_o(x) = \sum_{n\geq 0} u_{2n+1}x^{2n+1},$$
$$V(x) = \sum_{n\geq 0} v_{2n}x^{2n}.$$

We need the following lemma due to Atkinson, Walker and Linton [1].

Lemma 3.3 *Given two permutations $p = p_1p_2\cdots p_n$ and $q = q_1q_2\cdots q_n$ of the same multiset of $[n]$, we have*

$$|\mathcal{W}_{[1^{p_1}2^{p_2}\cdots n^{p_n}]}(132)| = |\mathcal{W}_{[1^{q_1}2^{q_2}\cdots n^{q_n}]}(132)|.$$

The following theorem gives a recurrence relation for u_{2n} and u_{2n+1}.

Theorem 3.4 *For $n \geq 0$, we have*

$$u_{2n+1} = \sum_{i+j=2n} u_iu_j, \tag{3.4}$$

which implies that

$$U_o(x) = x\left(U_o^2(x) + U_e^2(x)\right). \tag{3.5}$$

Proof. Assume that $\pi \in \mathcal{W}_{[1^2 2^2\cdots n^2(n+1)]}(132)$. Write π in the form $\sigma(n+1)\tau$. Since π is 132-avoiding, both σ and τ are 132-avoiding. Moreover, for any element r in σ and any element s in τ, we have $r \geq s$. Let k be the maximum number in τ. It can be seen that τ contains all the numbers in the multiset $\{1^2, 2^2, \ldots, n^2, (n+1)\}$ that are smaller than k, that is, τ contains all the elements in the multiset $\{1^2, 2^2, \ldots, (k-1)^2\}$.

There are two cases. If $|\tau|$ is even, then τ contains two occurrences of k. Thus τ is in $\mathcal{W}_{[1^2 2^2\cdots k^2]}(132)$, which is counted by u_{2k}. Moreover, σ is in $\mathcal{W}_{[(k+1)^2(k+2)^2\cdots n^2]}(132)$. It is easily seen that $|\mathcal{W}_{[(k+1)^2(k+2)^2\cdots n^2]}(132)| = |\mathcal{W}_{[1^2 2^2\cdots(n-k)^2]}(132)|$, which is counted by u_{2n-2k}.

If $|\tau|$ is odd, then we have $\tau \in \mathcal{W}_{[1^2 2^2 \cdots (k-1)^2 k]}(132)$ and $\sigma \in \mathcal{W}_{[k(k+1)^2(k+2)^2 \cdots n^2]}(132)$. In this case, $\mathcal{W}_{[1^2 2^2 \cdots (k-1)^2 k]}(132)$ is counted by u_{2k-1}. By Lemma 3.3, we see that $|\mathcal{W}_{[k(k+1)^2 \cdots n^2]}(132)| = |\mathcal{W}_{[k^2(k+1)^2 \cdots (n-1)^2 n]}(132)|$, which is counted by $u_{2n+1-2k}$. Combining the above two cases, we obtain (3.4).

Using (3.4), we obtain

$$
\begin{aligned}
U_o(x) &= \sum_{n\geq 0} u_{2n+1} x^{2n+1} \\
&= x \sum_{n\geq 0} \sum_{i+j=2n} u_i u_j x^{2n} \\
&= x \sum_{n\geq 0} \sum_{2i+2j=2n} u_{2i} u_{2j} x^{2n} + x \sum_{n\geq 0} \sum_{2i+1+2j+1=2n} u_{2i+1} u_{2j+1} x^{2n} \\
&= x \left(U_o^2(x) + U_e^2(x) \right),
\end{aligned}
$$

as claimed. ∎

The following theorem shows that v_{2n} can be expressed in terms of u_{2n} and u_{2n-1}.

Theorem 3.5 *For $n \geq 0$, we have*

$$v_{2n} = u_{2n} + u_{2n-1}, \tag{3.6}$$

which implies that

$$V(x) = U_e(x) + xU_o(x). \tag{3.7}$$

Proof. Clearly, (3.6) holds for $n = 0$ under the assumptions that $u_{-1} = 0$ and $u_0 = v_0 = 1$. So we assume that $n \geq 1$, and assume that $\pi = \pi_1\pi_2 \cdots \pi_{2n} \in \mathcal{W}_{[1^2 2^2 \cdots (n-1)^2 n(n+1)]}(132)$. There are two cases. If $n+1$ precedes n in π, then we have $\pi_1 = n+1$. Otherwise, $\pi_1(n+1)n$ forms a 132-pattern in π, a contradiction. Using the fact that $\pi_1 = n+1$, it is clear that $\pi \in \mathcal{W}_{[1^2 2^2 \cdots (n-1)^2 n(n+1)]}(132)$ if and only if $\pi_2\pi_3 \cdots \pi_{2n} \in \mathcal{W}_{[1^2 2^2 \cdots (n-1)^2 n]}(132)$. Notice that $\mathcal{W}_{[1^2 2^2 \cdots (n-1)^2 n]}(132)$ is counted by u_{2n-1}.

If n precedes $n+1$ in π, then there does not exist any 132-pattern of π that contains both n and $n+1$. In this case, we may treat $n+1$ as n. Such permutations form the set $\mathcal{W}_{[1^2 2^2 \cdots (n-1)^2 n^2]}(132)$, which is counted by u_{2n}. Combining the above two cases, we obtain (3.6), which yields (3.7). ∎

To compute the generating functions $U_e(x)$, $U_o(x)$ and $V(x)$, we still need one more relation, which is given below.

Theorem 3.6 *For $n \geq 1$, we have*

$$u_{2n} = 2 \sum_{2i+j=2n-1} u_{2i} u_j + \sum_{2i+1+j=2n-2} u_{2i+1} u_j - u_{2n-1}, \tag{3.8}$$

which implies that

$$U_e(x) = 1 + 2xU_e(x)U_o(x) - x^2U_e^2(x). \tag{3.9}$$

Proof. Assume that $\pi \in W_{[1^22^2\cdots n^2]}(132)$. Write π in the form $\sigma n \tau$ such that n appears in σ. Since π is 132-avoiding, both σ and τ are 132-avoiding. Moreover, for any element r in σ and any element s in τ, we have $r \geq s$.

Let k be the maximum number in τ. There are two cases. If $|\tau|$ is even, using the same argument as in Theorem 3.4, we deduce that $\tau \in \mathcal{W}_{[1^22^2\cdots k^2]}(132)$ and $\sigma \in \mathcal{W}_{[(k+1)^2\cdots(n-1)^2n]}(132)$. In this case, $\mathcal{W}_{[1^22^2\cdots(k-1)^2k^2]}(132)$ is counted by u_{2k} and $\mathcal{W}_{[(k+1)^2\cdots(n-1)^2n]}(132)$ is counted by $u_{2n-1-2k}$.

If $|\tau|$ is odd, it can be seen that τ is in $\mathcal{W}_{[1^22^2\cdots(k-1)^2k]}(132)$, which is counted by u_{2k-1}, and σ is in $\mathcal{W}_{[k(k+1)^2\cdots(n-1)^2n]}(132)$. By Lemma 3.3, we find that

$$|\mathcal{W}_{[k(k+1)^2\cdots(n-1)^2n]}(132)| = |\mathcal{W}_{[k^2\cdots(n-2)^2(n-1)n]}(132)|,$$

which is counted by v_{2n-2k}. Observing that σ is not empty, we have $2n - 2k > 0$.

Combining the above two cases, we get

$$u_{2n} = \sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-1} u_{2i+1}v_j - u_{2n-1}.$$

In view of relation (3.6), we obtain

$$\begin{aligned} u_{2n} &= \sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-1} u_{2i+1}u_j + \sum_{2i+1+j=2n-1} u_{2i+1}u_{j-1} - u_{2n-1} \\ &= 2\sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-2} u_{2i+1}u_j - u_{2n-1}. \end{aligned}$$

It remains to prove relation (3.9). Using (3.8), we have

$$\begin{aligned} U_e(x) &= 1 + \sum_{n\geq 1} u_{2n}x^{2n} \\ &= 1 + \sum_{n\geq 1}\left(2\sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-2} u_{2i+1}u_j - u_{2n-1}\right)x^{2n} \\ &= 1 + 2\sum_{n\geq 1}\sum_{2i+j=2n-1} u_{2i}u_jx^{2n} + \sum_{n\geq 1}\sum_{2i+1+j=2n-2} u_{2i+1}u_jx^{2n} - \sum_{n\geq 1} u_{2n-1}x^{2n} \\ &= 1 + 2xU_e(x)U_o(x) + x^2U_o^2(x) - xU_o(x). \end{aligned} \tag{3.10}$$

Substituting (3.5) into (3.10), we obtain

$$U_e(x) = 1 + 2xU_e(x)U_o(x) + x^2U_o^2(x) - x^2\left(U_o^2(x) + U_e^2(x)\right)$$

$$= 1 + 2xU_e(x)U_o(x) - x^2U_e^2(x),$$

as claimed. ∎

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Note that $Q(x) = U_e(x)$. By (3.9), we get

$$U_o(x) = \frac{x^2U_e^2(x) + U_e(x) - 1}{2xU_e(x)}. \tag{3.11}$$

Plugging (3.11) into (3.5) yields the following equation

$$(x^4 + 4x^2)U_e^4(x) - (2x^2 + 1)U_e^2(x) + 1 = 0. \tag{3.12}$$

Given the initial values of u_{2n}, we are led the solution of $U_e(x)$ as given by (3.1). ∎

To conclude, we note that the generating functions $U_o(x)$ and $V(x)$ are given as follows:

$$U_o(x) = \frac{1}{2x} - \frac{1 + \sqrt{1 - 12\,x^2}}{4\,x}U_e(x),$$
$$V(x) = \frac{1}{2} + \frac{3 - \sqrt{1 - 12\,x^2}}{4}U_e(x).$$

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