ORBITAL STABILITY OF SOLITARY WAVES FOR DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this paper, we show the orbital stability of solitons arising in the cubic derivative nonlinear Schrödinger equations. We consider the zero mass case that is not covered by earlier works [8, 3]. As this case enjoys L^2 scaling invariance, we expect the orbital stability in the sense up to scaling symmetry, in addition to spatial and phase translations. For the proof, we are based on the variational argument and extend a similar argument in [21]. Moreover, we also show a self-similar type blow up criteria of solutions with the critical mass 4π.

1. Introduction

We study the orbital stability of soliton solutions arising in the nonlinear Schrödinger equation with derivative (DNLS):

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \tag{1.1}$$

The well-posedness for the equation (1.1) is intensive studied. Especially, it was proved by Hayashi and Ozawa [9, 10, 11, 16] the local well-posedness in $H^1(\mathbb{R})$ and the global well-posedness when the initial data satisfies $\int_{\mathbb{R}} |u_0(x)|^2\,dx < 2\pi$. The results are analogous to that for the focusing quintic nonlinear Schrödinger equation. There are many low regularity local and global well-posedness results [17, 18, 4, 5, 12, 7, 14, 13]. Recently, Wu [20, 21] showed that global well-posedness holds as long as $\int_{\mathbb{R}} |u_0(x)|^2\,dx < 4\pi$. In [21] the author observed that the threshold 4π corresponds to the mass of a ground state. This observation draws our attention to study the orbital stability or instability of soliton solutions with the critical mass 4π.
As is shown in [8, 3], the equation (1.1) has two parameter family of solitons of the form

$$u_{\omega,c}(t,x) = \phi_{\omega,c}(x+ct)e^{i\omega t - i\frac{c}{2}(x+ct) + \frac{3}{4}i\int_{-\infty}^{x+ct}|\phi_{\omega,c}(y)|^2\,dy},$$

where $(\omega, c) \in \mathbb{R}\times\mathbb{R}$, and $\phi_{\omega,c}$ is a ground state solution to the elliptic equation

$$-\partial_{xx}\phi + \big(\omega - \frac{c^2}{4}\big)\phi + \frac{c}{2}\phi^3 - \frac{3}{16}\phi^5 = 0, \tag{1.2}$$

If $c^2 < 4\omega$, then $\phi_{\omega,c}$ shows an exponential decay:

$$\phi_{\omega,c}(x) = \Big(\frac{\sqrt{\omega}}{4\omega - c^2}\big[\cosh(\sqrt{(4\omega - c^2)}x) - \frac{c}{\sqrt{4\omega}}\big]\Big)^{-\frac{1}{2}},$$

2010 *Mathematics Subject Classification.* Primary 35Q55; Secondary 35A01.
Key words and phrases. derivative NLS, orbital stability, solitary wave, energy space.
Kwon is partially supported by NRF of Korea (No. 2010-0024017) and POSCO science fellowship. Wu is partially supported by the NSF of China (No. 11101042) and the Fundamental Research Funds for the Central Universities of China.

and the mass of $\phi_{\omega,c}$ is given by

$$\|\phi_{\omega,c}\|_{L^2}^2 = 8\tan^{-1}\sqrt{\frac{\sqrt{4\omega}+c}{\sqrt{4\omega}-c}} < 4\pi.$$

The orbital stability of those solitons was proved in [8] for $c<0$ and $c^2<4\omega$ and in [3] for any $c^2<4\omega$. Here, the orbit is given by the phase and spatial translation. See [15] for the related studies.

In this work, we consider the endpoint case, $c^2 = 4\omega$. It is called *the zero mass case* in view of (1.2). Let W be a ground state of the elliptic equation

$$-\Phi_{xx} + \Phi^3 - \frac{3}{16}\Phi^5 = 0. \tag{1.3}$$

Then, $W_c(x) = c^{\frac{1}{2}}W(cx)$ is also the ground state solution to

$$-\Phi_{xx} + \frac{c}{2}\Phi^3 - \frac{3}{16}\Phi^5 = 0,$$

and we have

$$W(x) = 2^{\frac{3}{2}}\left(4x^2+1\right)^{-\frac{1}{2}}, \qquad \|W\|_{L^2}^2 = \|W_c\|_{L^2}^2 = 4\pi.$$

The corresponding solitary wave solution to (1.1) with 4π mass is

$$R(t,x) = e^{\frac{3}{4}i\int_{-\infty}^{x+2t}|W(y)|^2\,dy}e^{-it-ix}W(x+2t). \tag{1.4}$$

We recall the mass, energy and momentum conservation laws:

$$\begin{aligned} M(u(t)) &= \int |u(t)|^2\,dx,\\ E(u(t)) &= \int |u_x(t)|^2 + \frac{3}{2}\mathrm{Im}|u(t)|^2u(t)\overline{u_x(t)} + \frac{1}{2}|u(t)|^6\,dx,\\ P(u(t)) &= \mathrm{Im}\int \overline{u(t)}u_x(t) - \frac{1}{2}\int |u(t)|^4\,dx. \end{aligned}$$

One may observe that $E(R)=P(R)=0$ and $M(R)=4\pi$. Similarly, we denote $R_\lambda(t,x) = \lambda^{\frac{1}{2}}R(\lambda^2t,\lambda x)$. Then R_λ is also a solution to (1.1). As opposed to the case of $c^2<4\omega$, the conservation laws do not restrict rescaling of solutions. Thus, our main theorem of the orbital stability includes scaling parameter, in addition to the phase and spatial translation.

Theorem 1. *For any $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)$ such that if*

$$\|u_0 - R(0)\|_{H^1} \le \delta, \tag{1.5}$$

then for any $t\in I=(-T_*,T^*)$ (the maximal lifespan), there exist $\theta(t)\in[0,2\pi), y(t)\in\mathbb{R}$, and $\lambda(t)\in[\lambda_0,\infty)$ for some constant $\lambda_0>0$, such that

$$\|u(t) - e^{i\theta}R_\lambda(t,\cdot-y)\|_{H^1} \le \varepsilon.$$

Moreover, from an extension of our argument we can also show a self-similar type blow-up criteria of solutions with the critical mass, which is equal to that of the ground state W.

Theorem 2. *Let $u_0\in H^1(\mathbb{R})$ with $\|u_0\|_{L^2} = \|W\|_{L^2}$. Suppose that the solution u to* (1.1) *blows up in the finite time T^*, then there exist $\theta(t)\in[0,2\pi), y(t)\in\mathbb{R}$, such that when $t\to T^*$,*

$$e^{-i\theta(t)}u_{\lambda(t)}\big(t,\cdot+y(t)\big) - R(t) \to 0, \qquad \text{strongly in } H^1(\mathbb{R}),$$

where $u_\lambda(t,x) = \lambda^{\frac{1}{2}}u(\lambda^2t,\lambda x)$, and $\lambda(t) = \|\partial_xW\|_{L^2}/\|\partial_xv(t)\|_{L^2}$.

The proof of the theorems are based on the following variational result. Let the quantities

$$S(w) =\|w_x\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^4}^4 - \frac{1}{16}\|w\|_{L^6}^6; \tag{1.6}$$

$$K(w) =6\|w\|_{L^4}^4 - \|w\|_{L^6}^6. \tag{1.7}$$

We note that $K(W) = 0$. Then we have the following rigidity of W.

Proposition 1. *Let $g \in H^1(\mathbb{R})$. For any $\varepsilon > 0$, there exists ε_0, such that if*

$$\big|S(g) - S(W)\big| + \big|K(g)\big| < \varepsilon_0, \tag{1.8}$$

then

$$\inf_{(\theta,y)\in\mathbb{R}^2} \big\|g - e^{i\theta}W(\cdot - y)\big\|_{\dot H^1} < \varepsilon.$$

We provide the proof of Proposition 1 in Section 2. We use the fact that W is an optimal function of a sharp Gagliardo-Nirenberg inequality (see [1]),

$$\|f\|_{L^6} \le C_{GN}\|f\|_{L^4}^{\frac{8}{9}}\|f_x\|_{L^2}^{\frac{1}{9}}, \tag{1.9}$$

where we denoted C_{GN} to be the sharp constant: $C_{GN} = 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$. Roughly speaking, Proposition 1 tells that if a function closely attains the equality of the sharp Gagliardo-Nirenberg inequality (1.9), then it is close to W up to the symmetries of spatial, phase translation and scaling.

The strategy to prove Theorem 1 and Theorem 2 is a variational argument. In addition, we combine it with the argument in [21]. To do this, we use the following gauge transformation. Let

$$v(t,x) := e^{-\frac{3}{4}i\int_{-\infty}^{x}|u(t,y)|^2\,dy}u(t,x), \tag{1.10}$$

then from (1.1), v is the solution of

$$\begin{cases} i\partial_t v + \partial_x^2 v = \dfrac{i}{2}|v|^2 v_x - \dfrac{i}{2}v^2\bar{v}_x - \dfrac{3}{16}|v|^4 v, & t\in\mathbb{R}, x\in\mathbb{R},\\ v(0,x) = v_0(x), \end{cases} \tag{1.11}$$

where $v_0(x) := e^{-\frac{3}{4}i\int_{-\infty}^{x}|u_0|^2\,dy}u_0$. We first show that there exists ε_0 such that $\delta \ll \varepsilon_0 \ll \varepsilon$, and

$$\|v(t)\|_{L^6} \ge \varepsilon_0 \text{ for any } t \in I.$$

This is done by a rigidity theorem (Proposition 1). Under this fact, we use a similar argument in [21] to show that $f(t) := \|v(t)\|_{L^4}^4/\|v(t)\|_{L^6}^3$, is close to $\sqrt{\frac{8}{3}\pi}$. This almost fix the ratio between $\|v(t)\|_{L^4}$ and $\|v(t)\|_{L^6}$. Then we use the conservation laws, to establish the relationships between $\|v(t)\|_{L^4}$, $\|v(t)\|_{L^6}$ and $\|v_x(t)\|_{L^2}$. Then after suitable transformations, the solution almost attains the equality of the sharp Gagliardo-Nirenberg (1.9). Using Proposition 1, we conclude main theorems.
In Section 2, we prove Proposition 1 and in Section 3, we prove Theorem 1 and 2.

2. Proof of Proposition 1

First, we recall the uniqueness of the non-trivial solution for (1.3). Indeed, the non-trivial solution for (1.3), which vanishes at infinity, is uniqueness up to the rotation and the spatial transformations.

Lemma 1. *If $w \in H^1(\mathbb{R}) \setminus \{0\}$ is a solution for (1.3), then there exists (θ, x_0) such that*

$$w(x) = e^{i\theta} W(x - x_0).$$

Proof. See for example Berestycki and Lions [2] for the standard argument. □

If w is the solution of (1.3), we have $K(w) = 0$. Indeed, it follows from integrating against $\frac{1}{2}w - x\partial_x w$ on the both side of (1.3) and then integration. Furthermore, set

$$d := \inf\{S(\phi) : \phi \in H^1(\mathbb{R}) \setminus \{0\}, K(\phi) = 0\}. \tag{2.1}$$

Then $d \leq S(W)$ due to $K(W) = 0$.
Moreover, using the fact $K(\phi) = 0$, we claim that $d > 0$. If we assume that $d = 0$, then there exists a sequence $\{g_n\}_{n=1}^\infty \subset H^1(\mathbb{R}) \setminus \{0\}$, such that

$$K(g_n) = 0, \qquad \text{and} \qquad S(g_n) \to 0.$$

This gives

$$-\frac{1}{12}K(g_n) + S(g_n) = \|\partial_x g_n\|_{L^2}^2 + \frac{1}{48}\|g_n\|_{L^6}^6 \to 0.$$

Hence, by interpolation, there exists N_0, such that $n \geq N_0$,

$$\|g_n\|_{L^\infty} \leq 1.$$

Now by the definition of K, for $n \geq N_0$,

$$\begin{aligned} 0 = K(g_n) &= \int \big(6|g_n|^4 - |g_n|^6\big)\,dx = \int |g_n|^4\big(6 - |g_n|^2\big)\,dx \\ &\geq \int |g_n|^4\big(6-1\big)\,dx = 5\int |g_n|^4\,dx. \end{aligned}$$

That is, $\int |g_n|^4\,dx = 0$. This implies that $g_n \equiv 0$. This contradicts with $g_n \neq 0$. Hence, we conclude $d > 0$.

Next, we shall prove that W is the unique minimizer (up to symmetries) which attains d. First of all, we prove the existence of the minimizer.

Proposition 2. *For any sequence $\{g_n\} \subset H^1(\mathbb{R})$ satisfying that*

$$S(g_n) \to d, \quad K(g_n) \to 0, \qquad \text{as } n \to \infty,$$

there exists a function G, such that

$$g_n \to G \qquad \text{in } \dot{H}^1(\mathbb{R}).$$

In particular, $S(G) = d$, and $K(G) = 0$.

Proof. By the profile decomposition with respect to H^1 Sobolev embedding (see [6] for example), there exist sequences $\{V^j\}_{j=1}^\infty, \{x_n^j\}_{j,n=1}^\infty, \{R_n^L\}$ such that, up to a subsequence, for each L

$$g_n = \sum_{j=1}^{L} V^j(\cdot - x_n^j) + R_n^L, \tag{2.2}$$

where $|x_n^j - x_n^k| \to \infty$, as $n \to \infty, j \neq k$, and

$$\lim_{L\to\infty}\Big[\lim_{n\to\infty}\|R_n^L\|_{L^4\cap L^6}\Big] = 0. \tag{2.3}$$

Moreover,

$$\|g_n\|_{L^4}^4 = \sum_{j=1}^{L} \|V^j\|_{L^4}^4 + \|R_n^L\|_{L^4}^4 + o_n(1), \tag{2.4}$$

$$\|g_n\|_{L^6}^6 = \sum_{j=1}^{L} \|V^j\|_{L^6}^6 + \|R_n^L\|_{L^6}^6 + o_n(1), \tag{2.5}$$

$$\|g_n\|_{\dot{H}^1}^2 = \sum_{j=1}^{L} \|V^j\|_{\dot{H}^1}^2 + \|R_n^L\|_{\dot{H}^1}^2 + o_n(1). \tag{2.6}$$

From (2.4)–(2.6), we have

$$S(g_n) = \sum_{j=1}^{L} S(V^j) + S(R_n^L) + o_n(1), \tag{2.7}$$

$$K(g_n) = \sum_{j=1}^{L} K(V^j) + K(R_n^L) + o_n(1). \tag{2.8}$$

Now we need the following lemma.

Lemma 2. *Let $f \in H^1(\mathbb{R}) \setminus \{0\}$, suppose that*

$$\|f_x\|_{L^2}^2 + \frac{1}{8}\|f\|_{L^4}^4 \le d, \tag{2.9}$$

then $K(f) \ge 0$.

Proof. We prove by contradiction. Assume that there exists a function $f \in H^1(\mathbb{R}) \setminus \{0\}$ satisfies (2.9), but $K(f) < 0$. Then for

$$\lambda = \frac{\sqrt{6}\|f\|_{L^4}^2}{\|f\|_{L^6}^3},$$

we have $\lambda < 1$ and $K(\lambda f) = 0$. Then from the definition of d, we have $S(\lambda f) \ge d$. However,

$$\|\lambda f_x\|_{L^2}^2 + \frac{1}{8}\|\lambda f\|_{L^4}^4 = S(\lambda f) - \frac{1}{16}K(\lambda f) \ge d.$$

Since $\lambda < 1$, this contradicts with (2.9). Thus we obtain the lemma. □

Now we finish the proof of proposition. We first observe that

$$\|\partial_x g_n\|_{L^2}^2 + \frac{1}{8}\|g_n\|_{L^4}^4 \to d, \qquad \text{as } n \to \infty. \tag{2.10}$$

Since $S(g_n) \to d, K(g_n) \to 0$, we obtain

$$\|\partial_x g_n\|_{L^2} + \frac{1}{8}\|g_n\|_{L^4}^4 = S(g_n) - \frac{1}{16}K(g_n) \to d.$$

Moreover, by (2.4) and (2.6), we have

$$\|\partial_x g_n\|_{L^2}^2 + \frac{1}{8}\|g_n\|_{L^4}^4 = \sum_{j=1}^{L} \big(\|\partial_x V^j\|_{L^2}^2 + \frac{1}{8}\|V^j\|_{L^4}^4\big) + \big(\|\partial_x R_n^L\|_{L^2}^2 + \frac{1}{8}\|R_n^L\|_{L^4}^4\big) + o_n(1). \tag{2.11}$$

Taking the limits $\lim\limits_{L\to\infty}\lim\limits_{n\to\infty}$ on both sides, we have

$$\|\partial_x V^j\|_{L^2}^2 + \frac{1}{8}\|V^j\|_{L^4}^4 \le d, \quad \text{for any } j = 1, 2, \cdots.$$

Thus, by Lemma 2, we have

$$K(V^j) \ge 0, \quad \text{for } j = 1, 2, \cdots. \tag{2.12}$$

Now taking the limits $\lim\limits_{L\to\infty}\lim\limits_{n\to\infty}$ on the both two sides of (2.8), and by (2.12) we have

$$K(V^j) = 0, \quad \text{for any } j = 1, 2, \cdots. \tag{2.13}$$

Then by the definition of d, we deduce that for any $j = 1, 2, \cdots$,

$$\text{either } S(V^j) \ge d, \quad \text{or} \quad V^j = 0.$$

However, $S(g_n) \to d$ and $d > 0$, so there exists exactly one j, say $j = 1$, such that

$$S(V^1) = d,$$

and $V^j = 0$ for other $j \ge 2$. Since $K(V^1) = 0$, we obtain the minimizer $G = V^1$ which attains d. Moreover, from (2.10) and (2.11), we find that the remainder term R_n (since $L = 1$, we may omit the superscript L),

$$\lim_{n\to\infty}\|R_n\|_{L^4\cap\dot H^1} = 0.$$

Thus we close the proof of the proposition. □

As mentioned before, $d \le S(W)$. In fact, we have the equality.

Lemma 3. *$d = S(W)$.*

Proof. Consider the set

$$\mathcal{M} := \{\phi \in H^1(\mathbb{R}) \setminus \{0\} : S(\phi) = d, K(\phi) = 0\}. \tag{2.14}$$

Then by Proposition 2.1, $\mathcal{M} \ne \emptyset$. By the Lagrangian multiplier, there exists λ, such that for any $\phi \in \mathcal{M}$, such that

$$S'(\phi) = \lambda K'(\phi). \tag{2.15}$$

Testing a function $\psi = \frac{1}{2}\phi - x\phi_x$, we have

$$S'(\phi)\psi = \lambda K'(\phi)\psi. \tag{2.16}$$

On one hand, since $S'(\phi) = 2(-\partial_{xx}\phi + |\phi|^2\phi - \frac{3}{16}|\phi|^4\phi)$, we have

$$\begin{aligned} S'(\phi)\psi &= 2\mathrm{Re}\int_{\mathbb{R}}(-\phi_{xx} + |\phi|^2\phi - \frac{3}{16}|\phi|^4\phi)\bar\psi\,dx \\ &= \frac{1}{8}\int_{\mathbb{R}}(6|\phi|^4 - |\phi|^6)\,dx = \frac{1}{8}K(\phi). \end{aligned}$$

Thus $S'(\phi)\psi = 0$ for any $\phi \in \mathcal{M}$. On the other hand, $K'(\phi) = 24|\phi|^2\phi - 6|\phi|^4\phi$, gives that

$$\begin{aligned} K'(\phi)\psi &= 6\mathrm{Re}\int_{\mathbb{R}}(4|\phi|^2\phi - |\phi|^4\phi)\bar\psi\,dx \\ &= \int_{\mathbb{R}}(18|\phi|^4 - 4|\phi|^6)\,dx = -6\int_{\mathbb{R}}|\phi|^4\,dx. \end{aligned}$$

Thus, for any $\phi \in \mathcal{M}$, $K'(\phi)\psi = -6\int_{\mathbb{R}}|\phi|^4\,dx \ne 0$. Therefore, from (2.16), we obtain that $\lambda = 0$. Thus, (2.16) yields that $S'(\phi) = 0$. Hence, using Lemma 2.1, we obtain that $\phi = e^{i\theta}W(\cdot - x_0)$ for some $\theta, x_0 \in \mathbb{R}$ and thus $d = S(W)$. This proves the lemma. □

From the proof of Lemma 3, we obtain

$$\begin{aligned}\mathcal{M} &= \{\phi \in H^1(\mathbb{R}) \setminus \{0\} : S(\phi) = S(W), K(\phi) = 0\} \\ &= \{e^{i\theta} W(\cdot - x_0) : \theta \in \mathbb{R}, x_0 \in \mathbb{R}\}.\end{aligned}$$

This rigidity implies that the function G obtained in Proposition 2 is equal to $e^{i\theta}W(\cdot - x_0)$ for some $\theta, x_0 \in \mathbb{R}$. Therefore, we conclude Proposition 1 from Proposition 2.

3. Proof of Theorem 1 and Theorem 2

We first prove Theorem 1. Instead of proving Theorem 1, we give a slightly more general result. This will be more useful in the proof of Theorem 2. To this end, we study the solution to (1.11). We rewrite conservation laws in terms of $v(t)$ variable.

$$M(v(t)) := \|v(t)\|_{L_x^2}^2 = M(v_0), \tag{3.1}$$

$$P(v(t)) := \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} \bar{v}(t) v_x(t)\, dx + \frac{1}{8}\int_{\mathbb{R}} |v(t)|^4\, dx = P(v_0), \tag{3.2}$$

$$E(v(t)) := \frac{1}{2}\|v_x(t)\|_{L_x^2}^2 - \frac{1}{32}\|v(t)\|_{L_x^6}^6 = E(v_0). \tag{3.3}$$

Theorem 3. *For any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ such that if*

$$E(v_0) = O(\delta), \quad P(v_0) = O(\delta), \quad \text{and} \quad M(v_0) = M(W) + O(\delta), \tag{3.4}$$

then the result in Theorem 1 holds.

It is obvious that (1.5) implies (3.4). Hence Theorem 1 is a consequence of Theorem 3. Let $\varepsilon > 0$ be given. We first claim the following important lemma.

Lemma 4. *Let ε_0 be the constant in Proposition 1. Under the assumption in Theorem 3, for sufficiently small $\delta > 0$, we have*

$$\|v(t)\|_{L^6} \geq \varepsilon_0.$$

Before giving the proof of the lemma, we provide a preliminary setting. To simplify notations regarding to the functional S, we set

$$E_0 = E(v_0), \quad P_0 = P(v_0), \quad M_0 = M(v_0).$$

Then under the assumption (3.4), we have

$$E_0, P_0 = O(\delta), \qquad \text{and } M_0 = M(W) + O(\delta), \tag{3.5}$$

where $O(\delta) \to 0$ as $\delta \to 0$.
We define the function w by

$$w(t,x) := e^{-it+ix} v(t, x - 2t), \qquad w_0 = e^{ix} v_0. \tag{3.6}$$

Then the assumption (1.5) becomes

$$\|w_0 - W\|_{H^1} \leq \delta. \tag{3.7}$$

Again, we can rewrite conservation laws in $w(t,x)$ variable. The mass, momentum and energy conservation laws (3.1)–(3.3) are changed as follows:

$$M(w(t)) := \|w(t)\|_{L_x^2}^2 = M_0, \tag{3.8}$$

$$\widetilde{P}(w(t)) := \mathrm{Im}\int_{\mathbb{R}} \bar{w}(t)w_x(t)\,dx - \|w(t)\|_{L_x^2}^2 + \frac{1}{4}\int_{\mathbb{R}} |w(t)|^4\,dx = P_0, \tag{3.9}$$

$$\widetilde{E}(w(t)) := \|w_x(t)\|_{L_x^2}^2 - 2\mathrm{Im}\int \bar{w}(t)w_x(t)\,dx + \|w(t)\|_{L_x^2}^2 - \frac{1}{16}\|w(t)\|_{L_x^6}^6 = E_0. \tag{3.10}$$

We also find that

$$S(w) = \widetilde{E}(w(t)) + 2\widetilde{P}(w(t)) + \widetilde{M}(w(t)) = E_0 + 2P_0 + M_0.$$

Thus by (3.5), we have

$$\begin{aligned} S(w(t)) &= M(W) + O(\delta) \\ &= S(W) + O(\delta). \end{aligned} \tag{3.11}$$

Proof of Lemma 4.
Fix $t \in I$. Note that

$$\|w(t)\|_{L_x^4} = \|v(t)\|_{L_x^4}; \quad \|w(t)\|_{L^6} = \|v(t)\|_{L^6}.$$

Thus, we have

$$\|w(t)\|_{L_x^4}^4 \le \|w(t)\|_{L^2}\|w(t)\|_{L^6}^3 \le \sqrt{M_0}\|w(t)\|_{L^6}^3.$$

Assume that there is a time t, such that $\|v(t)\|_{L^6} \le \varepsilon_0$. Then we have

$$|K(w(t))| = \big|6\|w(t)\|_{L^4}^4 - \|w(t)\|_{L^6}^6\big| \le 6\sqrt{M_0}\varepsilon_0^3 - \varepsilon_0^6 \le \varepsilon_0.$$

Combining this with (3.11), and using Proposition 1, we have

$$\big\|w(t) - e^{i\theta}W(\cdot - y)\big\|_{\dot{H}^1} < \varepsilon$$

for some $(\theta, y) \in \mathbb{R}^2$. Moreover, by the mass conservation law and (3.4), we have

$$\|w(t)\|_{L^2}^2 - \|W\|_{L^2}^2 = O(\delta).$$

By choosing δ small enough, we have

$$\big\|w(t) - e^{i\theta}W(\cdot - y)\big\|_{H^1} < 2\varepsilon.$$

It implies that

$$\|v(t)\|_{L^6} = \|w(t)\|_{L^6} \ge \frac{1}{2}\|W\|_{L^6}^6,$$

which leads the contradiction with $\|v(t)\|_{L^6} \le \varepsilon_0$.

□

Now we consider the relationship between $\|v(t)\|_{L^4}$ and $\|v(t)\|_{L^6}$. We denote

$$f(t) = \frac{\|v(t)\|_{L^4}^4}{\|v(t)\|_{L^6}^3}.$$

We first prove that

Proposition 3. *For any $t \in I$,*

$$\big|f(t)^2 - \frac{8}{3}\pi\big| \le O(\delta).$$

To prove this proposition, we adopt the argument in [21]. We sketch the proof when the argument is highly similar to [21]. Firstly, we have

Lemma 5. *For any $t \in I$,*

$$2C_{GN}^{-\frac{9}{2}} + O(\delta) \leq f(t) \leq \sqrt{M_0}.$$

Proof. From the Hölder's inequality, we have

$$\|v(t)\|_{L^4}^4 \leq \|v(t)\|_{L^2}\|v(t)\|_{L^6}^3 = \sqrt{M_0}\|v(t)\|_{L^6}^3,$$

and thus

$$f(t) \leq \sqrt{M_0}.$$

On the other hand, by using the similar argument in [21], we have

$$f(t) \geq 2C_{GN}^{-\frac{9}{2}} + \varepsilon(t),$$

where

$$\varepsilon(t) := 2C_{GN}^{-\frac{9}{2}} \frac{\|v(t)\|_{L^6}^{\frac{3}{2}} - \left(\|v(t)\|_{L^6}^6 + 16E_0\right)^{\frac{1}{4}}}{\left(\|v(t)\|_{L^6}^6 + 16E_0\right)^{\frac{1}{4}}}.$$

By the Mean Value Theorem, $E_0 = C\delta$, and $\|v(t)\|_{L^6} \geq \varepsilon_0$ by Lemma 4, we have

$$\varepsilon(t) = CE_0\|v(t)\|_{L^6}^{-6} = O(\delta).$$

This proves the lemma. □

Proof of Proposition 3.
We define

$$\phi(t,x) = e^{i\alpha x}v(t,x),$$

where the parameter α depends on t, and is given below. Then we have

$$E(\phi) = E(v) + 2\alpha \mathrm{Im}\int \bar{v}\, v_x\, dx + \alpha^2\|v\|_{L^2}^2.$$

By the mass, energy conservation laws (3.1) and (3.3), (1.9), we have for any $\alpha > 0$,

$$-2\alpha \mathrm{Im}\int \overline{v(t,x)}\, v_x(t,x)\, dx \leq \Big(\frac{1}{16} - C_{GN}^{-18} f(t)^{-4}\Big)\|v(t)\|_{L^6}^6 + \alpha^2 M_0 + E_0,$$

or

$$-\mathrm{Im}\int \overline{v(t,x)}\, v_x(t,x)\, dx \leq \frac{1}{2\alpha}\Big(\frac{1}{16} - C_{GN}^{-18} f(t)^{-4}\Big)\|v(t)\|_{L^6}^6 + \frac{1}{2}\alpha M_0 + \frac{1}{2\alpha}E_0. \tag{3.12}$$

By the momentum conservation law (3.2), we estimate

$$\frac{1}{4}\|v(t)\|_{L^4}^4 \leq \frac{1}{2\alpha}\Big(\frac{1}{16} - C_{GN}^{-18} f(t)^{-4}\Big)\|v(t)\|_{L^6}^6 + \frac{1}{2}\alpha M_0 + \frac{1}{2\alpha}E_0 + P_0. \tag{3.13}$$

Next, we claim that for any $t \in I$,

$$\Big(\frac{1}{16} - C_{GN}^{-18} f(t)^{-4}\Big)\|v(t)\|_{L^6}^6 \geq |E_0|. \tag{3.14}$$

To prove (3.14), for a contradiction, we assume there exists a time t_0 such that the negation of (3.14) holds. Then choosing $\alpha = \sqrt{|E_0|}$, we have

$$\frac{1}{4}\|v(t_0)\|_{L^4}^4 \leq \sqrt{|E_0|} + P_0 = O(\delta).$$

But by Lemma 5, $\|v(t)\|_{L^4}$ is on the level of ε_0^3, so suitably narrowing δ, we reach the contradiction.
Now, we choose

$$\alpha(t) = \sqrt{M_0^{-1}\Big(\frac{1}{16} - C_{GN}^{-18} f(t)^{-4}\Big)}\|v(t)\|_{L^6}^3.$$

By (3.13) and (3.14), we estimate $\alpha(t) \geq \sqrt{M_0^{-1}|E_0|}$ and

$$\begin{aligned}\|v(t)\|_{L^4}^4 \leq& \sqrt{M_0\Big(1 - 16C_{GN}^{-18} f(t)^{-4}\Big)}\|v(t)\|_{L^6}^3 + 2\alpha^{-1}E_0 + 4P_0;\\ \leq& \sqrt{M_0\Big(1 - 16C_{GN}^{-18} f(t)^{-4}\Big)}\|v(t)\|_{L^6}^3 + O(\delta). \end{aligned} \tag{3.15}$$

Since $\|v(t)\|_{L^6} \geq \varepsilon_0$, by (3.15), we find that

$$f \leq \sqrt{M_0\Big(1 - 16C_{GN}^{-18} f^{-4}\Big)} + O(\delta).$$

By Lemma 5, we obtain

$$f^6 \leq M_0 f^4 - 16M_0 C_{GN}^{-18} + O(\delta). \tag{3.16}$$

Note that the equation

$$X^3 - M_0X^2 + 16M_0C_{GN}^{-18} \leq 0$$

admits only one solution $X = \frac{8}{3}\pi$ when $M_0 = 4\pi$. Thus when $M_0 = 4\pi + O(\delta)$, by the continuity argument, we have $f^2 = \frac{8}{3}\pi + O(\delta)$. □

Now we use the scaling argument, let $\lambda(t) = \|W\|_{L^6}/\|v(t)\|_{L^6}$, and

$$v_\lambda(t,x) = \lambda^{\frac{1}{2}} v(\lambda^2 t, \lambda x).$$

Then from Lemma 4, $\lambda \leq \varepsilon_0\|W\|_{L^6}$, and

$$\|v_\lambda(t)\|_{L^6}^6 = \|W\|_{L^6}^6 = 96\pi;. \tag{3.17}$$

Since $f(t)$ is scaling invariant, i.e. $\frac{\|v_\lambda(t)\|_{L^4}^4}{\|v_\lambda(t)\|_{L^6}^3} = f(t)$, we have

$$\|v_\lambda(t)\|_{L^4}^4 = f(t)\|W\|_{L^6}^3 = 16\pi + O(\delta) = \|W\|_{L^4}^4 + O(\delta). \tag{3.18}$$

Let $w(t,x;\lambda)$ be defined as

$$w(t,x;\lambda) := e^{-it+ix} v_\lambda(t, x-2t). \tag{3.19}$$

Then

$$M\big(w(\lambda)\big) = M(v_\lambda) = M(v) = M_0; \tag{3.20}$$

$$\widetilde{P}\big(w(\lambda)\big) = P(v_\lambda) = \lambda P(v) = \lambda P_0; \tag{3.21}$$

$$\widetilde{E}\big(w(\lambda)\big) = E(v_\lambda) = \lambda^2 E(v) = \lambda^2 E_0. \tag{3.22}$$

From $\widetilde{P}$ in (3.9), we have

$$\operatorname{Im}\int_{\mathbb{R}} \bar{w}(t;\lambda)\partial_x w(t;\lambda)\,dx = \|w(t;\lambda)\|_{L_x^2}^2 - \frac{1}{4}\int_{\mathbb{R}} |w(t;\lambda)|^4\,dx + \lambda P_0.$$

Note that $\lambda \leq \varepsilon_0^{-1}\|W\|_{L^6}$, combining this with (3.18), we have

$$\operatorname{Im}\int_{\mathbb{R}} \bar{w}(t;\lambda)\partial_x w(t;\lambda)\,dx = O(\delta). \tag{3.23}$$

Inserting (3.23) into $\widetilde{E}(w_\lambda)$ in (3.10), and applying (3.17), we have

$$\begin{aligned}\|\partial_x w(t;\lambda)\|^2_{L^2_x} =&2\mathrm{Im}\int \bar{w}(t;\lambda)\partial_x w(t;\lambda)\,dx - \|w(t;\lambda)\|^2_{L^2_x} + \frac{1}{16}\|w(t;\lambda)\|^6_{L^6_x} + \lambda^2 E_0;\\ =&2\pi + O(\delta).\end{aligned} \tag{3.24}$$

Therefore, by (3.17), (3.18) and (3.24), we have

$$\begin{aligned}S(w(\lambda)) =&\|\partial_x w(\lambda)\|^2_{L^2} + \frac{1}{2}\|w(\lambda)\|^4_{L^4} - \frac{1}{16}\|w(\lambda)\|^6_{L^6}\\ =&4\pi + O(\delta) = S(W) + O(\delta);\\ K(w(\lambda)) =&6\|w(\lambda)\|^4_{L^4} - \|w(\lambda)\|^6_{L^6} = O(\delta).\end{aligned}$$

By Proposition 1, we get

$$\inf_{(\theta,y)\in\mathbb{R}^2} \left\|w(\lambda) - e^{i\theta}W(\cdot - y)\right\|_{\dot{H}^1} < \varepsilon.$$

By the mass conservation law, we further obtain

$$\inf_{(\theta,y)\in\mathbb{R}^2} \left\|w(\lambda) - e^{i\theta}W(\cdot - y)\right\|_{H^1} < \varepsilon.$$

Thus, by (3.19) and (1.10), we prove that

$$\|u(t) - e^{i\theta}R_\lambda(t,\cdot - y)\|_{H^1} \le \varepsilon.$$

This proves Theorem 3.

Proof of Theorem 2.
Set $\lambda(s) = \|\partial_x W\|_{L^2}/\|\partial_x v(s)\|_{L^2}$, and

$$v_{[s]}(t,x) = \lambda(s)^{\frac{1}{2}} v\big(\lambda(s)^2 t, \lambda(s)x\big). \tag{3.25}$$

Then by the conservation laws of v, we have

$$\begin{aligned}E(v_{[s]}(t)) =&E(v_{[s]}(0)) = \lambda(s)^2 E(v_0),\\ P(v_{[s]}(t)) =&P(v_{[s]}(0)) = \lambda(s)^2 P(v_0),\\ M(v_{[s]}(t)) =&M(v_{[s]}(0)) = M(v_0).\end{aligned}$$

So under the assumption of Theorem 2, if the solution v to (1.11) blows up in the finite time T^*, then $\|\partial_x v(s)\|_{L^2} \to \infty$ as $s \to T^*$. Thus, we have

$$\lambda(s) \to 0, \text{ as } s \to T^*.$$

Hence, when $s \to T^*$,

$$E(v_{[s]}(0)), P(v_{[s]}(0)) \to 0, \text{ and } M(v_{[s]}(0)) = M(W).$$

This implies from Theorem 3 that there exist $\theta_s(t) \in [0, 2\pi), y_s(t) \in \mathbb{R}$, and $\widetilde{\lambda}_s(t) \in [\lambda_0, \infty)$ such that when $s \to T^*$,

$$u_{[s]}(t) - e^{i\theta_s(t)}R_{\widetilde{\lambda}_s(t)}(t, \cdot - y_s(t)) \to 0 \quad \text{in } H^1(\mathbb{R}),$$

uniformly in t. Moreover, $\widetilde{\lambda}_s(s) = 1$. In particular, when $t = s$, we have

$$u_{[s]}(s) - e^{i\theta_s(s)}R(s, \cdot - y_s(s)) \to 0 \quad \text{in } H^1(\mathbb{R}).$$

In view of (3.25), we finish the proof of Theorem 2.

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