Log-concavity of the Fennessey-Larcombe-French Sequence

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Abstract. We prove the log-concavity of the Fennessey-Larcombe-French sequence based on its three-term recurrence relation, which was recently conjectured by Zhao. The key ingredient of our approach is a sufficient condition for log-concavity of a sequence subject to certain three-term recurrence.

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Suggested Running Head: Log-concavity of the FLF Sequence

1. Introduction

The objective of this paper is to prove the log-concavity conjecture of the Fennessey-Larcombe-French sequence, which was posed by Zhao [17] in the study of log-balancedness of combinatorial sequences.

Let us begin with an overview of Zhao's conjecture. Recall that a sequence $\{a_k\}_{k\geq 0}$ is said to be log-concave if

$$a_k^2 \geq a_{k+1}a_{k-1}, \quad \text{for } k \geq 1,$$

and it is log-convex if

$$a_k^2 \leq a_{k+1}a_{k-1}, \quad \text{for } k \geq 1.$$

We say that $\{a_k\}_{k\geq 0}$ is log-balanced if the sequence itself is log-convex while $\{\frac{a_k}{k!}\}_{k\geq 0}$ is log-concave.

The Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$ can be given by the following three-term recurrence relation [9]

$$n(n+1)^2V_{n+1} = 8n(3n^2+5n+1)V_n - 128(n-1)(n+1)^2V_{n-1}, \quad \text{for } n \geq 1, \tag{1.1}$$

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with the initial values $V_0 = 1$ and $V_1 = 8$. This sequence was introduced by Larcombe, French and Fennessey [8], in connection with a series expansion of the complete elliptic integral of the second kind, precisely,

$$\int_0^{\pi/2} \sqrt{1 - c^2 \sin^2\theta}\, d\theta = \frac{\pi\sqrt{1-c^2}}{2} \sum_{n=0}^{\infty} \left(\frac{1 - \sqrt{1-c^2}}{16} \right)^n V_n.$$

The Fennessey-Larcombe-French sequence is closely related to the Catalan-Larcombe-French sequence, which was first studied by E. Catalan [1] and later examined and clarified by Larcombe and French [7]. Let $\{P_n\}_{n\geq 0}$ denote the Catalan-Larcombe-French sequence, and the following three-term recurrence relation holds:

$$(n+1)^2 P_{n+1} = 8(3n^2 + 3n + 1)P_n - 128n^2 P_{n-1}, \quad \text{for } n \geq 1,$$

with $P_0 = 1$ and $P_1 = 8$. As a counterpart of V_n, the numbers P_n appear as coefficients in the series expansion of the complete elliptic integral of the first kind, precisely,

$$\int_0^{\pi/2} \frac{1}{\sqrt{1 - c^2 \sin^2\theta}}\, d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{1 - \sqrt{1-c^2}}{16} \right)^n P_n.$$

Many interesting properties have been found for the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence, and the reader may consult references [5, 6, 7, 8, 9, 15].

Recently, there has arisen an interest in the study of the log-behavior of the Catalan-Larcombe-French sequence. For instance, Xia and Yao [13] obtained the log-convexity of the Catalan-Larcombe-French sequence, and confirmed a conjecture of Sun [11]. By using a log-balancedness criterion due to Došlić [4], Zhao [16] proved the log-balancedness of the Catalan-Larcombe-French sequence.

Zhao further studied the log-behavior of the Fennessey-Larcombe-French sequence, and obtained the following result.

Theorem 1.1 ([17]). *Both $\{nV_n\}_{n\geq 1}$ and $\{\frac{V_n}{(n-1)!}\}_{n\geq 1}$ are log-concave.*

She also made the following conjecture.

Conjecture 1.2. *The Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 1}$ is log-concave.*

Note that the Hadamard product of two log-concave sequences without internal zeros is still log-concave, see [14, Proposition 2]. Since both $\{n\}_{n\geq 1}$ and $\{\frac{1}{(n-1)!}\}_{n\geq 1}$ are log-concave, Conjecture 1.2 implies Theorem 1.1.

In this paper, we obtain a sufficient condition for proving the log-concavity of a sequence satisfying a three-term recurrence. Then we give an affirmative answer to Conjecture 1.2 by using this criterion. By further employing a result of Wang and Zhu [12, Theorem 2.1], we derive the monotonicity of the sequence $\{\sqrt[n]{V_{n+1}}\}_{n\geq 1}$ from the log-concavity of $\{V_n\}_{n\geq 1}$.

2. Log-concavity derived from three-term recurrence

The aim of this section is to prove the log-concavity of the Fennessey-Larcombe-French sequence based on its three-term recurrence relation.

We first give a sufficient condition for log-concavity of a positive sequence subject to certain three-term recurrence. It should be mentioned that the log-behavior of sequences satisfying three-term recurrences has been extensively studied, see Liu and Wang [10], Chen and Xia [3], Chen, Guo and Wang [2], and Wang and Zhu [12]. However, most of these studies have focused on the log-convexity of such sequences instead of their log-concavity. Our criterion for determining the log-concavity of a sequence satisfying a three-term recurrence is as follows.

Proposition 2.1. *Let $\{S_n\}_{n\geq 0}$ be a positive sequence satisfying the following recurrence relation:*

$$a(n)S_{n+1} + b(n)S_n + c(n)S_{n-1} = 0, \quad \text{for } n \geq 1, \tag{2.1}$$

where $a(n), b(n)$ and $c(n)$ are real functions of n. Suppose that there exists an integer n_0 such that for any $n > n_0$,

(i) it holds $a(n) > 0$, and

(ii) either $b^2(n) < 4a(n)c(n)$ or $\frac{S_n}{S_{n-1}} \geq \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)}$.

Then the sequence $\{S_n\}_{n\geq n_0}$ is log-concave, namely, $S_n^2 \geq S_{n+1}S_{n-1}$ for any $n > n_0$.

Proof. Let $r(n) = \frac{S_n}{S_{n-1}}$. It suffices to show that $r(n) \geq r(n+1)$ for any $n > n_0$. On one hand, the conditions (i) and (ii) imply that

$$a(n)r^2(n) + b(n)r(n) + c(n) \geq 0, \quad \text{for } n > n_0.$$

Since $\{S_n\}_{n\geq 0}$ is a positive sequence, so is $\{r_n\}_{n\geq 1}$. Thus, the above inequality is equivalent to the following

$$a(n)r(n) + b(n) + \frac{c(n)}{r(n)} \geq 0, \quad \text{for } n > n_0. \tag{2.2}$$

On the other hand, dividing both sides of (2.1) by S_n, we obtain

$$a(n)r(n+1) + b(n) + \frac{c(n)}{r(n)} = 0. \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$a(n)r(n+1) \leq a(n)r(n), \quad \text{for } n > n_0.$$

By the condition (i), we have $r(n+1) \leq r(n)$ for any $n > n_0$. This completes the proof. ■

We are now able to give the main result of this section, which offers an affirmative answer to Conjecture 1.2.

Theorem 2.2. *Let $\{V_n\}_{n\geq 0}$ be the Fennessey-Larcombe-French sequence given by (1.1). Then, for any $n \geq 2$, we have $V_n^2 \geq V_{n-1}V_{n+1}$.*

Proof. By the recurrence relation (1.1), we have $V_1 = 8, V_2 = 144, V_3 = 2432$ and $V_4 = 40000$. It is easy to verify that $V_2^2 \geq V_1V_3$ and $V_3^2 \geq V_2V_4$.

We proceed to use Proposition 2.1 to prove that $V_n^2 > V_{n-1}V_{n+1}$ for $n > 3$, namely taking $n_0 = 3$. For the sequence $\{V_n\}_{n\geq 0}$, the corresponding polynomials $a(n), b(n), c(n)$ appearing in Proposition 2.1 are as follows:

$$\begin{aligned} a(n) &= n(n+1)^2, \\ b(n) &= -8n(3n^2+5n+1), \\ c(n) &= 128(n-1)(n+1)^2. \end{aligned}$$

It is clear that $a(n) > 0$ for any $n > 3$. By a routine computation, we get

$$b^2(n) - 4a(n)c(n) = 64(n^6+6n^5+15n^4+26n^3+25n^2+8n) > 0, \quad \text{for } n > 3.$$

It suffices to show that

$$\frac{V_n}{V_{n-1}} \geq \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)}, \quad \text{for } n > 3. \tag{2.4}$$

This inequality also implies the positivity of V_n since its right-hand side is positive for any $n > 3$. (Note that $b(n)$ is negative.) However, it is difficult to directly prove (2.4). The key idea of our proof is to find an intermediate function $h(n)$ such that

$$\frac{V_n}{V_{n-1}} \geq h(n) \geq \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)}, \quad \text{for } n > 3.$$

Let

$$h(n) = \frac{16(n^3-n^2+1)}{n^3-n^2}, \quad \text{for } n \geq 2, \tag{2.5}$$

and we shall show that this function fulfills our purpose. This will be done in two steps.

First, we need to prove that

$$h(n) - \frac{-b(n)+\sqrt{b^2(n)-4a(n)c(n)}}{2a(n)} \geq 0, \quad \text{for } n > 3. \tag{2.6}$$

A straightforward computation shows that the quantity on the left-hand side is equal to

$$\frac{32(4n^6+7n^5+n^4+n^3+9n^2+8n+2)}{(n^4-n^2)(n+1)(n^5+2n^4+n^2+8n+4+(n^2-n)\sqrt{n^6+6n^5+15n^4+26n^3+25n^2+8n})},$$

which is clearly positive for $n>3$.

Second, we need to prove that

$$\frac{V_n}{V_{n-1}} \geq h(n), \quad \text{for } n>3. \tag{2.7}$$

For convenience, let $g(n)=\frac{V_n}{V_{n-1}}$. We use induction on n to prove that $g(n)\geq h(n)$ for $n>3$. By the recurrence relation (1.1), we have

$$g(n+1)=\frac{8(3n^2+5n+1)}{(n+1)^2}-\frac{128(n-1)}{ng(n)}, \quad n\geq 1, \tag{2.8}$$

with the initial value $g(1)=8$. It is clear that $g(3)=152/9=h(3)$ and $g(4)=625/38>49/3=h(4)$ by (2.5) and (2.8). Assume that $g(n)>h(n)$, and we proceed to show that $g(n+1)>h(n+1)$. Note that

$$\begin{aligned} g(n+1)-h(n+1) &= \frac{8(3n^2+5n+1)}{(n+1)^2}-\frac{128(n-1)}{ng(n)}-\frac{16(n^3+2n^2+n+1)}{n(n+1)^2} \\ &= \frac{8(n^3+n^2-n-2)}{n(n+1)^2}-\frac{128(n-1)}{ng(n)} \\ &= \frac{8(n^3+n^2-n-2)g(n)-128(n-1)(n+1)^2}{n(n+1)^2g(n)}. \end{aligned}$$

By the induction hypothesis, we have $g(n)>h(n)>0$ and thus

$$\begin{aligned} g(n+1)-h(n+1) &> \frac{8(n^3+n^2-n-2)h(n)-128(n-1)(n+1)^2}{n(n+1)^2g(n)} \\ &= \frac{128(2n^2-n-2)}{n^3(n-1)(n+1)^2g(n)}>0. \end{aligned}$$

Combining (2.6) and (2.7), we obtain the inequality (2.4). This completes the proof. ■

Wang and Zhu [12, Theorem 2.1] showed that if $\{z_n\}_{n\geq 0}$ is a log-concave sequence of positive integers with $z_0>1$, then $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is strictly decreasing. Applying their criterion to the Fennessey-Larcombe-French sequence, we obtain immediately the following result.

Proposition 2.3. *The sequence* $\{\sqrt[n]{V_{n+1}}\}_{n\geq 1}$ *is strictly decreasing.*

Proof. Let $\{z_n\}_{n\geq 0}$ be the sequence given by $z_n=V_{n+1}$. It is clear that $z_0=V_1=8>1$. Moreover, by Theorem 2.2, the sequence $\{z_n\}_{n\geq 0}$ is log-concave. Thus, $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is strictly decreasing by [12, Theorem 2.1]. ■

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