Concentrating patterns of reaction-diffusion systems: A variational approach

Dedicated to Antonio Ambrosetti on the occasion of his 70th birthday

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Abstract

Our purpose is to motivate an analytical characterization aimed at predicting patterns for general reaction-diffusion systems, depending on the spatial distribution involved in the reaction terms. It is shown that there must be a pattern concentrating around the local minimum of the chemical potential distribution for small diffusion coefficients. A multiple concentrating result is also established to illustrate the mechanisms leading to emergent spatial patterns. The results of this paper were proved by using a general variational technique. This enables us to consider nonlinearities which grow either super quadratic or asymptotic quadratic at infinity.

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1 Introduction and main result

In this paper we describe a unified abstract setting for strongly indefinite singular limit problems. As applications a special attention is paid to concentrating patterns of reaction-diffusion systems.

General reaction-diffusion systems have been used to study mechanisms leading to emergent spatial patterns. They arise naturally in a variety of models from theoretical physics, chemistry and biology (see for example [20, 26] and references therein). A reaction-diffusion system consists of two or more coupled nonlinear partial differential equations (PDEs), which describe reactions and diffusion of chemicals or morphogens.

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In its most general form a reaction-diffusion model describing the time variation of two chemical concentrations U and V due to reaction and diffusion can be written in the form

$$
\begin{cases} \partial_t U = D_U \Delta_x U + f(U,V) \\ \partial_t V = D_V \Delta_x V + g(U,V) \end{cases} \tag{1.1}
$$

where D_U and D_V are the diffusion coefficients setting the pace of diffusion for chemicals U and V, respectively. The dynamics of the model is determined by the reaction kinetics $f(U,V)$ and $g(U,V)$, which are nonlinear functions of the concentrations. Specifically for a positive diffusion coefficients, let us consider the first equation in (1.1) as an example, $D_U \Delta_x U$ is the diffusion term which specifies that U will increase in proportion to the Laplacian of U. When the quantity of U is higher in neighboring areas, U will increase (this follows the Fick's first law). The nonlinear terms, f and g, are the reaction terms modeling chemical reaction with a replenishment or diminishment. These terms can be derived from chemical reaction formulae by using the law of mass action and other physical conditions. The parameters within f and g will generally govern the patterns in the model.

Alan Turing showed in 1952 that a simple system of coupled reaction-diffusion equations could give rise to finite wavelength spatial patterns due to a mechanism called diffusion-driven instability [27]. These so-called Turing patterns and other related chemical systems have ever since been under intensive theoretical studies and similar pattern forming mechanisms have been connected to various physical systems. Due to the large applicability of pattern generating mechanisms in several research fields, understanding the relationship between reaction-diffusion parameters and specific patterns becomes essential. So, the present work is intended as an attempt to motivate an analytic characterization aimed at predicting patterns for general reaction-diffusion systems, depending on the spatial distribution involved in the reaction terms. We will illustrate these ideas with some general pattern-generating reaction-diffusion systems involving variational structure.

For such, we will consider the following $2M$-component reaction-diffusion system

$$
\begin{cases} \partial_t u = \varepsilon^2 \Delta_x u - u - V(x)v + \partial_v H(u,v) \\ \partial_t v = -\varepsilon^2 \Delta_x v + v + V(x)u - \partial_u H(u,v) \end{cases} \tag{1.2}
$$

in spatial domain $\mathbb{R}^N$, where $(u,v) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^M$ models the concentration field of different chemicals. In such system the function $V : \mathbb{R}^N \to \mathbb{R}$ determines a relative spatial distribution of a chemical potential, and the nonlinear part (determined by the function $H : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$) gives a external physico-chemical force. It is worth noting that, in the second equation of (1.2), the diffusion coefficient is negative, which represents the phenomenon referred as the uphill diffusion (occurs during phase separation, a situation where the transport of particles in a medium occurs towards regions of higher concentration). This type of problems arises in a wide variety of applications. Specifically, the important feature of these systems for our purpose is the competition between different temporal growth rates and spatial ranges of diffusion for different chemicals in the system. For example the very simple 2-component equations for the concentrations $u(t,x)$ and $v(t,x)$ of two reaction and diffusion chemicals, the expressions (1.2) have been interpreted in terms of the interaction of an *inhibitor* u and an *activator* v (see for example Murrary [23]). The function H models the nonlinear response of intense electromagnetic waves propagating in various types of media. And this nonlinearity leads to equations for the envelope of the waves, in the form of "nonlinear Schrödinger" type equations.

It turns out that in the coupled system (1.2), the nonlinear function H is operated in such a way that different spatial modes of the electromagnetic fields are excited and parameters can be adjusted so that spatial patterns will appear. Hence, here in this context, we will focus on the asymptotic behavior of the solutions to (1.2) with small diffusion coefficients (that is the performance of these solutions as $\varepsilon \to 0$). This presents a sort of concentricity of the patterns generated by the reaction-diffusion process and its dependence on the parameters and the spatial distributions.

There is not much work on solutions of systems similar to (1.2). Brézis and Nirenberg [5] considered the 2-component system

$$\begin{cases} \partial_t u = \Delta_x u - v^5 + f(x) \\ \partial_t v = -\Delta_x v - u^3 + g(x) \end{cases} \quad \text{in } (0,T)\times\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f, g \in L^\infty(\Omega)$. Subject to the boundary conditions $u(t,x) = v(t,x) = 0$ on $(0,T)\times\partial\Omega$ and $u(0,x) = v(x,T) = 0$ on Ω, the authors obtained a solution (u,v) with $u \in L^4$ and $v \in L^6$ of (1.3) by using Schauders fixed point theorem. In [8], Clément, Felmer and Mitidieri considered the problem (unbounded Hamiltonian systems)

$$\begin{cases} \partial_t u = \Delta_x u + |v|^{q-2}v \\ \partial_t v = -\Delta_x v - |u|^{p-2}u \end{cases} \quad \text{in } (-T,T)\times\Omega, \tag{1.4}$$

where Ω is a smoothly bounded domain in R^N, and $\frac{N}{N+2} < \frac{1}{p} + \frac{1}{q} < 1$. By variational arguments, they proved that there exists $T_0 > 0$ such that for each $T > T_0$, (1.4) has at least one positive solution satisfying the 0-boundary condition: $u(t,\cdot)|_{\partial\Omega} = 0 = v(t,\cdot)|_{\partial\Omega}$ for all $t \in (-T,T)$, and the periodicity condition: $u(-T,\cdot) = u(T,\cdot)$, $v(-T,\cdot) = v(T,\cdot)$. Moreover, by passing to the limit as $T \to \infty$, they showed that (1.4) has at least one positive solution defined on $\mathbb{R}\times\Omega$ satisfying the 0-boundary condition and

$$\lim_{t\to\infty} u(t,x) = \lim_{t\to\infty} v(t,x) = 0 \quad \text{uniformly in } x \in \Omega.$$

For later developments, we mention that Bartsch and Ding [3] investigate the following $2M$-component (infinite dimensional Hamiltonian control [18, 24]) system

$$\begin{cases} \partial_t u = \Delta_x u - V(x)u + \partial_v H(t,x,u,v) \\ \partial_t v = -\Delta_x v + V(x)v - \partial_u H(t,x,u,v) \end{cases} \quad \text{in } \mathbb{R}\times\mathbb{R}^N. \tag{1.5}$$

The authors established a proper variational framework and proved the existence and multiplicity of solutions of homoclinic type to (1.5) under appropriate conditions on the nonlinearities (see also [12, 14]).

The model (1.2) is different from the above mentioned ones and we are interested in finding the pattern generalizing (mechanisms) dependence on the varying parameters and the spatial distributions of chemical potentials (to our best knowledge, we are not aware of an earlier work where such topic is considered). To give a better description of our framework, a few words regarding terminology are necessary. Let us first introduce for $r \geq 1$ the Banach space

$$B^r(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M}) := W^{1,r}\big(\mathbb{R}, L^r(\mathbb{R}^N, \mathbb{R}^{2M})\big) \cap L^r\big(\mathbb{R}, W^{2,r}(\mathbb{R}^N, \mathbb{R}^{2M})\big)$$

equipped with the norm

$$\|z\|_{B^r} := \left(\iint_{\mathbb{R}\times\mathbb{R}^N} \Big(|z|^r + |\partial_t z|^r + |\Delta_x z|^r \Big)\, dxdt \right)^{1/r}. \tag{1.6}$$

In the sequel, when no confusion can arise, we will use B^r for short. First we formulate the hypotheses on the potential V as

(V) *V is locally Hölder continuous and* $\max |V| < 1$.

In order to get asymptotic results, it is necessary to put some restrictions on H. It is required that the nonlinear function $H : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ has the form $H(\xi) = G(|\xi|) := \int_0^{|\xi|} g(s)s\, ds$. We shall make the following assumptions on g under consideration.

(H_1) $g \in C[0,\infty) \cap C^1(0,\infty)$ such that $g(0) = 0$, $g'(s) \geq 0$, $g'(s)s = o(s)$ as $s \to 0$, and

$$g'(s) \leq C s^{(4-N)/N} \quad \text{for all } s \geq 1, \text{ some } C > 0.$$

(H_2) The function $s \mapsto g(s) + g'(s)s$ is increasing on $[0,\infty)$.

(H_3) (i) There exists $\beta > 2$ such that $0 < \beta G(s) \leq g(s)s^2$ if $s \neq 0$;

(ii) there exist $\alpha > 0$ and $p \in (2, 2(N+2)/N)$ such that $g(s) \leq \alpha s^{p-2}$ for all $s \geq 1$.

Probably (H_2) can be replaced by other hypotheses on the growth of $\nabla^2 H(\xi)$ at both the origin and the infinity, however, we find the monotonicity is essentially related with our proofs. Assumption (H_3) is a super quadratic condition on G. Such an assumption can be replaced by the following asymptotic quadratic condition. With the notation $\widehat{G}(s) := \frac{1}{2}g(s)s^2 - G(s)$, we introduce

(H_3') (i) There exists $b > 1 + \sup|V|$ such that $g(s) \to b$ as $s \to \infty$;

(ii) $\widehat{G}(s) > 0$ if $s > 0$ and $\widehat{G}(s) \to +\infty$ as $s \to \infty$.

Let us mention that our listed above assumptions admit elements of

1. $H(\xi) = c|\xi|^p$ with $p \in (2, 2(N+2)/N)$ for the super quadratic case,
2. $H(\xi) = \frac{b}{2}|\xi|^2\big(1 - \frac{1}{\ln(e+|\xi|)}\big)$ for the asymptotically quadratic case,

as well as finite summations of them.

Involving the assumptions introduced above, our result comes as follows.

Theorem 1.1. *Assume that* (V), (H_1), (H_2) *and either* (H_3) *or* (H_3') *are satisfied. If* $\Lambda \subset \mathbb{R}^N$ *is an open bounded set such that*

$$\underline{c} := \min_{\Lambda} V < \min_{\partial\Lambda} V, \tag{1.7}$$

then for $\varepsilon > 0$ *small problem* (1.2) *has a solution* $\tilde{z}_\varepsilon = (u_\varepsilon, v_\varepsilon) \in B^r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$ *for all* $r \geq 2$ *such that*

(i) *there exists a family of points* $\{y_\varepsilon\}$ *in* Λ *with* $\lim_{\varepsilon\to 0} V(y_\varepsilon) = \underline{c}$ *such that*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N} \int_{\mathbb{R}} \int_{B_{\varepsilon\rho}(y_\varepsilon)} |\tilde{z}_\varepsilon|^2 \, dxdt > 0 \quad \textit{for every } \rho > 0$$

and

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \|\tilde{z}_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N \setminus B_{\varepsilon R}(y_\varepsilon))} = 0 \quad \textit{for every } t \in \mathbb{R};$$

(ii) *the transition sequence* $w_\varepsilon(t,x) = \tilde{z}_\varepsilon(t, \varepsilon x + y_\varepsilon)$ *converges in* $B^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$, *as* $\varepsilon \to 0$, *to a ground state solution of*

$$\begin{cases} \partial_t u = \Delta_x u - u - \underline{c}\, v + \partial_v H(u,v)\,, \\ \partial_t v = -\Delta_x v + v + \underline{c}\, u - \partial_u H(u,v)\,. \end{cases}$$

The important point to note here is no restriction on the global behavior of V is required other than (V), particularly, the behavior of V outside Λ is irrelevant. Due to this observation, we have an immediate consequence of our Theorem 1.1:

Corollary 1.2. *Assume that* (V), (H_1), (H_2) *and either* (H_3) *or* (H_3') *are satisfied. If there exist mutually disjoint bounded domains* Λ_j, $j = 1, ..., k$, *and constants* $c_1 < c_2 < \cdots < c_k$ *such that*

$$c_j = \min_{\Lambda_j} V < \min_{\partial\Lambda_j} V, \tag{1.8}$$

then for $\varepsilon > 0$ *small problem* (1.2) *has at least* k *solutions* $\tilde{z}_\varepsilon^j = (u_\varepsilon^j, v_\varepsilon^j) \in B^r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$ $(j = 1, ..., k)$ *for all* $r \geq 2$ *such that*

(i) *for each* Λ_j *there exists a family of points* $\{y_\varepsilon^j\}$ *in* Λ_j *with* $\lim_{\varepsilon\to 0} V(y_\varepsilon^j) = c_j$ *such that*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N} \int_{\mathbb{R}} \int_{B_{\varepsilon\rho}(y_\varepsilon^j)} |\tilde{z}_\varepsilon^j|^2 \, dxdt > 0 \quad \textit{for every } \rho > 0$$

and

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \|\tilde{z}_\varepsilon^j(t,\cdot)\|_{L^\infty(\mathbb{R}^N \setminus B_{\varepsilon R}(y_\varepsilon^j))} = 0 \quad \textit{for every } t \in \mathbb{R};$$

(ii) *each transition sequence* $w_\varepsilon^j(t,x) = \tilde{z}_\varepsilon^j(t, \varepsilon x + y_\varepsilon^j)$ *converges respectively in* $B^2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$*, as* $\varepsilon \to 0$*, to a ground state solution of*

$$\begin{cases} \partial_t u = \Delta_x u - u - c_j v + \partial_v H(u,v) \\ \partial_t v = -\Delta_x v + v + c_j u - \partial_u H(u,v). \end{cases}$$

Our results provides a natural and intrinsic characterization of the pattern generalizing dependence on the varying parameters and the spatial distributions of chemical potentials. The theorems express that there must be a pattern concentrating around the local minimum of the chemical potential distribution for small diffusion coefficients, moreover, it is the emergence of such local minimum that guarantees the existence of such concentrating phenomenon. Furthermore, if there exists distinguished regions of local minima, there will exist multiple patterns with different shapes concentrating separately in the very region the local minimum lies in.

Mathematically, problems like (1.2) with small coefficient is referred as singular perturbation problems. Such problems are generally characterized by dynamics operating on multiple scales. The researches of singular perturbation problems involving variational methods goes back to the semi-classical analysis on nonlinear Schrödinger equation arising in the *non-relativistic* quantum mechanics:

$$\hbar^2 \Delta w - V(x) w + f(w) = 0 \quad w \in H^1(\mathbb{R}^n)\,. \tag{1.9}$$

Initiated by Rabinowitz [25], the existence of positive solutions of (1.9) for small $\hbar > 0$ is proved whenever

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^n} V(x)\,.$$

And these solutions concentrate around the global minimum points of V when $\hbar \to 0$, as was shown by Wang [28]. It should be pointed out that M. Del Pino and P. Felmer in [10] firstly succeeded in proving a localized version of the concentration behaviour of semi-classical solutions. In [10], assuming $\inf V = V_0 > 0$ and (1.7) for some bounded domain Λ, the authors showed the existence of a single-peak solution which concentrates around the minimum points of V in Λ. Their approach depends on a penalization argument and Mountain-pass theorem. Note that, since the Schrödinger operator $-\Delta + V$ is bounded from below, techniques based on the Mountain-pass theorem are well applied to the investigation. See also [1, 2, 6, 7, 9, 11, 16] and their references for further related results.

There are at least three difficulties in extending the quoted results (on the elliptic case) to the system (1.2). Firstly, no uniqueness results seem to be known for the "limit problem"

$$\begin{cases} \partial_t u = \Delta_x u - u - v + \partial_v H(u, v) \\ \partial_t v = -\Delta_x v + v + u - \partial_u H(u, v) \end{cases} \quad \text{in } \mathbb{R} \times \mathbb{R}^N$$

and this is in some cases a crucial assumption in the single elliptic equation case (compare e.g. with [9, Assumption (f5)], [11, Assumption (f4)]). Secondly, as we will see in Section 3, the variational structure of system (1.2) is of *strongly indefinite* type (the energy functional is neither bounded from above nor from below, even on subspaces of finite dimension or codimension). Differ from the single elliptic equation case, the quadratic part of the energy functional has no longer a positive sign. At a technical level, this causes some difficulties; for instance, the penalization arguments as used in [10, 11] can not be applied to our problem; we have to provide a more delicate analysis. Moreover, from a conceptual point view, in the case of a system we also have to face the fact that the method based on the the Mountain-pass theorem breaks down. As a consequence, we need a deep insight into the linking structure of strongly indefinite functional. This difficulty was bypassed in [3, 12, 14], where a direct approach was proposed. In these papers either the case $V(x) \equiv 1$ or the periodic case V is T_j-periodic in x_j for $j = 1, \cdots, N$ are considered.

We mention that in the papers [21, 22] the Hamiltonian elliptic type systems are considered. Such systems also have the indefinite character of the energy functional, by a reduction argument, the authors showed multiple spike-layered solutions concentrating around the local minimum points of the potential functions. However in this context, when the parabolic system is considered, we have to face the t-Anisotropic Sobolev spaces $B^r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$ and the interpolation space between $B^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$ and $L^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$ (see the functional settings in Section 3). At this point, it is not clear whether the the cut-off method as used in [21, 22] can be applied to our problem.

The rest of the paper is devoted to the proof of Theorem 1.1 and Corollary 1.2. In the next section, we briefly introduce an abstract critical point theorem which can be applied to the study of indefinite functionals. The specific proof of the abstract critical point theorem will be presented in Section 4. Section 3 falls naturally into two parts which devotes to give a full proof of Theorem 1.1: the first part constitutes sufficient preparation to apply the abstract theorem; the second part provides the delicate analysis on the concentrating solutions to system (1.2). Due to the strongly indefiniteness, we have to recover a compactness condition at some certain minimax levels. All we do is to build a modification of the energy functional associated to (1.2). In such a way, the functional is proved to satisfy a compactness condition defined in Section 2. And then, for ε sufficiently small, a critical point associate to the modified functional is indeed a solution to the original system (1.2) thanks to some priori estimates of parabolic equations. The modification of the functional corresponds to a penalization technique "outside Λ", and this is why no other global assumptions are required for V. Finally, in the Appendix we collect some embedding and regularity results which are used in this paper.

2 An abstract critical point theorem

Before stating the main results of this section we shall introduce some notations and definitions. We denote by E a real Hilbert space, by $\langle\cdot,\cdot\rangle$ the scalar product in E, by $\|\cdot\|$ the norm in E. The dual of E is denoted by E^*. By $C^k(E,\mathbb{R})$ for $k \geq 1$ we denote the space of k-times Frechét differentiable functionals from E to $\mathbb{R}$. We shall denote by $\mathscr{L}(E)$ the space of bounded linear maps from E to E, endowed with the uniform operator norm, and by $\mathscr{L}_s(E)$ the same space endowed with the strong operator topology. The adjoint of A in $\mathscr{L}(E)$ is denoted by A^*, and by self-adjoint we mean $A = A^*$. The space E_w is the space E endowed with its weak topology. We denote weak convergence of a sequence in E with the symbol $\rightharpoonup$. Let G be a Lie group and let $\mathscr{T} : G \to U(E)$ be a representation of G on the group of the unitary linear transformations on E. We set $\mathscr{G} = \mathscr{T}(G)$ and sometimes, when no ambiguity is possible, we will use $\mathscr{G}$ instead of G to stand for the Lie group.

Definition 2.1. A subset $M \subset E$ is called $\mathscr{G}$-invariant if $g(M) = M$ for every $g \in \mathscr{G}$. A functional Φ on E is called $\mathscr{G}$-invariant if $\Phi \circ g = \Phi$ for every $g \in \mathscr{G}$. A map h from E to E is called $\mathscr{G}$-equivariant if $h \circ g = g \circ h$ for every $g \in \mathscr{G}$.

Let $\{A_\varepsilon\}_{\varepsilon>0} \subset \mathscr{L}(E)$ be a family of $\mathscr{G}$-equivariant self-adjoint operators. Let $\{\Psi_\varepsilon\}_{\varepsilon>0} \subset C^2(E,\mathbb{R})$ be a $\mathscr{G}$-invariant family, and set $\psi_\varepsilon := \nabla\Psi_\varepsilon : E \to E$. Considering a given splitting $E = X \oplus Y$ of E into $\mathscr{G}$-invariant orthogonal subspaces X and Y with associated bounded projections P^X and P^Y, we write $z^X := P^X z$ and $z^Y := P^Y z$ for $z \in E$. In the sequel, we are interested in finding critical points of the functionals

$$\Phi_\varepsilon : E \to \mathbb{R}, \quad \Phi_\varepsilon(z) := \frac{1}{2}\big(\|z^X\|^2 - \|z^Y\|^2\big) + \frac{1}{2}\langle A_\varepsilon z, z\rangle - \Psi_\varepsilon(z)$$

for ε small. Let $A_0 \in \mathscr{L}(E)$ be a $\mathscr{G}$-equivariant self-adjoint operator and Ψ_0 be a $\mathscr{G}$-invariant C^2 functional, and set $\psi_0 := \nabla\Psi_0 : E \to E$. Consider

$$\Phi_0 : E \to \mathbb{R}, \quad \Phi_0(z) := \frac{1}{2}\big(\|z^X\|^2 - \|z^Y\|^2\big) + \frac{1}{2}\langle A_0 z, z\rangle - \Psi_0(z),$$

as singular limit functional.

Since we are interested in the situation when ε is small, by setting $\mathcal{E} = [0,1]$, we will be concerned with $\{\Phi_\varepsilon\}_{\varepsilon\in\mathcal{E}} := \{\Phi_0\} \cup \{\Phi_\varepsilon\}_{\varepsilon\in(0,1]}$. Now we collect some hypotheses on $\{\Phi_\varepsilon\}_{\varepsilon\in\mathcal{E}}$ which we will impose in the various results:

$(A1)$ There exists $\theta \in (0,1)$ such that $\sup_{\varepsilon\in(0,1]} \|A_\varepsilon\| \le \theta$.

$(A2)$ $A_\varepsilon \to A_0$ in $\mathscr{L}_s(E)$ as $\varepsilon \to 0$.

$(N1)$ For each $\varepsilon \in \mathcal{E}$, Ψ_ε is non-negative and convex, and $\psi_\varepsilon : E_w \to E_w$ is sequentially continuous.

$(N2)$ For each $z \in E$, $\psi_\varepsilon(z) \to \psi_0(z)$ in E as $\varepsilon \to 0$.

$(N3)$ There exists $\kappa \in C(\mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ = [0,\infty)$), independent of ε, such that

$$\left|\Psi_\varepsilon''(z)[v,w]\right| \le \kappa(\|z\|)\cdot\|v\|\cdot\|w\|$$

for $z, v, w \in E$ and $\varepsilon \in \mathcal{E}$.

$(N4)$ For all $\varepsilon \in \mathcal{E}$ and $z \in E\setminus\{0\}$, $\widehat{\Psi}_\varepsilon(z) := \frac{1}{2}\Psi_\varepsilon'(z)z - \Psi_\varepsilon(z) > 0$, and $\widehat{\Psi}_\varepsilon : E_w \to \mathbb{R}$ is sequentially lower semi-continuous.

$(N5)$ Given arbitrarily $\varepsilon \in \mathcal{E}$, for every $z \in E\setminus\{0\}$ and $w \in E$ it holds that

$$\big(\Psi_\varepsilon''(z)[z,z] - \Psi_\varepsilon'(z)z\big) + 2\big(\Psi_\varepsilon''(z)[z,w] - \Psi_\varepsilon'(z)w\big) + \Psi_\varepsilon''(z)[w,w] > 0\,.$$

Remark 2.2. Condition $(A2)$ and $(N2)$ are quite natural in singular perturbation problems. Both are satisfied in the scaled equations when the parameter varies. Condition $(N3)$ and $(N4)$ are rather harmless, also $(N5)$ holds under rather general assumptions on the nonlinearity. They are technical assumptions which are deeply related to the proof of our results, and they will be easily checked if growth conditions on the nonlinearity are given.

We shall emphasis here the functionals Φ_ε for $\varepsilon \in \mathcal{E}$ are "strongly indefinite", that is X and Y are both infinite-dimensional, as it occurs in the study of solutions of unbounded Hamiltonian systems. Recall that a sequence $\{z_n\} \subset E$ is called to be a $(PS)_c$-sequence for functional $\Phi \in C^1(E,\mathbb{R})$ if $\Phi(z_n) \to c$ and $\Phi'(z_n) \to 0$, and is called to be $(C)_c$-sequence for Φ if $\Phi(z_n) \to c$ and $(1+\|z_n\|)\Phi'(z_n) \to 0$. It is clear that if $\{z_n\}$ is a $(PS)_c$-sequence with $\{\|z_n\|\}$ bounded then it is also a $(C)_c$-sequence. We remark that if Φ is $\mathscr{G}$-invariant then $\{g_n z_n\}$ is also a $(C)_c$-sequence (*resp.* $(PS)_c$-sequence) for any $\{g_n\} \subset \mathscr{G}$ provided $\{z_n\}$ is a $(C)_c$-sequence (*resp.* $(PS)_c$-sequence).

Definition 2.3. A $\mathscr{G}$-invariant functional $\Phi \in C^1(E,\mathbb{R})$ is said to satisfy the $\mathscr{G}$-weak $(C)_c$-condition if for each $(C)_c$-sequence $\{z_n\}$ there exists correspondingly $\{g_n\} \subset \mathscr{G}$ such that $\{g_n z_n\}$ possesses a subsequence converge weakly to a point in $E\setminus\{0\}$.

Next, we will be concerned with the functionals $\{\Phi_\varepsilon\}_{\varepsilon\in\mathcal{E}}$ satisfying the assumptions mentioned above.

Theorem 2.4. *Suppose the family* $\{\Phi_\varepsilon\}_{\varepsilon\in\mathcal{E}}$ *satisfies* $(A1)$-$(A2)$, $(N1)$-$(N5)$ *and*

($I1$) *there exist* $\rho, \tau > 0$*, both independent of* $\varepsilon \in \mathcal{E}$*, such that* $\Phi_\varepsilon\big|_{B_\rho^X} \geq 0$ *and* $\Phi_\varepsilon\big|_{S_\rho^X} \geq \tau$ *where* $B_\rho^X := B_\rho \cap X = \{z \in X : \|z\| \leq \rho\}$ *and* $S_\rho^X := \partial B_\rho^X = \{z \in X : \|z\| = \rho\}$*;*

($I2$) *for any* $e \in X \setminus \{0\}$ *set* $E_e = \mathbb{R}^+ e \oplus Y$*, either* $\sup_{z \in E_e} \Phi_0(z) = +\infty$ *or* $\Phi_0(z) \to -\infty$ *as* $z \in E_e$ *and* $\|z\| \to \infty$ *(here* $\mathbb{R}^+ = [0, \infty)$*).*

If Φ_ε *satisfies the* $\mathscr{G}$*-weak* $(C)_c$*-condition for each* $c \in \mathbb{R} \setminus \{0\}$ *and* $\varepsilon \neq 0$ *and*

$$c_0 = \inf_{e \in X} \sup_{z \in E_e} \Phi_0(z) < +\infty$$

is a critical value for Φ_0*, then*

(1) *for all* ε *small,* Φ_ε *admits a critical value*

$$c_\varepsilon = \inf_{e \in X} \sup_{z \in E_e} \Phi_\varepsilon(z)\,;$$

(2) c_ε *is the ground state energy of* Φ_ε *and* $c_\varepsilon \leq c_0 + o(1)$ *as* $\varepsilon \to 0$*.*

The proofs of Theorem 2.4 is quite technical and self-contained, so we will first give the applications to the singular perturbation problem of reaction-diffusion system and then show the proofs in Section 4.

Remark 2.5. Theorem 2.4 is the first unified abstract result concerned with strongly indefinite singular limit problem. The assumptions ($I1$) and ($I2$) are geometrical assumptions, which imply a linking structure of the strongly indefinite functionals. ($I2$) generalizes the requirements in [13, 15] when treating nonlinear Dirac equations, which allows to deal with a larger class of nonlinearities including the asymptotic quadratic ones.

Remark 2.6. In Theorem 2.4 the assumption c_0 is a critical value of Φ_0 is not difficult to verify in application. In fact the singular limit equation is, after the scaling transform, a autonomous equation such that the corresponding functional Φ_0 is invariant under actions of some Lie group containing $\mathscr{G}$ as a proper subgroup. Then the existence and characterization of c_0 can be derived from standard variational methods ([12], see also Section 3). The second conclusion in Theorem 2.4 seems to be optimal and essential in the study of singular perturbation problem (for Schrödinger equations see [10, 16] and references therein). and for Dirac equations see [13, 15])

Remark 2.7. As we mentioned above, Φ_0 is $\mathscr{G}$-invariant but we can not expect Φ_0 to satisfy the $\mathscr{G}$-weak $(C)_c$-condition since there exists another Lie group $\mathscr{G}'$ such that $\mathscr{G} \subsetneqq \mathscr{G}'$ and Φ_0 is $\mathscr{G}'$-invariant. As a matter of fact, in the applications, one will see that Theorem 2.4 is nontrivial since we are dealing with the situation Φ_0 does not satisfy the $\mathscr{G}$-weak $(C)_c$-condition as Φ_ε for $\varepsilon > 0$. And hence, the proof is not simply passing to the limit as ε vanishes.

3 Applications to singular perturbation problem of nonlinear reaction-diffusion systems

Now we consider the reaction-diffusion system (1.2) where V, H satisfy the assumptions (V), (H_1), (H_2) and (H_3) (or (H_3')). Making the change of variable $x \to \varepsilon x$, (1.2) becomes

$$\begin{cases} \partial_t u = \Delta_x u - u - V_\varepsilon(x)v + H_v(u,v) \\ -\partial_t v = \Delta_x v - v - V_\varepsilon(x)u + H_u(u,v) \end{cases} \tag{3.1}$$

where Δ_x denotes the Laplacian acts on the space variable x and $V_\varepsilon(x) = V(\varepsilon x)$. Setting

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \ \mathcal{J}_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \text{ and } A = \mathcal{J}_0(-\Delta_x + 1),$$

and let $L := \mathcal{J}\partial_t + A$. Then (3.1) can be rewritten as

$$\mathcal{J}\partial_t z = -Az - V_\varepsilon(x)z + g(|z|)z \quad \text{for } z = (u,v),$$

or in a more abstract representation

$$Lz + V_\varepsilon(x)z = g(|z|)z \quad \text{for } z = (u,v). \tag{3.2}$$

In this way, (3.1) can be regarded as an infinite dimensional Hamiltonian system. Here and subsequently, we fix the potential V and a bounded nonempty open set $\Lambda \subset \mathbb{R}^N$ such that

$$\underline{c} = \min_{\Lambda} V < \min_{\partial\Lambda} V. \tag{3.3}$$

Without loss of generality we can assume that the boundary of Λ is smooth, and that $0 \in \Lambda$.

3.1 Functional settings

We initially introduce some functional spaces we shall need in the sequel. If $1 \le q \le \infty$ we set $L^q := L^q(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$, and by $|\cdot|_q$ we denote the usual norm defined on L^q. Denoted by $(\cdot,\cdot)_2$ the usual L^2-inner product.

Now we consider L acts on the Hilbert space $L^2 := L^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$. It is quite standard to see that L is a self-adjoint operator with domain

$$\mathcal{D}(L) = B^2 := W^{1,2}\big(\mathbb{R}, L^2(\mathbb{R}^N, \mathbb{R}^{2M})\big) \cap L^2\big(\mathbb{R}, W^{2,2}(\mathbb{R}^N, \mathbb{R}^{2M})\big).$$

Let $\sigma(L)$ and $\sigma_e(L)$ be respectively the spectrum and essential spectrum of L, we have the following result (cf. [12, Lemma 8.7]).

Proposition 3.1. *$\sigma(L) = \sigma_e(L) \subset \mathbb{R} \setminus (-1, 1)$, moreover, $\sigma(L)$ is symmetric with respect to* 0.

As a direct consequence of Proposition 3.1, the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^+ \oplus L^-, \quad z = z^+ + z^-, \tag{3.4}$$

so that L is positive definite (*resp.* negative definite) in L^+ (*resp.* L^-). In order to construct the energy functionals whose critical points are the solutions of (3.2) we introduce $E := \mathcal{D}(|L|^{1/2})$ be equipped with the inner product

$$\langle z_1, z_2 \rangle = \big(|L|^{1/2} z_1, |L|^{1/2} z_2\big)_2$$

and the induced norm $\|z\| = \langle z, z \rangle^{1/2}$, where $|L|$ and $|L|^{1/2}$ denote respectively the absolute value of L and the square root of $|L|$. As an interpolation space between B^2 and L^2, E (being a Hilbert space) has the decomposition

$$E = E^+ \oplus E^-, \quad \text{where } E^\pm = E \cap L^\pm$$

which is orthogonal with respect to both $(\cdot, \cdot)_2$ and $\langle \cdot, \cdot \rangle$. We write $z = z^+ + z^-$ for $z \in E$ according to this decomposition and introduce the following bilinear form

$$a(z_1, z_2) = \big\langle z_1^+, z_2^+ \big\rangle - \big\langle z_1^-, z_2^- \big\rangle \quad \text{for } z_1, z_2 \in E.$$

The bilinear form $a(\cdot, \cdot)$ is symmetric and continuous in E. Observe that if $z_1, z_2 \in B^2$

$$a(z_1, z_2) = \int_{\mathbb{R}} \int_{\mathbb{R}^N} L z_1 \cdot z_2 \, dx dt.$$

Under the assumption (H_1), we see that there are positive constants c_1, c_2 such that

$$|\nabla H(z)| \le c_1 |z| + c_2 |z|^{(N+4)/N} \quad \text{for any } z \in \mathbb{R}^{2M}.$$

Remark that E is continuously embedded in L^r for $r \in [2, 2(N+2)/N]$ if $N \ge 2$, and compactly embedded in L^r_{loc} for $r \in [1, 2(N+2)/N)$ if $N \ge 2$ (cf. [12, Lemma 8.5]). Standard arguments show that the functional

$$J_\varepsilon(z) = \frac{1}{2} a(z, z) + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^N} V_\varepsilon(x) |z|^2 \, dx dt - \int_{\mathbb{R}} \int_{\mathbb{R}^N} G(|z|) \, dx dt, \quad z \in E$$

is 2-times Frechét differentiable and that its critical points correspond to the solutions of (3.2) (see Lemma A.5 in Appendix).

Since $\sigma(L) \subset \mathbb{R} \setminus (-1, 1)$, one has

$$|z|_2^2 \le \|z\|^2 \quad \text{for all } z \in E. \tag{3.5}$$

The decomposition of E induces also a natural decomposition of L^r, hence there is $d_r > 0$ such that

$$d_r |z^\pm|_r^r \le |z|_r^r \quad \text{for all } z \in E. \tag{3.6}$$

It is to be expected that critical points of J_ε can be found by applying Theorem 2.4, and that the asymptotic behaviour (which can be characterized by concentration phenomenon) of these critical points will follow subsequently. However, one may find the abstract theorem can not apply directly to J_ε due to the lack of $\mathscr{G}$-weak $(C)_c$-condition. In what follows, let us initially give a modification of J_ε which guarantees that the assumptions in the abstract theorem are all satisfied.

Choose $s_0 > 0$ be the value at which $g(s_0) + g'(s_0)s_0 = \frac{1-|V|_\infty}{2}$. Let us consider $\tilde{g} \in C[0,\infty) \cap C^1(0,\infty)$ such that

$$\frac{d}{ds}\big(\tilde{g}(s)s\big) = \begin{cases} g(s) + g'(s)s & \text{if } s \le s_0, \\ \dfrac{1-|V|_\infty}{2} & \text{if } s > s_0. \end{cases}$$

Such $\tilde{g}$ exists thanks to our assumptions (H_1) and (H_2). Define

$$f(x,s) = \chi_\Lambda(x)g(s) + \big(1 - \chi_\Lambda(x)\big)\tilde{g}(s),$$

where $\Lambda \subset \mathbb{R}^N$ is the bounded domain fixed to satisfy (3.3) and χ_Λ denotes its characteristic function. One should keep in mind here that Λ has to be rescaled when we consider the rescaled system (3.1). Write

$$F(x,s) = \int_0^s f(x,\tau)\tau\,d\tau \quad \text{and} \quad \widehat{F}(x,s) = \frac{1}{2}f(x,s)s^2 - F(x,s).$$

It is standard to check that (H_1) and (H_2) implies $F : [0,\infty) \to [0,\infty)$ is a Caratheodory function satisfying

$(F1)$ *$f_s(x,s) > 0$ exists every where, and $f(x,s)s = o(s)$ uniformly in x as $s \to 0$;*

$(F2)$ *$0 \le f(x,s)s \le g(s)s$ for all $x \in \mathbb{R}^N$ and $s \ge 0$;*

$(F3)$ *$0 < 2F(x,s) \le f(x,s)s^2 \le \frac{1-|V|_\infty}{2}s^2$ for all $x \notin \Lambda$ and $s > 0$;*

$(F4)$ *(i) if (H_3) is satisfied, then $0 < \beta F(x,s) \le f(x,s)s^2$ for all $x \in \Lambda$ and $s > 0$;*

(ii) if (H_3') is satisfied, then $\widehat{F}(x,s) > 0$ for all $s > 0$;

$(F5)$ *$\frac{d}{ds}\big(f(x,s)s\big) \ge 0$ for all x and $s > 0$;*

$(F6)$ *either (H_3) or (H_3') is satisfied, $\widehat{F}(x,s) \to \infty$ as $s \to \infty$ uniformly in x.*

For simplicity of notation, we let $f_\varepsilon(x,s)$ and $F_\varepsilon(x,s)$ stand for $f(\varepsilon x,s)$ and $F(\varepsilon x,s)$ respectively. Now, let us define the modified functional $\Phi_\varepsilon : E \to \mathbb{R}$ as

$$\begin{aligned}\Phi_\varepsilon(z) &= \frac{1}{2}a(z,z) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} V_\varepsilon(x)|z|^2\,dxdt - \int_{\mathbb{R}}\int_{\mathbb{R}^N} F_\varepsilon(x,|z|)\,dxdt \\ &= \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} V_\varepsilon(x)|z|^2\,dxdt - \Psi_\varepsilon(z).\end{aligned}$$

Then, we see that $\Phi_\varepsilon \in C^2(E,\mathbb{R})$ and critical points of Φ_ε correspond to solutions of

$$Lz + V_\varepsilon(x)z = f_\varepsilon(x,|z|)z.$$

Taking the singular limit into account, we find that the limit system is

$$Lz + V_0\, z = g(|z|)z,$$

where $V_0 := V(0)$ and that the associated functional is

$$\begin{aligned}\Phi_0(z) &= \frac{1}{2}a(z,z) + \frac{V_0}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} |z|^2\,dxdt - \int_{\mathbb{R}}\int_{\mathbb{R}^N} G(|z|)\,dxdt \\ &= \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{V_0}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} |z|^2\,dxdt - \Psi_0(z).\end{aligned}$$

To apply the abstract theorem stated in the preceding section we now analyze the relevant properties of the variational functionals involved. As introduced above we have $E = E^+ \oplus E^-$, and let A_ε denote the self-adjoint operator defined by $z \mapsto |L|^{-1}V_\varepsilon(\cdot)z$ for $z \in E$. Analogously, A_0 can be defined by $z \mapsto |L|^{-1}V_0 z$. It is all clear we have Φ_ε and Φ_0 are in the forms that we introduced in Section 2, so our strategy is to check all the assumptions appeared in Theorem 2.4 are satisfied.

3.1.1 *The group action*

Denote by $\star$ the action of $\mathscr{G} := \mathbb{R}$ on E that arises from translation: for $z \in E$ and $g \in \mathscr{G}$ define $(g\star z)(t,x) = z(t-g,x)$. From the fact V and $\tilde{H}$ are independent of t that we have Φ_ε is $\mathscr{G}$-invariant for all $\varepsilon > 0$. Moreover, if denote by $\bar{\star}$ the action of $\mathscr{G}' := \mathbb{R}\times\mathbb{R}^N$ on E by $(g'\bar{\star}z)(t,x) = z(t-g_1, x-g_2)$ for $g' = (g_1,g_2) \in \mathscr{G}'$, we soon have Φ_0 is invariant under the action of $\mathscr{G}'$.

3.1.2 *The quadratic part*

Recall the definitions of $A_\varepsilon, A_0 \in \mathscr{L}(E)$. By virtue of (3.5), we easily have

$$\begin{aligned}\|A_\varepsilon\| &= \sup\big\{\, \langle A_\varepsilon z, z\rangle : \, u \in E,\ \|z\| = 1\big\} \\ &= \sup\big\{\big(V_\varepsilon(\cdot)z, z\big)_2 : \, z \in E,\ \|z\| = 1\big\} \\ &\le |V|_\infty \cdot \sup\big\{(z,z)_2 : \, z \in E,\ \|z\| = 1\big\} \\ &\le |V|_\infty < 1.\end{aligned}$$

This guarantees the condition $(A1)$. To check $(A2)$, that is the convergency of $\{A_\varepsilon\}$ in the strong operator topology of $\mathscr{L}(E)$, let us remark that $V_\varepsilon(x) \to V_0$ uniformly on bounded sets of x as $\varepsilon \to 0$. Therefore, for each $z \in E$ we deduce that

$$\begin{aligned}\|A_\varepsilon z - A_0 z\| &= \sup_{\|w\|=1} \langle (A_\varepsilon - A_0)z, w\rangle \\ &= \sup_{\|w\|=1} \big((V_\varepsilon(\cdot) - V_0)z, w\big)_2 \\ &\le \sup_{\|w\|=1} \big|(V_\varepsilon(\cdot) - V_0)z\big|_2 \cdot |w|_2 \\ &\le \big|(V_\varepsilon(\cdot) - V_0)z\big|_2 = o(1)\end{aligned}$$

as $\varepsilon \to 0$. And thus we obtain $(A2)$.

3.1.3 *The nonlinear part*

The required properties (see $(N1)$-$(N5)$) of the nonlinear part will be checked based on $(F1)$-$(F6)$. Recall the notation $\mathcal{E} := [0, 1]$. Firstly, by $(F2)$, we observe that

$$G(|z|) = \int_0^{|z|} g(s)s\,ds \ge \int_0^{|z|} f_\varepsilon(x, s)s\,ds = F_\varepsilon(x, |z|),$$

which implies $\Psi_0(z) \ge \Psi_\varepsilon(z) \ge 0$ for all $z \in E$. Note that $\frac{d}{ds}\big(g(s)s\big) \ge 0$ and $\frac{d}{ds}\big(f_\varepsilon(x, s)s\big) \ge 0$ for all $x \in \mathbb{R}^N$, we have $\Psi_\varepsilon''(z)[w, w] \ge 0$ for $z, w \in E$ and $\varepsilon \in \mathcal{E}$.

Recall the assumptions we have required on H (see (H_1), (H_3) or (H_3')). Also recall the embeddings $E \hookrightarrow L^r$ for $r \in [2, 2(N+2)/N)$. It holds that if $z_n \rightharpoonup z$ in E then $\{z_n\}$ is bounded in L^r and converges to z in L^r_{loc}, for $r \in [1, 2(N+2)/N)$. Moreover, recall we have assumed that $0 \in \Lambda$, we infer that $\chi_\Lambda(\varepsilon x) \to 1$ a.e. on $\mathbb{R}^N$ as $\varepsilon \to 0$. Therefore it is easy to check $(N1)$ and $(N2)$ are satisfied for Ψ_ε, $\varepsilon \in \mathcal{E}$. One should keep in mind here that the map $|L|^{-1} : E^* \to E$ is the isomorphism induced from the Riesz representation theorem.

$(N3)$ is much more obvious. Indeed, the modified nonlinearities satisfy

$$\begin{aligned}|f_\varepsilon(x, s)| &\le \big|\chi_\Lambda(\varepsilon x)g(s)\big| + \big|(1 - \chi_\Lambda(\varepsilon x))\tilde{g}(s)\big| \\ &\le |g(s)| + |\tilde{g}(s)|\end{aligned}$$

for all $z \in \mathbb{R}^{2M}$. Therefore, by (H_1) and the embedding $E \hookrightarrow L^{2(N+2)/N}$, we have

$$\big|\Psi_\varepsilon''(z)[v, w]\big| \le C_1\|v\| \cdot \|w\| + C_2\|z\|^{4/N} \cdot \|v\| \cdot \|w\|$$

and $(N3)$ is satisfied.

It remains to prove $(N4)$ and $(N5)$. The verification for Ψ_0 is similar to and simpler than that for Ψ_ε, $\varepsilon > 0$, so we only check the latter. First note that $(F3)$ and $(F4)$ implies

$$\widehat{F}(x, s) := \frac{1}{2}f(x, s)s^2 - F(x, s) > 0 \quad \text{for all } x \in \mathbb{R}^N \text{ and } s > 0.$$

An easy calculation shows

$$\widehat{\Psi}_\varepsilon(z) = \int_{\mathbb{R}} \int_{\mathbb{R}^N} \frac{1}{2} f_\varepsilon(x, |z|)|z|^2 - F_\varepsilon(x, |z|)\, dxdt > 0$$

provided $z \neq 0$, and the sequentially lower semi-continuity comes from Fatou's lemma. Next, to see that $(N5)$ holds, we remark that

$$\Psi'_\varepsilon(z)w = \int_{\mathbb{R}} \int_{\mathbb{R}^N} f_\varepsilon(x, |z|) z \cdot w\, dxdt$$

and

$$\Psi''_\varepsilon(z)[v, w] = \int_{\mathbb{R}} \int_{\mathbb{R}^N} f_\varepsilon(x, |z|) v \cdot w + \partial_s f_\varepsilon(x, |z|)|z| \frac{z \cdot v}{|z|} \cdot \frac{z \cdot w}{|z|}\, dxdt$$

for any $z, v, w \in E$. Then, we deduce that

$$\begin{aligned} &\big(\Psi''_\varepsilon(z)[z, z] - \Psi'_\varepsilon(z)z\big) + 2\big(\Psi''_\varepsilon(z)[z, w] - \Psi'_\varepsilon(z)w\big) + \Psi''_\varepsilon(z)[w, w] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^N} f_\varepsilon(x, |z|)|w|^2 + \partial_s f_\varepsilon(x, |z|)|z| \Big(|z| + \frac{z \cdot w}{|z|}\Big)^2 dxdt \end{aligned} \tag{3.7}$$

And by $(F1)$, we soon obtain $(N5)$ from the above formula.

3.1.4 *The geometric structure and $\mathscr{G}$-weak compactness*

Recall the assumption (H_1) and the definition of F, we remark that there exists $C > 0$ such that

$$|G(|z|)| \leq \frac{1 - |V|_\infty}{4} |z|^2 + C|z|^{2(N+2)/N} \tag{3.8}$$

and

$$|F(x, |z|)| \leq \frac{1 - |V|_\infty}{4} |z|^2 + C|z|^{2(N+2)/N} \tag{3.9}$$

for all $(x, z) \in \mathbb{R} \times \mathbb{R}^{2M}$. Hence, we have:

Lemma 3.2. *There are $\rho, \tau > 0$, both independent of $\varepsilon \in \mathcal{E}$, such that $\Phi_\varepsilon|_{B_\rho^+} \geq 0$ and $\Phi_\varepsilon|_{S_\rho^+} \geq \tau$ where*

$$B_\rho^+ := B_\rho \cap E^+ = \{z \in E^+ : \|z\| \geq \rho\},$$

$$S_\rho^+ := \partial B_\rho^+ = \{z \in E^+ : \|z\| = \rho\}.$$

Proof. For convenience set $2^* = 2(N+2)/N$. Notice that $|z|_{2^*} \le C\|z\|$ for $z \in E$ by the embedding $E \hookrightarrow L^{2^*}$. The conclusion follows easily because, for $z \in E^+$,

$$\begin{aligned}\Phi(z) &= \frac{1}{2}\|z\|^2 + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} V_\varepsilon(x)|z|^2\,dxdt - \Psi_\varepsilon(z) \\ &\ge \frac{1}{2}\|z\|^2 - \frac{|V|_\infty}{2}|z|_2^2 - \left(\frac{1-|V|_\infty}{4}|z|_2^2 + C|z|_{2^*}^{2^*}\right) \\ &\ge \frac{1-|V|_\infty}{4}\|z\|^2 - C'\|z\|^{2^*}\end{aligned}$$

with $C, C' > 0$ independent of ε. $\square$

The preceding lemma shows $(I1)$ is satisfied for the family $\{\Phi_\varepsilon\}_{\varepsilon\in\mathcal{E}}$. The proof of $(I2)$ will be separated into the following two lemmas.

Lemma 3.3. *For the super quadratic nonlinearity, that is (H_3) is satisfied, for $e \in E^+ \setminus \{0\}$, $\Phi_0(z) \to -\infty$ provided $z \in E_e$ and $\|z\| \to \infty$ (recall $E_e := \mathbb{R}^+ e \oplus E^-$).*

Lemma 3.4. *For the asymptotically quadratic nonlinearity, that is (H_3') is satisfied, either $\sup_{z\in E_e}\Phi_0(z) = +\infty$ or $\Phi_0(z) \to -\infty$ as $z \in E_e$ and $\|z\| \to \infty$.*

Proof of Lemma 3.3. First remark that (H_1) and $(H_3)(i)$ implies that for any $\delta > 0$ there exists $c_\delta > 0$ such that

$$G(|z|) \ge c_\delta |z|^\beta - \delta|z|^2 \quad \text{for all } z \in \mathbb{R}^{2M}.$$

Let $e \in E^+ \setminus \{0\}$, by virtue of (3.6), we have for $z = se + v \in E_e$

$$\begin{aligned}\Phi_0(z) &= \frac{1}{2}\|se\|^2 - \frac{1}{2}\|v\|^2 + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} V_\varepsilon(x)|se+v|^2\,dxdt - \Psi_0(se+v) \\ &\le \frac{s^2}{2}\|e\|^2 - \frac{1}{2}\|v\|^2 + \frac{|V|_\infty}{2}|se+v|_2^2 + \delta|se+v|_2^2 - c_\delta|se+v|_\beta^\beta \\ &\le \frac{1+|V|_\infty+2\delta}{2}s^2\|e\|^2 - \frac{1-|V|_\infty-2\delta}{2}\|v\|^2 - c_\delta s^\beta |e|_\beta^\beta.\end{aligned}$$

By noting that $\beta > 2$, let δ be chosen small enough, we have the assertion proved. $\square$

Proof of Lemma 3.4. Let us first assume $\sup_{z\in E_e}\Phi_0(z) = C < \infty$. It is clear that $C > 0$ (by Lemma 3.2). Suppose contrarily, for some sequence $\{z_n\} \subset E_e$ with $\|z_n\| \to \infty$, there exists $C_0 > 0$ such that $\Phi_0(z_n) \ge -C_0$ for all n. Then, setting $v_n = z_n/\|z_n\|$, we have $\|v_n\| = 1$, $v_n \rightharpoonup v$, $v_n^- \rightharpoonup v^-$, $v_n^+ \to v^+ \in \mathbb{R}^+ e$ and

$$-\frac{C_0}{\|z_n\|} \le \frac{\Phi_0(z_n)}{\|z_n\|} \le \frac{1}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 + \frac{|V|_\infty}{2}|v_n|_2^2. \tag{3.10}$$

Remark that $v^+ \neq 0$. Indeed, if not, it follows from (3.10) that

$$\frac{1-|V|_\infty}{2}\|v_n^-\|^2 \leq \frac{1+|V|_\infty}{2}\|v_n^+\|^2 + \frac{C_0}{\|z_n\|} \to 0 \quad \text{as } n\to\infty,$$

which is a contradiction.

First, a direct calculation shows for $\lambda > 0$

$$\begin{aligned}\frac{d}{d\lambda}\Phi_0(\lambda v_n) &= \frac{1}{\lambda}\Phi_0(\lambda v_n)(\lambda v_n) = \frac{1}{\lambda}\big(2\Phi_0(\lambda v_n) - 2\widehat{\Psi}_0(\lambda v_n)\big)\\ &\leq \frac{2C}{\lambda} - \frac{2}{\lambda}\iint_{\mathbb{R}\times\mathbb{R}^N}\widehat{G}(\lambda|v_n|)\,dxdt.\end{aligned}$$

Meanwhile, for $\delta > 0$ small, we infer

$$\begin{aligned}&\iint_{\mathbb{R}\times\mathbb{R}^N}\widehat{G}(\lambda|v_n|)\\ &\geq \iint_{\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:\,|v_n|\geq\delta\}}\widehat{G}(\lambda|v_n|)\,dxdt\\ &\geq \tilde{G}_\delta(\lambda)\cdot \textit{meas}\big\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:\ |v_n|\geq\delta\big\},\end{aligned}\tag{3.11}$$

where $\tilde{G}_\delta(\lambda) := \inf\big\{\widehat{G}(|z|) :\ z\in\mathbb{R}^{2M},\ |z|\geq\lambda\delta\big\}$. We claim that *meas*$\big\{(t,x)\in\mathbb{R}\times\mathbb{R}^N :\ |v_n|\geq\delta\big\}\geq r_0$ with some $r_0>0$ for all n provided δ is fixed small enough. Indeed, if such r_0 does not exist, we then have $v_n \rightharpoonup 0$ in E_e. However, this contradicts with the fact $v^+\neq 0$. Now, from (3.11) and $(H_3')(ii)$, we deduce that $\tilde{G}_\delta(\lambda)\to+\infty$ as $\lambda\to\infty$ and

$$\begin{aligned}\frac{d}{d\lambda}\Phi_0(\lambda v_n) &\leq \frac{2C}{\lambda} - 2r_0\frac{\tilde{G}_\delta(\lambda)}{\lambda}\\ &\leq \frac{2C}{\lambda} - \frac{3C}{\lambda}\\ &= -\frac{C}{\lambda}\end{aligned}$$

for all n and $\lambda\geq\lambda_0$, some $\lambda_0>0$ large. Thus we have

$$\begin{aligned}\Phi_0(z_n) = \Phi_0(\|z_n\|v_n) &= \int_0^{\|z_n\|}\frac{d}{d\lambda}\Phi_0(\lambda v_n)\\ &\leq \Phi_0(\lambda_0 v_n) + \int_{\lambda_0}^{\|z_n\|} -\frac{C}{\lambda}\,d\lambda\\ &\leq C - C\int_{\lambda_0}^{\|z_n\|}\frac{1}{\lambda}\,d\lambda \to -\infty\end{aligned}$$

as $n\to\infty$, which is absurd. $\square$

The next lemma will be devoted to show the $\mathscr{G}$-weak compactness of the modified functional Φ_ε for each $\varepsilon > 0$. We remark that, by virtue of $(F4)(i)$,

$$\widehat{F}(x,s) \geq \frac{\beta-2}{2\beta} f(x,s)s^2 \geq \frac{\beta-2}{2} F(x,s) > 0 \tag{3.12}$$

for all $x \in \Lambda$ and $s > 0$ provided $(H_3)(i)$ is satisfied. This jointly with $(H_3)(ii)$ yields (written $\sigma = p/(p-1)$)

$$\big(f(x,s)s\big)^\sigma \leq a_1 f(x,s)s^2 \leq a_2 \widehat{F}(x,s) \tag{3.13}$$

for all $|z| \geq r_1$ and $x \in \Lambda$, where r_1 is chosen small enough such that

$$|f(x,s)| \leq \frac{1-|V|_\infty}{4} \quad \text{for all } s \leq r_1,\ x \in \mathbb{R}^N. \tag{3.14}$$

Lemma 3.5. *For each $\varepsilon > 0$, $c \in \mathbb{R} \setminus \{0\}$, Φ_ε satisfies the $\mathscr{G}$-weak $(C)_c$-condition.*

Proof. We begin by proving any $(C)_c$ sequence of Φ_ε is bounded in E. In fact, let $\{z_n\}$ be a sequence such that

$$\Phi_\varepsilon(z_n) \to c \quad \text{and} \quad (1 + \|z_n\|)\Phi'_\varepsilon(z_n) \to 0$$

as $n \to \infty$, the representation of Φ_ε implies that there is $C > 0$ such that

$$C \geq \Phi_\varepsilon(z_n) - \frac{1}{2}\Phi'_\varepsilon(z_n)z_n = \int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{F}_\varepsilon(x,|z_n|)\,dxdt > 0 \tag{3.15}$$

and

$$\begin{aligned} o(1) &= \Phi_\varepsilon(z_n)(z_n^+ - z_n^-) \\ &= \|z_n\|^2 + \int_{\mathbb{R}}\int_{\mathbb{R}^N} V_\varepsilon(x) z_n \cdot (z_n^+ - z_n^-)\,dxdt \\ &\quad - \int_{\mathbb{R}}\int_{\mathbb{R}^N} f_\varepsilon(x,|z_n|) z_n \cdot (z_n^+ - z_n^-)\,dxdt. \end{aligned} \tag{3.16}$$

Case 1. The super quadratic nonlinearity

By the definition of F and (3.16), we soon obtain

$$\begin{aligned} &\|z_n\|^2 - |V|_\infty \int_{\mathbb{R}}\int_{\mathbb{R}^N} |z_n| \cdot |z_n^+ - z_n^-|\,dxdt \\ \leq &\int_{\mathbb{R}}\int_{\mathbb{R}^N} f_\varepsilon(x,|z_n|)|z_n| \cdot |z_n^+ - z_n^-|\,dxdt + o(1) \\ \leq &\int_{\mathbb{R}}\int_{\Lambda_\varepsilon} f_\varepsilon(x,|z_n|)|z_n| \cdot |z_n^+ - z_n^-|\,dxdt \\ &+ \frac{1-|V|_\infty}{2} \int_{\mathbb{R}}\int_{\mathbb{R}^N} |z_n| \cdot |z_n^+ - z_n^-|\,dxdt + o(1), \end{aligned} \tag{3.17}$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$. Thus, from (3.13) and (3.14), we easily check that

$$\begin{aligned}
&\frac{1-|V|_\infty}{4}\|z_n\|^2 \\
&\le \iint_{\{(t,x)\in\mathbb{R}\times\Lambda_\varepsilon:\,|z_n|\ge r_1\}} f_\varepsilon(x,|z_n|)|z_n|\cdot|z_n^+ - z_n^-|\,dxdt + o(1) \\
&\le \bigg(\iint_{\{(t,x)\in\mathbb{R}\times\Lambda_\varepsilon:\,|z_n|\ge r_1\}} \big(f_\varepsilon(x,|z_n|)|z_n|\big)^\sigma\,dxdt\bigg)^{1/\sigma}|z_n^+ - z_n^-|_p + o(1).
\end{aligned}$$

It follows from (3.13), (3.15) and E embeds continuously into L^p, we find

$$\frac{1-|V|_\infty}{4}\|z_n\|^2 \le C_1\|z_n\| + o(1).$$

Then $\{z_n\}$ is bounded in E as desired.

Case 2. The asymptotically quadratic nonlinearity

Assume contrarily that $\|z_n\| \to \infty$ as $n\to\infty$ and set $v_n = z_n/\|z_n\|$. Then $|v_n|_2^2 \le C_2$ and $|v_n|_{2^*}^2 \le C_3$, where $2^* := 2(N+2)/N$. It follows from (3.5) and (3.16) that

$$\begin{aligned}
o(1) = \|z_n\|^2\bigg(&\|v_n\|^2 + \int_{\mathbb{R}}\int_{\mathbb{R}^N} V_\varepsilon(x)v_n\cdot(v_n^+ - v_n^-)\,dxdt \\
&- \int_{\mathbb{R}}\int_{\mathbb{R}^N} f_\varepsilon(x,|z_n|)v_n\cdot(v_n^+ - v_n^-)\,dxdt\bigg) \\
\ge \|z_n\|^2\bigg(&\big(1-|V|_\infty\big) - \int_{\mathbb{R}}\int_{\mathbb{R}^N} f_\varepsilon(x,|z_n|)v_n\cdot(v_n^+ - v_n^-)\,dxdt\bigg).
\end{aligned}$$

And thus

$$\liminf_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}^N} f_\varepsilon(x,|z_n|)v_n\cdot(v_n^+ - v_n^-)\,dxdt \ge \ell := 1-|V|_\infty. \tag{3.18}$$

To get a contradiction, let us first set

$$d(r) := \inf\big\{\widehat{F}(x,s) : x\in\mathbb{R}^N, \text{ and } s > r\big\},$$

$$\Omega_n(\rho,r) := \big\{(t,x)\in\mathbb{R}\times\mathbb{R}^N : \rho\le|z_n(t,x)|\le r\big\},$$

and

$$c_\rho^r := \inf\left\{\frac{\widehat{F}(x,s)}{s^2} : x\in\mathbb{R}^N, \text{ and } \rho\le s\le r\right\}.$$

By $(F6)$, $d(r)\to\infty$ as $r\to\infty$ and by definition

$$\widehat{F}_\varepsilon\big(x,|z_n(t,x)|\big) \ge c_\rho^r|z_n(t,x)|^2 \quad \text{for all } (t,x)\in\Omega_n(\rho,r).$$

From (3.15), we have

$$C \geq \iint_{\Omega_n(0,\rho)} \widehat{F}(\varepsilon x, |z_n|)\, dxdt + c_\rho^r \iint_{\Omega_n(\rho,r)} |z_n|^2\, dxdt$$
$$+ d(r) \cdot \textit{meas}\{\Omega_n(r,\infty)\}.$$

Remark that the above estimate shows $\textit{meas}\{\Omega_n(r,\infty)\} \leq C/d(r) \to 0$ as $r \to \infty$ uniformly in n, and for any fixed $0 < \rho < r$

$$\iint_{\Omega_n(\rho,r)} |v_n|^2\, dxdt = \frac{1}{\|z_n\|^2} \iint_{\Omega_n(\rho,r)} |z_n|^2\, dxdt \leq \frac{C}{c_\rho^r \|z_n\|^2} \to 0$$

as $n \to \infty$.

Now let us choose $0 < \delta < \ell/3$. By $(F1)$ there is $\rho_\delta > 0$ such that

$$f_\varepsilon(x,s) < \frac{\delta}{C_2}$$

for all $x \in \mathbb{R}^N$ and $s \in [0, \rho_\delta]$. Consequently,

$$\iint_{\Omega_n(0,\rho_\delta)} f_\varepsilon(x,|z_n|)|v_n| \cdot |v_n^+ - v_n^-|\, dxdt \leq \frac{\delta}{C_2} |v_n|_2^2 \leq \delta$$

for all n. Recall that, by (H_1), $(H_3')(i)$ and the definition of F, there exists $\tilde{C} > 0$ such that $0 \leq f(x,z) \leq \tilde{C}$ for all (x,z). Using Hölder inequality we can take r_δ so large that

$$\begin{aligned}
&\iint_{\Omega_n(r_\delta,\infty)} f_\varepsilon(x,|z_n|)|v_n| \cdot |v_n^+ - v_n^-|\, dxdt \\
&\leq \tilde{C} \iint_{\Omega_n(r_\delta,\infty)} |v_n| \cdot |v_n^+ - v_n^-|\, dxdt \\
&\leq \tilde{C} \cdot \textit{meas}\{\Omega_n(r_\delta,\infty)\}^{1/(N+2)} \cdot |v_n|_2 \cdot |v_n^+ - v_n^-|_{2^*} \\
&\leq \tilde{C}' \cdot \textit{meas}\{\Omega_n(r_\delta,\infty)\}^{1/(N+2)} \leq \delta
\end{aligned}$$

for all n. Moreover, there is n_0 such that

$$\begin{aligned}
&\iint_{\Omega_n(\rho_\delta,r_\delta)} f_\varepsilon(x,|z_n|)|v_n| \cdot |v_n^+ - v_n^-|\, dxdt \\
&\leq \tilde{C} \iint_{\Omega_n(\rho_\delta,r_\delta)} |v_n| \cdot |v_n^+ - v_n^-|\, dxdt \\
&\leq \tilde{C} \cdot |v_n|_2 \left(\iint_{\Omega_n(\rho_\delta,r_\delta)} |v_n|^2\, dxdt \right)^{1/2} \leq \delta
\end{aligned}$$

for all $n \geq n_0$. Therefore, for n large enough, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^N} f_\varepsilon(x,|z_n|)|v_n| \cdot |v_n^+ - v_n^-|\, dxdt \leq 3\delta < \ell,$$

which is impossible. Hence, the boundedness of $\{z_n\}$ is verified.

The next step is concerned with the $\mathscr{G}$-weak compactness of the $(C)_c$-sequence $\{z_n\}$, $c \neq 0$. Keep in mind that $\varepsilon > 0$ is now fixed (which implies Λ_ε is a bounded domain in $\mathbb{R}^N$), let us choose $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\overline{\Lambda}_\varepsilon \subset \operatorname{supp}\varphi$ and

$$\varphi(x) = \begin{cases} 1 & x \in \overline{\Lambda}_\varepsilon, \\ 0 & x \notin N_1(\overline{\Lambda}_\varepsilon), \end{cases}$$

where $N_1(\overline{\Lambda}_\varepsilon) := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \overline{\Lambda}_\varepsilon) < 1\}$. Denote by $z_n' = \varphi \cdot z_n$, we have $\{z_n'\}$ is a bounded sequence in E since $\{z_n\}$ is bounded. We claim that there exist $\{g_n\} \subset \mathscr{G} := \mathbb{R}$ and $t_0, \delta_0 > 0$ such that

$$\int_{g_n - t_0}^{g_n + t_0} \int_{N_1(\overline{\Lambda}_\varepsilon)} |z_n'|^2 \, dxdt \geq \delta_0 \quad \text{for all } n \geq 1. \tag{3.19}$$

Then, from the compact embedding $E \hookrightarrow L^2_{\mathrm{loc}}$, we have the new sequence $\{g_n \star z_n\}$ possesses a subsequence converge weakly to a point in $E \setminus \{0\}$ (here we used the inequality $|z_n| \geq |z_n'|$).

To prove (3.19), let us assume by contradiction that

$$\lim_{n\to\infty} \sup_{g\in\mathbb{R}} \int_{g-r}^{g+r} \int_{N_1(\overline{\Lambda}_\varepsilon)} |z_n'|^2 \, dxdt = 0 \quad \text{for any } r > 0.$$

Since Λ_ε is bounded, jointly with the definition of z_n', we see that $\{z_n'\}$ is vanishing. By Lion's concentration compactness principle [19], we have $|z_n'|_q \to 0$ for all $q \in (2, 2(N+2)/N)$. By virtue of (3.14), (3.16) and the definition of F, we have

$$\begin{aligned} &\frac{1 - |V|_\infty}{4} \|z_n\|^2 \\ &\leq \iint_{\{(t,x)\in\mathbb{R}\times\Lambda_\varepsilon : \, |z_n| \geq r_1\}} f_\varepsilon(x, |z_n|)|z_n| \cdot |z_n^+ - z_n^-| \, dxdt + o(1) \\ &\leq \iint_{\{(t,x)\in\mathbb{R}\times N_1(\overline{\Lambda}_\varepsilon) : \, |z_n'| \geq r_1\}} f_\varepsilon(x, |z_n'|)|z_n'| \cdot |z_n^+ - z_n^-| \, dxdt + o(1). \end{aligned}$$

Remark that the above estimate holds for both super and asymptotic quadratic nonlinearities, moreover, there must exist $C_0 > 0$ and $p_0 \in (2, 2(N+2)/N)$ such that

$$|f(x,s)| \leq C_0 s^{p_0 - 2} \quad \text{for all } x \in \mathbb{R}^N \text{ and } s \geq r_1.$$

Indeed, one may choose $p_0 = p$ for the super quadratic nonlinearity and $p_0 = q$ for any $q \in (2, 2(N+2)/N)$ for the asymptotic quadratic nonlinearity. Then, by Hölder inequality and the fact $|z_n'|_q \to 0$ for all $q \in (2, 2(N+2)/N)$, we have $z_n \to 0$ in E as $n \to \infty$ which implies $\Phi_\varepsilon(z_n) \to 0$. This contradicts our assumption: $\{z_n\}$ is a $(C)_c$-sequence with $c \neq 0$. $\square$

3.1.5 *The autonomous limit system*

What remains is to check under the assumptions (H_1), (H_2) and either (H_3) or (H_3'), the functional Φ_0 admits a critical value

$$c_0 = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in E_e} \Phi_0(z) < \infty$$

where $E_e = \mathbb{R}^+ e \oplus E^-$.

In what follows, let us consider the autonomous system

$$\begin{cases} \partial_t u = \Delta_x u - u - \mu v + H_v(u,v) \\ -\partial_t v = \Delta_x v - v - \mu u + H_u(u,v) \end{cases} \tag{3.20}$$

for $\mu \in (-1,1)$. Remark that $H(\xi) = \int_0^{|\xi|} g(s)s\,ds$ for $\xi \in \mathbb{R}^{2M}$, it is evident that (3.20) can be rewritten as

$$Lz + \mu z = g(|z|)z$$

for $z = (u,v)$. The solutions to (3.20) are critical points of the $\mathscr{G}'$-invariant energy functional

$$\mathscr{T}_\mu := \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{\mu}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} |z|^2\,dxdt - \Psi_0(z)$$

defined for $z = z^+ + z^- \in E = E^+ \oplus E^-$. It is obvious, for the case $\mu = V_0$, we have $\mathscr{T}_{V_0}$ coincide with Φ_0. For notation convenience, let us denote

$$\mathscr{K}_\mu := \big\{ z \in E\setminus\{0\} : \mathscr{T}_\mu'(z) = 0 \big\} \quad \text{and} \quad \gamma_\mu := \inf\big\{ \mathscr{T}_\mu(z) : z \in \mathscr{K}_\mu \big\}.$$

We state, in the two following propositions, some known results about the existence of solutions of (3.20) that will be used in the sequel.

Proposition 3.6. *Let (H_1) and (H_3) hold. Then the super quadratic nonlinear system* (3.20) *has a nontrivial solution z which lies in $B^r(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ for all $r \geq 2$.*

Proposition 3.7. *Let (H_1) and (H_3') hold. Then the asymptotic quadratic nonlinear system* (3.20) *has a nontrivial solution z which lies in $B^r(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ for all $r \geq 2$.*

Proposition 3.6 and Proposition 3.7 are obtained just collecting the results in [12] (see e.g. Theorem 8.1 and Theorem 8.6). The proofs of the above two propositions can be done, by applying the linking theorems associate with the strongly indefinite functionals.

For fixed $v \in E^+$, let $\phi_v : E^- \to \mathbb{R}$ be defined by $\phi_v(w) = \mathscr{T}_\mu(v+w)$. We infer

$$\phi_v(w) \leq \frac{1+|\mu|}{2}\|v\|^2 - \frac{1-|\mu|}{2}\|w\|^2. \tag{3.21}$$

Moreover, we have, for any $w, z \in E^-$,

$$\phi_v''(w)[z,z] = -\|z\|^2 - \mu \int_{\mathbb{R}} \int_{\mathbb{R}^N} |z|^2 \, dxdt - \Psi_0''(v+w)[z,z] \\ \le -(1-|\mu|)\|z\|^2. \tag{3.22}$$

This is due to the fact that Ψ_0 is convex. A direct consequence of (3.21) and (3.22), there exists a unique bounded C^1 mapping $\mathscr{J}_\mu : E^+ \to E^-$ such that

$$\mathscr{T}_\mu\big(v + \mathscr{J}_\mu(v)\big) = \max_{w \in E^-} \mathscr{T}_\mu(v+w).$$

We omit the proof for the boundedness and C^1 property of $\mathscr{J}_\mu$ since a specific proof under more abstract settings will be presented in Section 4.

Let us consider a new functional defined by

$$\mathscr{R}_\mu : E^+ \to \mathbb{R}, \quad \mathscr{R}_\mu(v) = \mathscr{T}_\mu\big(v + \mathscr{J}_\mu(v)\big).$$

We remark that critical points of $\mathscr{R}_\mu$ and $\mathscr{T}_\mu$ are in one-to-one correspondence via the injective map $v \mapsto v + \mathscr{J}_\mu(v)$ from E^+ into E (this will also be proved in Section 4). Denoted by

$$\Gamma_\mu = \big\{ \nu \in C([0,1], E^+) : \nu(0) = 0, \, \mathscr{R}_\mu(\nu(1)) < 0 \big\},$$

and consider the minimax schemes

$$d_\mu^1 = \inf_{\nu \in \Gamma_\mu} \max_{t \in [0,1]} \mathscr{R}_\mu(\nu(t)) \quad \text{and} \quad d_\mu^2 = \inf_{v \in E^+ \setminus \{0\}} \max_{t \ge 0} \mathscr{R}_\mu(tv),$$

we have the following useful result.

Lemma 3.8. *For the autonomous system* (3.20)*, assume* (H_1)*,* (H_2) *and either* (H_3) *or* (H_3')*, there holds:*

(1) $\gamma_\mu > 0$ *is attained, and* $\gamma_\mu = d_\mu^1 = d_\mu^2$*;*

(2) *if* $\mu_1 > \mu_2$*, then* $\gamma_{\mu_1} > \gamma_{\mu_2}$*.*

Proof. To show (1), let $\{z_n\} \subset \mathscr{K}_\mu$ such that $\mathscr{T}_\mu(z_n) \to \gamma_\mu$. Clearly $\{z_n\}$ is a $(C)_{\gamma_\mu}$-sequence, and hence is bounded.

Claim. $\inf\{\|z\| : z \in \mathscr{K}_\mu\} > 0$.

Indeed, for $z \in \mathscr{K}_\mu$ we have

$$0 = \|z\|^2 + \mu \int_{\mathbb{R}} \int_{\mathbb{R}^N} z \cdot (z^+ - z^-) \, dxdt - \Psi_0(z)(z^+ - z^-).$$

Using (H_1), for $\delta > 0$ small,

$$(1-|\mu|)\|z\|^2 \le \Psi_0(z)(z^+ - z^-) \le \delta |z|_2^2 + C_\delta |z|_{2^*}^{2^*}$$

where $2^* = 2(N+2)/N$. This implies $\|z\|^2 \le C'_\delta \|z\|^{2^*}$ or equivalently $C''_\delta \le \|z\|^{2^*-2}$, and hence the claim is proved.

It is quite standard to check $\gamma_\mu \ge 0$, and if $\gamma_\mu = 0$ we soon have

$$(3.23)\quad (1-|\mu|)\|z_n\|^2 \le \Psi_0(z_n)(z_n^+ - z_n^-) = \int_{\mathbb{R}}\int_{\mathbb{R}^N} g(|z_n|)z_n \cdot (z_n^+ - z_n^-)\, dxdt,$$

and

$$(3.24)\qquad o(1) = \mathscr{T}_\mu(z_n) = \mathscr{T}_\mu(z_n) - \frac{1}{2}\mathscr{T}'_\mu(z_n)z_n = \int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{G}(|z_n|)\, dxdt.$$

For the super quadratic nonlinearity, argue similarly as Lemma 3.5 (see *Case 1*), we have

$$\begin{aligned}
&\frac{1-|\mu|}{4}\|z_n\|^2 \\
\le& \Big(\iint_{\{(t,x)\in\mathbb{R}\times\Lambda_\varepsilon:\, |z_n|\ge r_1\}} \big(g(|z_n|)|z_n|\big)^\sigma\, dxdt \Big)^{1/\sigma} |z_n^+ - z_n^-|_p + o(1) \\
\le& C\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{G}(|z_n|)\, dxdt \Big)^{1/\sigma} \|z_n\| + o(1).
\end{aligned}$$

Together with (3.24), we deduce $\|z_n\| \to 0$ as $n \to \infty$ which is a contradiction. For the asymptotic quadratic nonlinearity, it follows from (3.23) and $\inf_{n\ge1}\|z_n\| > 0$ that $\{z_n\}$ is non-vanishing. Since $\mathscr{T}_\mu$ is $\mathscr{G}'$-invariant, up to a translation, we can assume $z_n \rightharpoonup z_0 \in \mathscr{K}_\mu$. Since, by assumption (H_3'), $\widehat{G}(|z|) \ge 0$ for all $z \in \mathbb{R}^{2M}$, one has (by Fauto's lemma) $\iint \widehat{G}(|z_0|)\, dxdt = 0$. This contrary to that $z_0 \ne 0$.

The proof above gives that $\{z_n\}$ is an non-vanishing sequence in $\mathscr{K}_\mu$ such that $\mathscr{T}_\mu(z_n) \to \gamma_\mu$. By the concentration-compactness principle and the $\mathscr{G}'$-invariance of $\mathscr{T}_\mu$. a standard argument shows γ_μ is attained.

By noting that γ_μ is also the ground state energy of $\mathscr{R}_\mu$, it is not difficult to check that $\gamma_\mu \le d_\mu^1 \le d_\mu^2$. To prove $d_\mu^2 \le \gamma_\mu$ we note that: for $v \in E^+ \setminus \{0\}$, the function $t \mapsto \mathscr{R}_\mu(tv)$ has at most one nontrivial critical point $t = t(v) > 0$ which (if exists) will be the maxima point (this fact will be proved in Lemma 4.6 in Section 4). So, denoted by

$$\mathscr{M}_\mu := \big\{ t(v)v :\, v \in E^+ \setminus \{0\},\ t(v) < \infty \big\},$$

we have $\mathscr{M}_\mu \ne \emptyset$ due to γ_μ is attained. Meanwhile, we notice

$$d_\mu^2 = \inf_{v\in\mathscr{M}_\mu} \mathscr{R}_\mu(v).$$

Remark that, for $z \in \mathscr{K}_\mu$ with $\mathscr{T}_\mu(z) = \gamma_\mu$, we soon have $\mathscr{R}_\mu(tz^+) \to -\infty$ thanks to Lemma 3.3 and Lemma 3.4 and $z^+ \in \mathscr{M}_\mu$ with $\mathscr{R}(z^+) = \gamma_\mu$. Therefore, we deduce $d_\mu^2 \le \mathscr{R}(z^+) = \gamma_\mu$.

Finally (2) comes directly because, let $z \in \mathscr{K}_{\mu_1}$ be such that $\mathscr{T}_{\mu_1}(z) = \gamma_{\mu_1}$, we obtain z^+ is a critical point of $\mathscr{R}_{\mu_1}$ and $\gamma_{\mu_1} = \mathscr{R}_{\mu_1}(z^+) = \max_{t\geq 0} \mathscr{R}_{\mu_1}(tz^+)$. Let $\tau > 0$ be such that $\mathscr{R}_{\mu_2}(\tau z^+) = \max_{t\geq 0} \mathscr{R}_{\mu_2}(tz^+)$, we deduce

$$\begin{aligned}
\gamma_{\mu_1} &= \mathscr{R}_{\mu_1}(z^+) = \max_{t\geq 0} \mathscr{R}_{\mu_1}(tz^+) \\
&\geq \mathscr{R}_{\mu_1}(\tau z^+) = \mathscr{T}_{\mu_1}\big(\tau z^+ + \mathscr{J}_{\mu_1}(\tau z^+)\big) \\
&\geq \mathscr{T}_{\mu_1}\big(\tau z^+ + \mathscr{J}_{\mu_2}(\tau z^+)\big) \\
&= \mathscr{T}_{\mu_2}\big(\tau z^+ + \mathscr{J}_{\mu_2}(\tau z^+)\big) + \frac{\mu_1 - \mu_2}{2}\big|\tau z^+ + \mathscr{J}_{\mu_2}(\tau z^+)\big|_2^2 \\
&= \mathscr{R}_{\mu_2}(\tau z^+) + \frac{\mu_1 - \mu_2}{2}\big|\tau z^+ + \mathscr{J}_{\mu_2}(\tau z^+)\big|_2^2 \\
&\geq \gamma_{\mu_2} + \frac{\mu_1 - \mu_2}{2}\big|\tau z^+ + \mathscr{J}_{\mu_2}(\tau z^+)\big|_2^2,
\end{aligned}$$

which ends the proof. □

Remarkably, the definition of $\mathscr{J}_\mu$ implies

$$\mathscr{R}_\mu(te) = \mathscr{T}_\mu\big(te + \mathscr{J}_\mu(te)\big) = \max_{w\in E^-} \mathscr{T}_\mu(te + w)$$

and therefore

$$\sup_{t\geq 0} \mathscr{R}_\mu(te) = \sup_{t\geq 0} \max_{w\in E^-} \mathscr{T}_\mu(te + w) = \sup_{z\in Ee} \mathscr{T}_\mu(z).$$

By taking infimum with respect to $e \in E^+ \setminus \{0\}$, we have

$$\gamma_\mu = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in Ee} \mathscr{T}_\mu(z).$$

And therefore, for the case $\mu = V_0$, we have Φ_0 admits the critical value

$$c_0 = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in Ee} \Phi_0(z) < \infty$$

as required.

3.2 Proof of Theorem 1.1

From the arguments in the preceding subsection, applying Theorem 2.4, we obtain the following proposition:

Proposition 3.9. *Assume (V), (H_1), (H_2) and either (H_3) or (H_3'). For all $\varepsilon > 0$ small, the modified functional Φ_ε admits a critical value which can be characterized by*

$$c_\varepsilon = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in E_e} \Phi_\varepsilon(z).$$

Moreover, c_ε is the ground state for each Φ_ε and $c_\varepsilon \leq c_0 + o(1)$ as $\varepsilon \to 0$ with

$$c_0 = \inf_{e\in E^+\setminus\{0\}} \sup_{z\in E_e} \Phi_0(z).$$

Next, we shall devote to show the concentration phenomenon with additionally assuming

$$\underline{c} = \min_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

It is worth pointing out that the method of proving Proposition 3.9 carries more information on the ranges of c_ε. In fact the proof strongly depended on the assumption that $0 \in \Lambda$, and however, the value of $V(0)$ is irrelevant. Let $\{x_\varepsilon\}$ be a family of points in Λ_ε so chosen that $V_\varepsilon(x_\varepsilon) = \underline{c}$, and consider the associated equation

$$Lz + \hat{V}_\varepsilon(x)z = f_\varepsilon(x + x_\varepsilon, |z|)z \tag{3.25}$$

with energy functional $\hat{\Phi}_\varepsilon : E \to \mathbb{R}$ being written as

$$\begin{aligned}\hat{\Phi}_\varepsilon(z) = &\frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^N} \hat{V}_\varepsilon(x)|z|^2\,dxdt\\ &- \int_{\mathbb{R}}\int_{\mathbb{R}^N} F_\varepsilon(x + x_\varepsilon, |z|)\,dxdt\end{aligned}$$

where $\hat{V}_\varepsilon(x) = V\big(\varepsilon(x + x_\varepsilon)\big)$. Noting that let $z_\varepsilon \in E$ be the solution to

$$Lz + V_\varepsilon(x)z = f(x, |z|)z$$

with energy $\Phi_\varepsilon(z_\varepsilon) = c_\varepsilon$ and set $w_\varepsilon(t, x) = z_\varepsilon(t, x + x_\varepsilon)$, it is a simple matter to see that w_ε solves (3.25) with $\hat{\Phi}_\varepsilon(w_\varepsilon) = \Phi_\varepsilon(z_\varepsilon) = c_\varepsilon$. Remark that $\hat{V}_\varepsilon(x) \to V_\varepsilon(x_\varepsilon) = \underline{c}$ uniformly on bounded sets of x as $\varepsilon \to 0$, it is clear that Theorem 2.4 works for the new family $\{\hat{\Phi}_\varepsilon\}_{\varepsilon>0} \cup \{\mathscr{T}_{\underline{c}}\}$. Summarizing, we have the following characterization on the ranges of c_ε.

Lemma 3.10. *Let c_ε be given in Proposition* 3.9*. There holds*

$$\limsup_{\varepsilon\to0} c_\varepsilon \le \gamma_{\underline{c}}.$$

Thanks to the above observation, there is no loss of generality in assuming $V_0 := V(0) = \underline{c}$. For ease of notations, let us denote

$$\mathscr{K}_\varepsilon := \big\{z \in E \setminus \{0\} : \Phi_\varepsilon'(z) = 0\big\}, \quad \mathscr{L}_\varepsilon := \big\{z \in \mathscr{K}_\varepsilon : \Phi_\varepsilon(z) = c_\varepsilon\big\},$$

and

$$\mathscr{A} := \big\{x \in \Lambda : V(x) = V_0\big\}.$$

Then we have:

Lemma 3.11. *Under the assumptions of Theorem* 1.1*, for all $\varepsilon > 0$ small and $z_\varepsilon \in \mathscr{L}_\varepsilon$, the time-dependent process $|z_\varepsilon(t, \cdot)|$ possesses a (global) maximum $x_\varepsilon \in \Lambda_\varepsilon$ such that*

$$\lim_{\varepsilon\to0} V(\varepsilon x_\varepsilon) = c_\varepsilon.$$

Moreover, by setting $w_\varepsilon(t, x) = z_\varepsilon(t, x + x_\varepsilon)$, we must have $|w_\varepsilon|$ decays uniformly at infinity and $\{w_\varepsilon\}$ converges in $B^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$ to a ground state solution to

$$Lz + \underline{c}\, z = g(|z|)z.$$

Proof. Our proof starts with the observation that the family $\{z_\varepsilon\}_{\varepsilon>0}$ is bounded (see an argument of Lemma 3.5). In what follows, the proof will be divided into six steps.

Step 1. $\{z_\varepsilon\}$ is non-vanishing.

Suppose contrarily that

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N}\int_{t-R}^{t+R}\int_{B_R(x)}|z_\varepsilon|^2\,dxdt\to 0\quad\text{as } \varepsilon\to 0$$

for all $R>0$. Then, by Lion's concentration compactness principle [19], we have $|z_\varepsilon|_q\to 0$ for all $q\in(2,2(N+2)/N)$. Noting that, as argued in Lemma 3.5, there must exist $C_0>0$ and $p_0\in(2,2(N+2)/N)$ such that

$$|f(x,s)|\le C_0 s^{p_0-2}\quad\text{for all } x\in\mathbb{R}^N \text{ and } s\ge r_1$$

for $r_1>0$ fixed small enough. We soon have

$$\frac{1-|V|_\infty}{4}\|z_\varepsilon\|^2\le\iint_{\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:\,|z_\varepsilon|\ge r_1\}}f_\varepsilon(x,|z_\varepsilon|)|z_\varepsilon|\cdot|z_\varepsilon^+-z_\varepsilon^-|=o(1),$$

which implies $\Phi_\varepsilon(z_\varepsilon)\to 0$, a contradiction.

Step 2. $\{\chi_{\Lambda_\varepsilon}\cdot z_\varepsilon\}$ is non-vanishing, that is: there exist $(t_\varepsilon,x_\varepsilon)\in\mathbb{R}\times\overline{\Lambda}_\varepsilon$ and constants $R,\delta>0$ such that

$$\int_{t_\varepsilon-R}^{t_\varepsilon+R}\int_{B_R(x_\varepsilon)}|\chi_{\Lambda_\varepsilon}\cdot z_\varepsilon|^2\ge\delta.$$

Indeed, if $\{\chi_{\Lambda_\varepsilon}\cdot z_\varepsilon\}$ vanishes, by virtue of *Step 1* we have $\{(1-\chi_{\Lambda_\varepsilon})\cdot z_\varepsilon\}$ is non-vanishing. Then there exist $(t_\varepsilon,x_\varepsilon)\in\mathbb{R}\times(\mathbb{R}^N\setminus\overline{\Lambda}_\varepsilon)$ and constants $R,\delta>0$ such that

$$\int_{t_\varepsilon-R}^{t_\varepsilon+R}\int_{B_R(x_\varepsilon)}|z_\varepsilon|^2\ge\delta.$$

Let us denote by $w_\varepsilon(t,x)=z_\varepsilon(t,x+x_\varepsilon)$, then w_ε satisfies

$$Lw_\varepsilon+\hat{V}_\varepsilon(x)w_\varepsilon=f_\varepsilon(x+x_\varepsilon,|w_\varepsilon|)w_\varepsilon \tag{3.26}$$

where $\hat{V}_\varepsilon(x):=V\big(\varepsilon(x+x_\varepsilon)\big)$. Additionally, $w_\varepsilon\rightharpoonup w\ne 0$ in E and $w_\varepsilon\to w$ in L^q_{loc} for $q\in[1,2(N+2)/N)$. Remark that $\{\chi_{\Lambda_\varepsilon}\cdot z_\varepsilon\}$ vanishes implies $\chi_{\Lambda_\varepsilon}\cdot z_\varepsilon\rightharpoonup 0$ in L^q for all $q\in(2,2(N+2)/N)$. Now assume without loss of generality that $V(\varepsilon x_\varepsilon)\to V_\infty$, using $\psi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{2M})$ as a test function in (3.26), one gets

$$\begin{aligned}0&=\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\Big(Lw_\varepsilon+\hat{V}_\varepsilon(x)w_\varepsilon-f_\varepsilon(x+x_\varepsilon,|w_\varepsilon|)w_\varepsilon\Big)\cdot\psi\,dxdt\\&=\int_{\mathbb{R}}\int_{\mathbb{R}^N}\Big(Lw+V_\infty w-(1-\chi_\infty)\tilde{g}(|w|)w\Big)\cdot\psi\,dxdt\end{aligned}$$

where χ_∞ is either a characteristic function of a half-space of $\mathbb{R}^N$ provided

$$\limsup_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \partial\Lambda_\varepsilon) < +\infty$$

or $\chi_\infty \equiv 0$ (since Λ is an open set with smooth boundary, this can be see by the fact $\chi_\Lambda(\varepsilon(\cdot + x_\varepsilon))$ converges pointwise a.e. on $\mathbb{R}^N$ to $\chi_\infty(\cdot)$ and $x_\varepsilon \in \mathbb{R}^N \setminus \overline{\Lambda}_\varepsilon$). Hence w solves

$$Lw + V_\infty w = (1-\chi_\infty)\tilde{g}(|w|)w. \tag{3.27}$$

However, using the test function $w^+ - w^-$ in (3.27), we have (with $(F3)$)

$$\begin{aligned} 0 &= \|w\|^2 + V_\infty \int_{\mathbb{R}}\int_{\mathbb{R}^N} w\cdot(w^+ - w^-)\,dxdt \\ &\quad - \int_{\mathbb{R}}\int_{\mathbb{R}^N} (1-\chi_\infty)\tilde{g}(|w|)w\cdot(w^+ - w^-)\,dxdt \\ &\geq \|w\|^2 - |V|_\infty\|w\|^2 - \frac{1-|V|_\infty}{2}\|w\|^2 \\ &= \frac{1-|V|_\infty}{2}\|w\|^2. \end{aligned}$$

Therefore, we have $w = 0$ a contradiction.

Step 3. Let $x_\varepsilon \in \mathbb{R}^N$ and $R, \delta > 0$ be such that

$$\int_{\mathbb{R}}\int_{B_R(x_\varepsilon)} |\chi_{\Lambda_\varepsilon}\cdot z_\varepsilon|^2\,dxdt \geq \delta.$$

Then $\varepsilon x_\varepsilon \to \mathscr{A}$.

First, by virtue of *Step 2*, we can conclude such x_ε do exist and we can choose $x_\varepsilon \in \Lambda_\varepsilon$ (i.e. $\varepsilon x_\varepsilon \in \Lambda$). Suppose that, up to a subsequence, $\varepsilon x_\varepsilon \to x_0 \in \overline{\Lambda}$ as $\varepsilon \to 0$. Again, set $w_\varepsilon(t,x) = z_\varepsilon(t, x + x_\varepsilon)$, we have $w_\varepsilon \rightharpoonup w \neq 0$ in E and w satifies

$$Lw + V(x_0)w = f_\infty(x, |w|)w, \tag{3.28}$$

where $f_\infty(x,s) = \chi_\infty g(s) + (1-\chi_\infty)\tilde{g}(s)$ and χ_∞ is either a characteristic function of a half-space of $\mathbb{R}^N$ provided

$$\limsup_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \partial\Lambda_\varepsilon) < +\infty$$

or $\chi_\infty \equiv 1$ (this is because $x_\varepsilon \in \Lambda_\varepsilon$). Denote by S_∞ the associate energy functional to (3.28):

$$S_\infty := \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{V(x_0)}{2}|z|_2^2 - \Psi_\infty(z),$$

where

$$\Psi_\infty(z) := \int_{\mathbb{R}}\int_{\mathbb{R}^N} F_\infty(x, |z|)\,dxdt.$$

By noting that $\Psi_\infty(z) \le \Psi_0(z)$ (thanks to the definition of $\tilde{g}$), we have

$$S_\infty(z) \ge \mathscr{T}_{V(x_0)}(z) = \mathscr{T}_{V_0}(z) + \frac{V(x_0) - V_0}{2}|z|_2^2 \quad \text{for all } z \in E.$$

Remark that Ψ_∞ is convex, and furthermore, as argued in the preceding subsection, for $z \in E \setminus \{0\}$ and $w \in E$ we have

$$\big(\Psi_\infty''(z)[z,z] - \Psi_\infty'(z)z\big) + 2\big(\Psi_\infty''(z)[z,w] - \Psi_\infty'(z)w\big) + \Psi_\infty''(z)[w,w] > 0.$$

Let us define (as in proving Lemma 3.8) $h_\infty : E^+ \to E^-$ and $I_\infty : E^+ \to \mathbb{R}$ by

$$S_\infty\big(v + h_\infty(v)\big) = \max_{z \in E^-} S_\infty(v + z),$$

$$I_\infty(v) = S_\infty\big(v + h_\infty(v)\big).$$

Since we already have $w \neq 0$ is a critical point of S_∞, we then infer w^+ is a critical point of I_∞ and $I_\infty(w^+) = \max_{t \ge 0} I_\infty(tw^+)$ (the proof is similar to the case in Lemma 3.8). Let $\tau > 0$ be such that $\mathscr{R}_{V_0}(\tau w^+) = \max_{t\ge 0} \mathscr{R}_{V_0}(tw^+)$, we deduce

$$\begin{aligned}
S_\infty(w) &= I_\infty(w^+) = \max_{t \ge 0} I_\infty(tw^+) \\
&\ge I_\infty(\tau w^+) = S_\infty\big(\tau w^+ + h_\infty(\tau w^+)\big) \\
&\ge S_\infty\big(\tau w^+ + \mathscr{J}_{V_0}(\tau w^+)\big) \ge \mathscr{T}_{V_0}\big(\tau w^+ + \mathscr{J}_{V_0}(\tau w^+)\big) \\
&\quad + \frac{V(x_0) - V_0}{2}\big|\tau w^+ + \mathscr{J}_{V_0}(\tau w^+)\big|_2^2 \\
&= \mathscr{R}_{V_0}(\tau w^+) + \frac{V(x_0) - V_0}{2}\big|\tau w^+ + \mathscr{J}_{V_0}(\tau w^+)\big|_2^2 \\
&\ge \gamma_{V_0} + \frac{V(x_0) - V_0}{2}\big|\tau w^+ + \mathscr{J}_{V_0}(\tau w^+)\big|_2^2 .
\end{aligned} \tag{3.29}$$

On the other hand, by Fatou's lemma, we find

$$\begin{aligned}
\liminf_{\varepsilon \to 0} c_\varepsilon &= \liminf_{\varepsilon \to 0} \left(\Phi_\varepsilon(z_\varepsilon) - \frac{1}{2}\Phi_\varepsilon'(z_\varepsilon)z_\varepsilon\right) \\
&= \liminf_{\varepsilon \to 0} \int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{F}_\varepsilon(x, |z_\varepsilon|)\, dxdt \\
&= \liminf_{\varepsilon \to 0} \int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{F}_\varepsilon(x + x_\varepsilon, |w_\varepsilon|)\, dxdt \\
&\ge \int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{F}_\infty(x, w)\, dxdt \\
&= S_\infty(w) - \frac{1}{2}S_\infty'(w)w = S_\infty(w),
\end{aligned}$$

where $\widehat{F}_\infty(x,s) := \frac{1}{2}f_\infty(x,s)s^2 - F_\infty(x,s)$ for $(x,s) \in \mathbb{R}^N \times \mathbb{R}^+$. Therefore, together with (3.29), we have $\liminf_{\varepsilon\to 0} c_\varepsilon \ge \gamma_{V_0}$ and $\liminf_{\varepsilon \to 0} c_\varepsilon > \gamma_{V_0}$ provided $V(x_0) \neq V_0$. Recall we have assumed that $V_0 = \underline{c}$, and from the fact $c_\varepsilon \le \gamma_{\underline{c}} + o(1)$

as $\varepsilon \to 0$, we soon conclude that $c_\varepsilon \to \gamma_{V_0}$, and moreover, $x_0 \in \mathscr{A}$ and $\chi_\infty \equiv 1$ (that is $f_\infty(x,s) \equiv g(s)$).

Step 4. Let w_ε be defined in *Step 3*, then $w_\varepsilon \to w$ in E.

It suffices to prove that there is a subsequence $\{w_{\varepsilon_j}\}$ such that $w_{\varepsilon_j} \to w$ in E. Recall that, as the argument shows, w is a ground state solution to

$$Lw + V_0 w = g(|w|)w \tag{3.30}$$

and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \widehat{F}_\varepsilon(x + x_\varepsilon, |w_\varepsilon|)\, dxdt = \int_{\mathbb{R}} \int_{\mathbb{R}^N} \widehat{G}(|w|)\, dxdt.$$

Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \le 1$, $\eta(s) = 0$ if $s \ge 2$. Define $\tilde{w}_j(t,x) = \eta(2|(t,x)|/j)w(t,x)$ (here, and in the sequel, by $|(t,x)|$ we mean the Euclid norm on $\mathbb{R} \times \mathbb{R}^N$). One has

$$\|\tilde{w}_j - w\| \to 0 \quad \text{and} \quad |\tilde{w}_j - w|_q \to 0 \quad \text{as } j \to \infty \tag{3.31}$$

for $q \in [2, 2(N+2)/N]$. Set $B_d := \big\{(t,x) \in \mathbb{R} \times \mathbb{R}^N : |(t,x)| \le d\big\}$ for $d > 0$. We have that there possesses a subsequence $\{w_{\varepsilon_j}\}$ such that: for any $\delta > 0$ there exists $r_\delta > 0$ satisfying

$$\limsup_{j\to\infty} \iint_{B_j \setminus B_r} |w_{\varepsilon_j}|^q\, dxdt \le \delta$$

for all $r \ge r_\delta$ (see an argument of [12, Lemma 5.7]). Here we shall use

$$q = \begin{cases} p & \text{for the super quadratic case,} \\ 2 & \text{for the asymptotically quadratic case,} \end{cases}$$

where $p \in (2, 2(N+2)/N)$ is the constant in condition (H_2). Denote $v_j = w_{\varepsilon_j} - \tilde{w}_j$, we remark that $\{v_j\}$ is bounded in E and

$$\begin{aligned} \lim_{j\to\infty} \bigg| \int_{\mathbb{R}} \int_{\mathbb{R}^N} & F_{\varepsilon_j}(x + x_{\varepsilon_j}, |w_{\varepsilon_j}|) - F_{\varepsilon_j}(x + x_{\varepsilon_j}, |v_j|) \\ & - F_{\varepsilon_j}(x + x_{\varepsilon_j}, |\tilde{w}_j|)\, dxdt \bigg| = 0 \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} \lim_{j\to\infty} \bigg| \int_{\mathbb{R}} \int_{\mathbb{R}^N} & \Big[f_{\varepsilon_j}(x + x_{\varepsilon_j}, |w_{\varepsilon_j}|) w_{\varepsilon_j} - f_{\varepsilon_j}(x + x_{\varepsilon_j}, |v_j|) v_j \\ & - f_{\varepsilon_j}(x + x_{\varepsilon_j}, |\tilde{w}_j|) \tilde{w}_j \Big] \cdot \varphi\, dxdt \bigg| = 0 \end{aligned} \tag{3.33}$$

uniformly in $\varphi \in E$ with $\|\varphi\| \le 1$ (analysis similar to that in the proof of [12, Lemma 7.10]). Using the decay of w and the fact that $\hat{V}_{\varepsilon_j}(x) \to V_0$, $F_{\varepsilon_j}(x +$

$x_{\varepsilon_j}, |w|) \to G(|w|)$ as $j \to \infty$ uniformly on any bounded set of x, one checks easily the following

$$\int_{\mathbb{R}}\int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x)w_{\varepsilon_j} \cdot \tilde{w}_j \, dxdt \to \int_{\mathbb{R}}\int_{\mathbb{R}^N} V_0 \cdot |w|^2 \, dxdt,$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}^N} F_{\varepsilon_j}(x + x_{\varepsilon_j}, |\tilde{w}_j|) \, dxdt \to \int_{\mathbb{R}}\int_{\mathbb{R}^N} G(|w|) \, dxdt.$$

Recall that w_{ε_j} solves

$$Lw_{\varepsilon_j} + \hat{V}_{\varepsilon_j}(x)w_{\varepsilon_j} = f_\varepsilon(x + x_\varepsilon, |w_{\varepsilon_j}|)w_{\varepsilon_j}, \tag{3.34}$$

denote $\hat{\Phi}_\varepsilon$ to be the associate energy functional of (3.34), we obtain

$$\begin{aligned}\hat{\Phi}_{\varepsilon_j}(v_j) &= \hat{\Phi}_{\varepsilon_j}(w_{\varepsilon_j}) - S_\infty(w) \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}^N} F_{\varepsilon_j}(x + x_{\varepsilon_j}, |w_{\varepsilon_j}|) - F_{\varepsilon_j}(x + x_{\varepsilon_j}, |v_j|) \\ &\quad - F_{\varepsilon_j}(x + x_{\varepsilon_j}, |\tilde{w}_j|) \, dxdt + o(1) \\ &= o(1)\end{aligned}$$

as $j \to \infty$, which implies that $\hat{\Phi}_{\varepsilon_j}(v_j) \to 0$. Similarly,

$$\begin{aligned}\hat{\Phi}'_{\varepsilon_j}(v_j)\varphi &= \int_{\mathbb{R}}\int_{\mathbb{R}^N} \Big[f_{\varepsilon_j}(x + x_{\varepsilon_j}, |w_{\varepsilon_j}|)w_{\varepsilon_j} - f_{\varepsilon_j}(x + x_{\varepsilon_j}, |v_j|)v_j \\ &\quad - f_{\varepsilon_j}(x + x_{\varepsilon_j}, |\tilde{w}_j|)\tilde{w}_j \Big] \cdot \varphi \, dxdt + o(1) \\ &= o(1)\end{aligned}$$

as $j \to \infty$ uniformly in $\|\varphi\| \le 1$, which implies $\hat{\Phi}'_{\varepsilon_j}(v_j) \to 0$. Therefore,

$$o(1) = \hat{\Phi}_{\varepsilon_j}(v_j) - \frac{1}{2}\hat{\Phi}'_{\varepsilon_j}(v_j)v_j = \int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{F}_{\varepsilon_j}\big(x + x_{\varepsilon_j}, |v_j|\big) \, dxdt. \tag{3.35}$$

Owning to (F_6) and the regularity result (see Corollary A.4 and Lemma A.5), one has $\{|v_j|_\infty\}$ is bounded and for any fixed $r > 0$ there holds

$$\int_{\mathbb{R}}\int_{\mathbb{R}^N} \widehat{F}_{\varepsilon_j}\big(x + x_{\varepsilon_j}, |v_j|\big) \, dxdt \ge C_r \iint_{\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:\, |v_j|\ge r\}} |v_j|^2 \, dxdt$$

for some constant C_r depends only on r. Hence

$$\iint_{\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:\, |v_j|\ge r\}} |v_j|^2 \, dxdt \to 0$$

as $j \to \infty$ for any fixed $r > 0$. Notice $\{|v_j|_\infty\}$ is bounded, as a consequence, we get

$$\begin{aligned}(1 - |V|_\infty)\|v_j\|^2 &\le \|v_j\|^2 + \int_{\mathbb{R}}\int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x) v_j \cdot (v_j^+ - v_j^-) \\ &= \hat{\Phi}'_{\varepsilon_j}(v_j)(v_j^+ - v_j^-) \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}^N} f_{\varepsilon_j}\big(x + x_{\varepsilon_j}, |v_j|\big) v_j \cdot (v_j^+ - v_j^-) \\ &\le o(1) + \frac{1 - |V|_\infty}{2}\|v_j\|^2 \\ &\quad + C_\infty \iint_{\{(t,x)\in\mathbb{R}\times\mathbb{R}^N :\, |v_j| \ge r\}} |v_j| \cdot |v_j^+ - v_j^-| \\ &\le o(1) + \frac{1 - |V|_\infty}{2}\|v_j\|^2,\end{aligned}$$

that is, $\|v_j\| \to 0$ as $j \to \infty$. Together with (3.31) we get $w_{\varepsilon_j} \to w$ in E.

Step 5. $w_\varepsilon \to w$ in $B^2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ as $\varepsilon \to 0$.

To prove this, we only need to show that $|L(w_\varepsilon - w)|_2 \to 0$ as $\varepsilon \to 0$ (this can be seen from that $|Lz|_2$ for $z \in B^2$ defines a equivalent norm on B^2). By (3.30) and (3.34), we obtain

$$L(w_\varepsilon - w) = f_\varepsilon\big(x + x_\varepsilon, |w_\varepsilon|\big) w_\varepsilon - g(|w|)w - \big(\hat{V}_\varepsilon(x) w_\varepsilon - V_0 w\big).$$

Using the result in *Step 4* and the uniform L^∞ estimate, it is easy to check that $|L(w_\varepsilon - w)|_2 \to 0$ as $\varepsilon \to 0$.

Step 6. $w_\varepsilon(t,x) \to 0$ as $|(t,x)| \to \infty$ uniformly for all small ε.

To show this, let us remark that: for $w_\varepsilon = (w_\varepsilon^1, w_\varepsilon^2) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^{2M}$ solves (3.34), if denoted by $\widehat{w}_\varepsilon(t,x) = \big(w_\varepsilon^1(t,x), w_\varepsilon^2(-t,x)\big)$, it is clear that $\widehat{w}_\varepsilon$ satisfies a equation of the form

$$\partial_t \widehat{w}_\varepsilon - \Delta_x \widehat{w}_\varepsilon + \widehat{w}_\varepsilon = \widehat{f}_\varepsilon(t,x) \quad \text{in } \mathbb{R}\times\mathbb{R}^N.$$

By virtue of Lemma A.5, we have $\widehat{f}_\varepsilon \in L^q$ for all $q \ge 2$. According to *Step 5* and the interpolation theory, we have $w_\varepsilon \to w$ in $B^r(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ for all $q \ge 2$. So, an easy calculation shows $\widehat{f}_\varepsilon \to \widehat{f}_0$ in L^q for some f_0 and all $q \ge 2$. Then an trivial application of Corollary A.4 shows that $|\widehat{w}_\varepsilon(t,x)| \to 0$ as $|(t,x)| \to \infty$, which yields the uniformly decay property of $\{w_\varepsilon\}$ as required.

Now, by collecting all the results proved in *Step 1-6*, we have the lemma proved. □

Now, we are ready to prove our main results.

Proof of Theorem 1.1. We follow the notation used in Lemma 3.11 and define

$$\tilde{z}_\varepsilon(t,x) = z_\varepsilon(t, x/\varepsilon) \quad \text{and} \quad y_\varepsilon = \varepsilon x_\varepsilon.$$

Then $\tilde{z}_\varepsilon = (\tilde{u}, \tilde{v})$ is a solution of

$$\begin{cases} \partial_t u = \varepsilon^2 \Delta_x u - u - V(x)v + f(x, |z|)v \\ -\partial_t v = \varepsilon^2 \Delta_x v - v - V(x)u + f(x, |z|)u \\ z = (u, v) \in B^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M}) \end{cases}$$

for ε small. Since y_ε is the maximum point, and due to the fact

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \|z_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^N \setminus B_R(x_\varepsilon))} = 0,$$

we have

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \|\tilde{z}_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^N \setminus B_{\varepsilon R}(y_\varepsilon))} = 0. \tag{3.36}$$

Noting that $y_\varepsilon \to \mathscr{A}$ as $\varepsilon \to 0$, the assumption

$$\min_{\Lambda} V < \min_{\partial\Lambda} V$$

and (3.36) implies: for $\varepsilon > 0$ sufficiently small, there holds $|\tilde{z}_\varepsilon(t, x)| < s_0$ provided $x \notin \Lambda$ (here $s_0 > 0$ is so chosen that $g(s_0) + g'(s_0)s_0 = \frac{1-|V|_\infty}{2}$). Therefore, by the definition of F, we have $F(x, |\tilde{z}_\varepsilon|) = H(\tilde{z}_\varepsilon)$ when $\varepsilon > 0$ is small enough. Note that we have actually proved that $\tilde{z}_\varepsilon$ is a solution to (3.1) for all small ε, and consequently the proof of the theorem is thereby completed by combing (3.36) with Lemma 3.11. $\square$

4 Proof of the abstract theorem

In this section, we devote to prove Theorem 2.4, and assume in the proofs that all the hypotheses in Theorem 2.4 hold. Observe that $\Psi_\varepsilon(z) \geq 0$ for all $z \in E$ and that $(N1)$ yields

$$\Psi''_\varepsilon(z)[w, w] \geq 0 \quad \text{for any } w \in E.$$

This can be seen by the fact $\Psi_\varepsilon \in C^2(E, \mathbb{R})$ and Ψ_ε is convex for each $\varepsilon \in \mathcal{E}$. Note that by $(I1)$ we have $\Phi_\varepsilon(0) \geq 0$, which means $\Psi_\varepsilon(0) = 0$. And condition $(N3)$ implies

$$\Psi_\varepsilon(z) = \int_0^1 \int_0^t \Psi''_\varepsilon(sz)[z, z]\, dsdt \leq C(\kappa, \|z\|)\|z\|^2, \quad \forall z \in E \tag{4.1}$$

where $C(\kappa, \|z\|) > 0$ is a constant depending only on the function κ and $\|z\|$.

Consider $\varepsilon \in \mathcal{E}$ being fixed, we define the nonlinear functional $\phi_v : Y \to \mathbb{R}$ by

$$\phi_v(w) = \Phi_\varepsilon(v + w) \quad \text{for } v \in X.$$

Notice that $(A1)$ and $(A2)$ implies $\sup_{\varepsilon\in\mathcal{E}}\|A_\varepsilon\|\le\theta<1$, we infer

$$\phi_v(w)\le\frac{1+\theta}{2}\|v\|^2-\frac{1-\theta}{2}\|w\|^2. \tag{4.2}$$

Moreover, by taking derivatives, we have

$$\begin{aligned}\phi_v''(w)[z,z]&=-\|z\|^2+\langle A_\varepsilon z,z\rangle-\Psi_\varepsilon''(v+w)[z,z]\\&\le-(1-\theta)\|z\|^2\end{aligned} \tag{4.3}$$

for any $w,z\in Y$.

As a consequence of (4.2) and (4.3), ϕ_v is strictly concave and $\phi_v(w)\to-\infty$ as $\|w\|\to\infty$. It follows from the weak upper semi-continuity of ϕ_v that there is unique strict maximum point $h_\varepsilon(v)$ for ϕ_v, which we can easily confirm to be the only critical point of ϕ_v on Y. The uniquely defined map $h_\varepsilon:X\to Y$ can be seen as a reduction of Φ_ε on X satisfying

$$\Phi_\varepsilon\big(v+h_\varepsilon(v)\big)=\phi_v\big(h_\varepsilon(v)\big)=\max_{w\in Y}\phi_v(w)=\max_{w\in Y}\Phi_\varepsilon(v+w). \tag{4.4}$$

It follows from (4.4)

$$\begin{aligned}0\le\,&\Phi_\varepsilon\big(v+h_\varepsilon(v)\big)-\Phi_\varepsilon(v)\\=&-\frac12\|h_\varepsilon(v)\|^2+\frac12\left\langle A_\varepsilon(v+h_\varepsilon(v)),v+h_\varepsilon(v)\right\rangle-\Psi_\varepsilon\big(v+h_\varepsilon(v)\big)\\&-\frac12\left\langle A_\varepsilon v,v\right\rangle+\Psi_\varepsilon(v)\\\le&-\frac12\|h_\varepsilon(v)\|^2+\frac{\theta}{2}\|h_\varepsilon(v)\|^2+\frac{\theta}{2}\|v\|^2+\frac{\theta}{2}\|v\|^2+\Psi_\varepsilon(v)\end{aligned}$$

for $v\in X$. Hence

$$\|h_\varepsilon(v)\|^2\le\frac{2\theta}{1-\theta}\|v\|^2+\frac{2}{1-\theta}\Psi_\varepsilon(v)$$

and the boundedness of Ψ_ε (see (4.1)) implies that of h_ε. If $v\in X$ and $g\in\mathscr{G}$, we have by invariance of Φ_ε and by (4.4)

$$\begin{aligned}\Phi_\varepsilon\big(gv+h_\varepsilon(gv)\big)&=\Phi_\varepsilon\big(v+g^{-1}h_\varepsilon(gv)\big)\le\Phi_\varepsilon\big(v+h_\varepsilon(v)\big)\\&=\Phi_\varepsilon\big(gv+gh_\varepsilon(v)\big)\le\Phi_\varepsilon\big(gv+h_\varepsilon(gv)\big).\end{aligned}$$

Therefore, we conclude

$$\Phi_\varepsilon\big(gv+gh_\varepsilon(v)\big)=\Phi_\varepsilon\big(gv+h_\varepsilon(gv)\big)$$

which together with (4.4) implies that $g\circ h_\varepsilon=h_\varepsilon\circ g$, i.e. h_ε is $\mathscr{G}$-equivariant.

Next we define $\pi:X\times Y\to Y$ by

$$\pi(v,w)=P^Y\circ\mathcal{R}\circ\Phi_\varepsilon'(v+w)=P^Y\circ\nabla\Phi_\varepsilon(v+w)$$

where P^Y is the projection and $\mathcal{R} : E^* \to E$ denotes the isomorphism induced from the Riesz representation theorem. Observe that, for every $v \in X$, we derive from the definition of h_ε:

$$0 = \phi_v'\big(h_\varepsilon(v)\big)w = \Phi_\varepsilon'\big(v + h_\varepsilon(v)\big)w \quad \text{for any } w \in Y.$$

This implies

$$\pi\big(v, h_\varepsilon(v)\big) = 0 \quad \text{for } v \in X. \tag{4.5}$$

Notice that $\partial_w \pi(v,w) = P^Y \circ \mathcal{R} \circ \Phi_\varepsilon''(v+w)\big|_Y$ is a bounded linear operator on Y. And from (4.3), we infer $\partial_w \pi(v,w)$ is an isomorphism with

$$\big\|\partial_w \pi(v,w)^{-1}\big\| \le \frac{1}{1-\theta} \quad \forall v \in X. \tag{4.6}$$

Therefore (4.5) and (4.6) together with the implicit function theorem yield the uniquely defined map $h_\varepsilon : X \to Y$ is C^1 smooth with

$$h_\varepsilon'(v) = -\partial_w \pi\big(v, h_\varepsilon(v)\big)^{-1} \circ \partial_v \pi\big(v, h_\varepsilon(v)\big), \quad \forall v \in X$$

where $\partial_v \pi(v,w) = P^Y \circ \mathcal{R} \circ \Phi_\varepsilon''(v+w)\big|_X$.

Now set

$$I_\varepsilon : X \to \mathbb{R}, \quad I_\varepsilon(v) = \Phi_\varepsilon\big(v + h_\varepsilon(v)\big).$$

We have $I_\varepsilon \in C^1(X,\mathbb{R})$ is $\mathscr{G}$-invariant. And we can conclude from the above arguments that:

Proposition 4.1. *Suppose $(A1)$-$(A2)$, $(N1)$ and $(N3)$ are satisfied. Then $I_\varepsilon \in C^1(X,\mathbb{R})$ for each $\varepsilon \in \mathcal{E}$ and critical points of I_ε and Φ_ε are in one-to-one correspondence via the injective map $v \mapsto v + h_\varepsilon(v)$ from X to E. Moreover, if $\{v_n\} \subset X$ is a $(C)_c$-sequence of I_ε, then $\{v_n + h_\varepsilon(v_n)\}$ is a $(C)_c$-sequence of Φ_ε.*

Remark 4.2. The second part of the above proposition may seen to be not so obvious, however, by taking the derivative of I_ε we have

$$\begin{aligned} I_\varepsilon'(v)w &= \Phi_\varepsilon'\big(v + h_\varepsilon(v)\big)\big(w + h_\varepsilon'(v)w\big) \\ &= \Phi_\varepsilon'\big(v + h_\varepsilon(v)\big)(w + y) \end{aligned}$$

for all $v, w \in X$ and $y \in Y$. And hence $\|I_\varepsilon'(v)\|_{X^*} = \big\|\Phi_\varepsilon'\big(v + h_\varepsilon(v)\big)\big\|_{E^*}$ which implies nothing but the second conclusion in Proposition 4.1. Let us mention here that a reduction of a strongly indefinite functional to a functional on E^+ is well known under strong differentiability condition, see for example [21, 22]. In [21, 22] a reduction in two steps has been performed: first to E^+ and then to a Nehari manifold on E^+. However, since we are interested in the geometric situation $(I2)$, the so-called Nehari manifold is not defined for all direction in E^+, this context requires somewhat different arguments.

In order to state our next result, we shall give another observation. Due to the fact $h_\varepsilon(v)$ is the unique critical point of ϕ_v on Y that, by setting $z = w - h_\varepsilon(v)$ for $w \in Y$ and $l(t) := \Phi_\varepsilon\big(v + h_\varepsilon(v) + tz\big)$, we have $l(1) = \Phi_\varepsilon(v + w)$, $l(0) = \Phi_\varepsilon\big(v + h_\varepsilon(v)\big)$ and $l'(0) = 0$. So, by $l(1) - l(0) = \int_0^1 (1-s)l''(s)\,ds$, we deduce

$$\begin{aligned}
&\Phi_\varepsilon(v+w) - \Phi_\varepsilon\big(v + h_\varepsilon(v)\big) \\
&= \int_0^1 (1-s)\Phi_\varepsilon''\big(v + h_\varepsilon(v) + sz\big)[z,z]\,ds \\
&= -\int_0^1 (1-s)\big(\|z\|^2 - \langle A_\varepsilon z, z\rangle\big)\,ds \\
&\quad - \int_0^1 (1-s)\Psi_\varepsilon''\big(v + h_\varepsilon(v) + sz\big)[z,z]\,ds.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\Phi_\varepsilon\big(v + h_\varepsilon(v)\big) - \Phi_\varepsilon(v+w) \\
&= \frac{1}{2}\|z\|^2 - \frac{1}{2}\langle A_\varepsilon z, z\rangle + \int_0^1 (1-s)\Psi_\varepsilon''\big(v + h_\varepsilon(v) + sz\big)[z,z]\,ds
\end{aligned} \tag{4.7}$$

for all $v \in X$ and $w \in Y$.

Lemma 4.3. *Suppose* $(A1)$-$(A2)$ *and* $(N1)$-$(N3)$. *Then* $h_\varepsilon(v) \to h_0(v)$ *in* Y *as* $\varepsilon \to 0$ *for* $v \in X$.

Proof. For ease of notations, set $z_\varepsilon = v + h_\varepsilon(v)$, $w = v + h_0(v)$ and $v_\varepsilon = z_\varepsilon - w$. It sufficient to show $\|v_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Taking into account that

$$\Phi_\varepsilon(z) = \Phi_0(z) + \frac{1}{2}\langle (A_\varepsilon - A_0)z, z\rangle - \big(\Psi_\varepsilon(z) - \Psi_0(z)\big), \quad \forall z \in E$$

we infer

$$\begin{aligned}
&\big(\Phi_\varepsilon(z_\varepsilon) - \Phi_\varepsilon(w)\big) + \big(\Phi_0(w) - \Phi_0(z_\varepsilon)\big) \\
&= \frac{1}{2}\langle (A_\varepsilon - A_0)z_\varepsilon, z_\varepsilon\rangle - \frac{1}{2}\langle (A_\varepsilon - A_0)w, w\rangle + \big(\Psi_0(z_\varepsilon) - \Psi_0(w)\big) \\
&\quad - \big(\Psi_\varepsilon(z_\varepsilon) - \Psi_\varepsilon(w)\big).
\end{aligned} \tag{4.8}$$

Remark that

$$\Psi_0(z_\varepsilon) - \Psi_0(w) = \Psi_0(w)v_\varepsilon + \int_0^1 (1-s)\Psi_0''(w + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\,ds, \tag{4.9}$$

$$\Psi_\varepsilon(z_\varepsilon) - \Psi_\varepsilon(w) = \Psi_\varepsilon'(w)v_\varepsilon + \int_0^1 (1-s)\Psi_\varepsilon''(w + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\,ds, \tag{4.10}$$

and by (4.7)

$$\Phi_\varepsilon(z_\varepsilon) - \Phi_\varepsilon(w) \tag{4.11}$$
$$= \frac{1}{2}\|v_\varepsilon\|^2 - \frac{1}{2}\langle A_\varepsilon v_\varepsilon, v_\varepsilon\rangle + \int_0^1 (1-s)\Psi''_\varepsilon\big(z_\varepsilon - sv_\varepsilon\big)[v_\varepsilon, v_\varepsilon]\, ds,$$

$$\Phi_0(w) - \Phi_0(z_\varepsilon) \tag{4.12}$$
$$= \frac{1}{2}\|v_\varepsilon\|^2 - \frac{1}{2}\langle A_0 v_\varepsilon, v_\varepsilon\rangle + \int_0^1 (1-s)\Psi''_0\big(w + sv_\varepsilon\big)[v_\varepsilon, v_\varepsilon]\, ds.$$

We derived from (4.8)-(4.12) and the fact Ψ_ε is convex for all $\varepsilon \in \mathcal{E}$ that

$$\begin{aligned}
&\|v_\varepsilon\|^2 - \frac{1}{2}\langle (A_\varepsilon + A_0)v_\varepsilon, v_\varepsilon\rangle \\
\leq &\frac{1}{2}\langle (A_\varepsilon - A_0)z_\varepsilon, z_\varepsilon\rangle - \frac{1}{2}\langle (A_\varepsilon - A_0)w, w\rangle + \Psi'_0(w)v_\varepsilon - \Psi'_\varepsilon(w)v_\varepsilon \\
= &\frac{1}{2}\langle (A_\varepsilon - A_0)v_\varepsilon, v_\varepsilon\rangle + \langle (A_\varepsilon - A_0)w, v_\varepsilon\rangle + \Psi'_0(w)v_\varepsilon - \Psi'_\varepsilon(w)v_\varepsilon.
\end{aligned}$$

This implies

$$\|v_\varepsilon\|^2 - \langle A_\varepsilon v_\varepsilon, v_\varepsilon\rangle \leq \langle (A_\varepsilon - A_0)w, v_\varepsilon\rangle + \langle \psi_0(w) - \psi_\varepsilon(w), v_\varepsilon\rangle,$$

and, from $(A2)$ and $(N2)$, we have

$$(1-\theta)\|v_\varepsilon\|^2 \leq o(1)\|v_\varepsilon\|,$$

and thereby the proof is completed. □

As a corollary of Lemma 4.3, we shall give a first relationship between I_ε for $\varepsilon > 0$ and I_0 that is:

Corollary 4.4. *Suppose* $(A1)$-$(A2)$ *and* $(N1)$-$(N3)$ *are satisfied.* $I_\varepsilon(v) \to I_0(v)$ *as* $\varepsilon \to 0$ *for* $v \in X$.

Proof. As in the proof of Lemma 4.3, we set $z_\varepsilon = v + h_\varepsilon(v)$, $w = v + h_0(v)$ and $v_\varepsilon = z_\varepsilon - w$ for $v \in X$.

Recall the definition of I_ε for $\varepsilon \in \mathcal{E}$, by virtue of Lemma 4.3, we only need to show that $\Psi_\varepsilon(z_\varepsilon) \to \Psi_0(w)$ as $\varepsilon \to 0$. This can be seen by the fact

$$\langle A_\varepsilon z_\varepsilon, z_\varepsilon\rangle = \langle A_0 w, w\rangle + \langle (A_\varepsilon - A_0)w, w\rangle + O(\|v_\varepsilon\|)$$

and $\|v_\varepsilon\| = o(1)$ as $\varepsilon \to 0$.

From (4.1), we have

$$\Psi_\varepsilon(z_\varepsilon) = \int_0^1 \Psi'_\varepsilon(tz_\varepsilon)z_\varepsilon\, dt = \int_0^1\int_0^t \Psi''_\varepsilon(sz_\varepsilon)[z_\varepsilon, z_\varepsilon]\, dsdt, \tag{4.13}$$

$$\Psi_0(w) = \int_0^1 \Psi_0'(tw)w\,dt = \int_0^1 \int_0^t \Psi_0''(sw)[w,w]\,dsdt. \tag{4.14}$$

By virtue of $(N3)$, we easily conclude the family $\{f_\varepsilon\}$, where

$$f_\varepsilon : [0,1] \to \mathbb{R}, \quad f_\varepsilon(t) = \Psi_\varepsilon'(tz_\varepsilon)z_\varepsilon,$$

is uniformly bounded and equicontinuous. Then, by Arzelà-Ascoli theorem, the family $\{f_\varepsilon\}$ is compact in $C[0,1]$. Notice that $z_\varepsilon \to w$ in E as $\varepsilon \to 0$, a standard argument shows $f_\varepsilon(t) \to f_0(t)$ pointwise on $[0,1]$ as $\varepsilon \to 0$. Hence we have f_ε converges to f_0 in the $C[0,1]$ topology as ε shrinks. This together with (4.13) and (4.14), we see that $\Psi_\varepsilon(z_\varepsilon)$ converges to $\Psi_0(w)$ as $\varepsilon \to 0$. $\square$

Next we shall give geometric structures of I_ε for $\varepsilon \neq 0$. Recall that we assumed

$$c_0 = \inf_{e \in X \setminus \{0\}} \sup_{z \in E_e} \Phi_0(z), \tag{4.15}$$

to be a critical value for Φ_0, then our result will be:

Proposition 4.5. *Under the assumptions of Theorem* 2.4*, for* $\varepsilon > 0$ *small enough,* I_ε *possesses the mountain-pass structure:*

(1) $I_\varepsilon(0) = 0$ *and there exist* $r > 0$ *and* $\tau > 0$ *(both independent of* ε*) such that* $I_\varepsilon|_{S_r^X} \geq \tau$.

(2) *there exists* $v_0 \in X$ *(independent of* ε*) such that* $\|v_0\| > r$ *and* $I_\varepsilon(v_0) < 0$.

Moreover,

$$c_\varepsilon' = \inf_{\nu \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon\big(\nu(t)\big) \tag{4.16}$$

is a critical value for I_ε*, where*

$$\Gamma_\varepsilon = \big\{\nu \in C([0,1],X) : \nu(0) = 0,\ I_\varepsilon\big(\nu(1)\big) < 0\big\}.$$

Before giving the proof of Proposition 4.5, we shall give some equivalent characterizations of the critical value of Φ_0 defined in (4.15) which are essential in our proofs. Set

$$c_0' = \inf_{\nu \in \Gamma_0} \max_{t \in [0,1]} I_0\big(\nu(t)\big),$$

and

$$c_0'' = \inf_{e \in X \setminus \{0\}} \sup_{t \geq 0} I_0(te),$$

where $\Gamma_0 := \big\{\nu \in C([0,1],X) : \nu(0) = 0,\ I_0(\nu(1)) < 0\big\}$.

Lemma 4.6. *Suppose* $(A1)$-$(A2)$*,* $(N1)$-$(N5)$ *and* $(I1)$-$(I2)$ *are satisfied. If* $c_0 < \infty$ *is a critical value of* Φ_0*, then* $c_0 = c_0' = c_0''$.

Proof. Obviously, by $(I2)$ and $c_0 < \infty$, it follows from the definition of I_0 that $c_0' \le c_0'' \le c_0$. So, in what follows, we shall prove $c_0 \le c_0'$.

Claim 1. *If* $v \in X \setminus \{0\}$ *satisfies* $I_0'(v)v = 0$, *then* $I_0''(v)[v, v] < 0$.

In order to verify Claim 1, we fist do some basic calculations. Recall that $h_0(v)$ is the unique critical point of ϕ_v on Y for $v \in X$, we have

$$-\langle h_0(v), y\rangle + \big\langle A_0\big(v + h_0(v)\big), y\big\rangle - \Psi_0'\big(v + h_0(v)\big)y = 0 \quad \forall y \in Y. \tag{4.17}$$

Set $z = v + h_0(v)$ and $w = h_0'(v)v - h_0(v)$, then

$$\begin{aligned} I_0'(v)v &= \|v\|^2 - \big\langle h_0(v), h_0'(v)v\big\rangle + \big\langle A_0\big(v + h_0(v)\big), v + h_0'(v)v\big\rangle \\ &\quad - \Psi_0'\big(v + h_0(v)\big)\big(v + h_0'(v)v\big) \\ &= \|v\|^2 + \big\langle A_0\big(v + h_0(v)\big), v\big\rangle - \Psi_0'\big(v + h_0(v)\big)v \\ &= \|v\|^2 - \langle h_0(v), z^Y + y\rangle + \big\langle A_0\big(v + h_0(v)\big), z + y\big\rangle \\ &\quad - \Psi_0'\big(v + h_0(v)\big)(z + y) \\ &= \Phi_0'(z)(z + y) \end{aligned} \tag{4.18}$$

for all $y \in Y$. Since (4.17) is valid for all $v \in X$, by taking derivative with respect to v, we deduce

$$\begin{aligned} 0 \equiv &- \big\langle -h_0'(v)v, y\big\rangle + \big\langle A_0\big(v + h_0'(v)v\big), y\big\rangle \\ &- \Psi_0''\big(v + h_0(v)\big)\big[\big(v + h_0'(v)v\big), y\big] \end{aligned} \tag{4.19}$$

for all $y \in Y$. So, choose $y = z^Y + w = h_0'(v)v$ in (4.19), we infer

$$\begin{aligned} I_0''(v)[v, v] &= \|v\|^2 + \langle A_0(z + w), v\rangle - \Psi_0''(z)[z + w, v] \\ &= \|v\|^2 - \|z^Y + w\|^2 - \langle A_0(z + w), z + w\rangle \\ &\quad - \Psi_0''(z)[z + w, z + w] \\ &= \Phi_0''(z)[z + w, z + w]. \end{aligned}$$

Taking into account that $\Phi_0'(z)z = I_0'(v)v = 0$ (which follows from (4.18)), we can conclude

$$\begin{aligned} I_0''(v)[v, v] &= \Phi_0''(z)[z + w, z + w] \\ &= \Phi_0''(z)[z, z] + 2\Phi_0''(z)[z, w] + \Phi_0''(z)[w, w] \\ &= \|z^X\|^2 - \|z^Y\|^2 + \langle A_0 z, z\rangle - \Psi_0''(z)[z, z] \\ &\quad + 2\big(-\langle z^Y, w\rangle + \langle A_0 z, w\rangle - \Psi_0''(z)[z, w]\big) \\ &\quad + \big(-\|w\|^2 + \langle A_0 w, w\rangle - \Psi_0''(z)[w, w]\big) \\ &= \big(\Psi_0'(z)z - \Psi_0''(z)[z, z]\big) + 2\big(\Psi_0'(z)w - \Psi_0''(z)[z, w]\big) \\ &\quad - \Psi_0''(z)[w, w] - \|w\|^2 + \langle A_0 w, w\rangle \\ &< 0 \end{aligned}$$

due to $(N5)$ and $z \neq 0$.

Let $v \in X \setminus \{0\}$, we find the function $t \mapsto I_0(tv)$ has at most one nontrivial critical point $t = t(v) > 0$ which (if exists) will be the maxima point. Denoted by

$$\mathscr{M} = \{t(v)v : v \in X \setminus \{0\},\ t(v) < +\infty\},$$

we have $\mathscr{M} \neq \emptyset$ since c_0 is a critical value of Φ_0. We also observe that

$$c_0'' = \inf_{z \in \mathscr{M}} I_0(z),$$

moreover, by (I_2) and $\mathscr{M} \neq \emptyset$, we infer $\Gamma_0 \neq \emptyset$.

Claim 2. $c_0'' = c_0$.

Let $e \in \mathscr{M}$, then $\Phi_0'\big(e + h_0(e)\big)\big|_{E_e} = 0$ (recall $E_e = \mathbb{R}^+ e \oplus Y$). Hence $c_0 \le \max_{z \in E_e} \Phi_0(z) = I_0(e)$, which implies $c_0 \le c_0''$.

Claim 3. $c_0'' \le c_0'$.

We only need to show that given $\nu \in \Gamma_0$ there exists $\bar{t} \in [0,1]$ such that $\nu(\bar{t}) \in \mathscr{M}$. Assuming contrarily we have $\nu([0,1]) \cap \mathscr{M} = \emptyset$. As a result of $(I1)$,

$$I_0'(\nu(t))\nu(t) > 0 \quad \text{for } t > 0 \text{ small.}$$

Since the function $t \mapsto I_0'(\nu(t))\nu(t)$ is continuous and $I_0'(\nu(t))\nu(t) \neq 0$ for all $t \in (0,1]$, we have

$$I_0'(\nu(t))\nu(t) > 0 \quad \text{for all } t \in [0,1].$$

Then we find by $(N4)$

$$\begin{aligned} I_0(\nu(t)) &= \frac{1}{2} I_0(\nu(t))\nu(t) + \widehat{\Psi}_0\big(\nu(t) + h_0(\nu(t))\big) \\ &\ge \frac{1}{2} I_0(\nu(t))\nu(t) > 0 \end{aligned}$$

for all $t \in (0,1]$ which is absurd.

Combing Claim 1, Claim 2 and Claim 3, we have the assertion proved. $\square$

Proof of Proposition 4.5. Since we have $I_\varepsilon(v) \ge \Phi_\varepsilon(v)$ for all $v \in X$, (1) follows easily from (I_1).

To check (2), let $w = w^X + w^Y \in E = X \oplus Y$ be a critical point of Φ_0 with $\Phi_0(w) = c_0$. Then, by virtue of Proposition 4.1, we have $w^Y = h_0\big(w^X\big)$. A direct consequence of Lemma 4.6 is

$$c_0 = I_0\big(w^X\big) = \max_{t \ge 0} I_0\big(tw^X\big),$$

and, by $(I2)$, we can conclude there exists $t_0 > 0$ (large enough) such that

$$I_0\big(t_0 w^X\big) < -1.$$

As a result of Corollary 4.4, we have

$$
\begin{aligned}
I_\varepsilon\big(t_0 w^X\big) &= I_0\big(t_0 w^X\big) + o(1) \\
&\le -\frac{1}{2} + o(1)
\end{aligned}
$$

as $\varepsilon \to 0$. Therefore, there is $\varepsilon_0 > 0$ such that $I_\varepsilon\big(t_0 w^X\big) < 0$ for all $\varepsilon \in (0, \varepsilon_0]$.

As a consequence of the mountain-pass structure, we obtain a $(C)_{c'_\varepsilon}$-sequence for I_ε denoted by $\{v_\varepsilon^n\}_{n=1}^\infty$. By Proposition 4.1 and $\mathscr{G}$-weak $(C)_{c'_\varepsilon}$-condition for Φ_ε, we conclude that there exists $v_\varepsilon \neq 0$ such that $I'_\varepsilon(v_\varepsilon) = 0$. Moreover, from $(N4)$, we have

$$
\begin{aligned}
c'_\varepsilon &= \lim_{n\to\infty} \Big(I_\varepsilon(v_\varepsilon^n) - \frac{1}{2} I'_\varepsilon(v_\varepsilon^n) v_\varepsilon^n \Big) \\
&= \lim_{n\to\infty} \Big(\Phi_\varepsilon\big(v_\varepsilon^n + h_\varepsilon(v_\varepsilon^n)\big) - \frac{1}{2} \Phi'_\varepsilon\big(v_\varepsilon^n + h_\varepsilon(v_\varepsilon^n)\big)\big(v_\varepsilon^n + h_\varepsilon(v_\varepsilon^n)\big) \Big) \\
&= \lim_{n\to\infty} \Big(\frac{1}{2} \Psi'_\varepsilon\big(v_\varepsilon^n + h_\varepsilon(v_\varepsilon^n)\big)\big(v_\varepsilon^n + h_\varepsilon(v_\varepsilon^n)\big) - \Psi_\varepsilon\big(v_\varepsilon^n + h_\varepsilon(v_\varepsilon^n)\big) \Big) \\
&= \lim_{n\to\infty} \widehat{\Psi}_\varepsilon\big(v_\varepsilon^n + h_\varepsilon(v_\varepsilon^n)\big) \ge \widehat{\Psi}_\varepsilon\big(v_\varepsilon + h_\varepsilon(v_\varepsilon)\big) \\
&= I_\varepsilon(v_\varepsilon) - \frac{1}{2} I'_\varepsilon(v_\varepsilon) v_\varepsilon = I_\varepsilon(v_\varepsilon) > 0.
\end{aligned}
\tag{4.20}
$$

Now set

$$
c''_\varepsilon := \inf_{e \in X\setminus\{0\}} \sup_{t\ge 0} I_\varepsilon(te),
$$

and recall we have already defined in Theorem 2.4 that

$$
c_\varepsilon := \inf_{e\in X\setminus\{0\}} \sup_{z\in E_e} \Phi_\varepsilon(z).
$$

Let us repeat the proof of Claim 1 in Lemma 4.6, from which we can conclude: let $e \in X \setminus \{0\}$, the function $t \mapsto I_\varepsilon(te)$ has at most one nontrivial critical point $t = t(e) > 0$ which (if exists) will be the maximum point.

Noting that (4.20) and $v_\varepsilon \in X \setminus \{0\}$ is a critical point of I_ε implies

$$
c''_\varepsilon \le \sup_{t\ge 0} I_\varepsilon(tv_\varepsilon) = I_\varepsilon(v_\varepsilon) \le c'_\varepsilon < \infty.
$$

And on the other hand, it is not difficult to check $c'_\varepsilon \le c''_\varepsilon$. Hence we have $c'_\varepsilon = c''_\varepsilon$. Meanwhile, $c_\varepsilon = c''_\varepsilon$ is much obvious since the definition of h_ε implies

$$
I_\varepsilon(te) = \Phi_\varepsilon\big(te + h_\varepsilon(te)\big) = \max_{w\in Y} \Phi_\varepsilon(te + w)
$$

and therefore

$$
\sup_{t\ge 0} I_\varepsilon(te) = \sup_{t\ge 0} \max_{w\in Y} \Phi_\varepsilon(te + w) = \sup_{z\in Ee} \Phi_\varepsilon(z).
$$

By taking infimum with respect to $e \in X \setminus \{0\}$, we have $c_\varepsilon = c'_\varepsilon = c''_\varepsilon$.

Due to the above observation, if we have proved

$$I_\varepsilon(v_\varepsilon) \geq c''_\varepsilon, \tag{4.21}$$

then we can conclude $I_\varepsilon(v_\varepsilon) = c'_\varepsilon$ immediately from (4.20).

In fact, let us set

$$\mathscr{M}_\varepsilon := \big\{ t(v)v : \ v \in X \setminus \{0\}, \ 0 < t(v) < \infty \text{ such that } I'_\varepsilon(t(v)v)v = 0 \big\},$$

we infer that

$$c''_\varepsilon = \inf_{z \in \mathscr{M}_\varepsilon} I_\varepsilon(z).$$

Since $v_\varepsilon \in \mathscr{M}_\varepsilon$, (4.21) is obviously valid. And the proof is thereby completed. $\square$

As a by-product of the proof of Proposition 4.5 we have

Lemma 4.7. *Let $\varepsilon \in (0, \varepsilon_0]$ be such that Proposition* 4.5 *is valid. Then $c_\varepsilon = c'_\varepsilon = c''_\varepsilon$ characterize the ground state energy of Φ_ε.*

To complete our proof of Theorem 2.4, in what follows, we need to show the asymptotic behaviour of the critical values found in Proposition 4.5.

Lemma 4.8. *Let $\varepsilon \in (0, \varepsilon_0]$ be such that Proposition* 4.5 *is valid. Then $c_\varepsilon \leq c_0 + o(1)$ as $\varepsilon \to 0$.*

Proof. Again, Let $w = w^X + w^Y \in E = X \oplus Y$ be the critical point of Φ_0 such that $\Phi_0(w) = I_0\big(w^X\big) = c_0$. Set $t_0 > 0$ such that $I_0\big(t_0 w^X\big) \leq -1$. By virtue of Lemma 4.6 and Lemma 4.7, it sufficient to prove

$$I_\varepsilon(tw^X) = I_0(tw^X) + o(1) \quad \text{uniformly in } t \in [0, t_0] \tag{4.22}$$

as $\varepsilon \to 0$.

To this end, we only need to show the family $\{f_\varepsilon\} \subset C[0, t_0]$

$$f_\varepsilon(t) := I_\varepsilon(tw^X)$$

is uniformly bounded and equicontinuous. This can be seen from Corollary 4.4 that if $\{f_\varepsilon\}$ is compact in the $C[0, t_0]$ topology then (4.22) is valid.

Clearly, $f_\varepsilon \in C^1$ and the uniformly boundedness of $\{f_\varepsilon\}$ and $\{f'_\varepsilon\}$ on $[0, t_0]$ comes easily from $(A1)$-$(A2)$ and $(N3)$. So, by Arzelà-Ascoli theorem, we have $\{f_\varepsilon\}$ is compact in the $C[0, t_0]$. And therefore, by (4.22), we conclude

$$\begin{aligned} c_\varepsilon &\leq \sup_{t \geq 0} I_\varepsilon(tw^X) = \sup_{t \in [0, t_0]} I_\varepsilon(tw^X) = \sup_{t \in [0, t_0]} I_0(tw^X) + o(1) \\ &= \sup_{t \geq 0} I_0(tw^X) + o(1) = I_0(w^X) + o(1) \\ &= c_0 + o(1) \end{aligned}$$

as $\varepsilon \to 0$. $\square$

Now, combining Proposition 4.5, Lemma 4.7 and Lemma 4.8 we summarize the following result which together with Proposition 4.1 gives the complete proof of Theorem 2.4.

Proposition 4.9. *Under the assumptions of Theorem* 2.4*, for* $\varepsilon > 0$ *small,* I_ε *possesses a nontrivial critical value which can be characterized by*

$$c_\varepsilon = \inf_{e \in X \setminus \{0\}} \sup_{t \geq 0} I_\varepsilon(te).$$

Moreover, $c_\varepsilon \leq c_0 + o(1)$ *as* $\varepsilon \to 0$.

A Appendix

We devote this appendix to some embedding results of t-Anisotropic Sobolev spaces and regularity results that were used in the text. For the following embedding theorem, we refer the readers to [29, Theorem 1.4.1].

Definition A.1. A domain $\Omega \subset \mathbb{R}^N$ is said to have the property of uniform inner cone, if there is a finite cone C such that every point $x \in \Omega$ is the vertex of a finite cone C_x congruent with C.

We remark that C_x need not to be obtained from C by parallel translation, but simply by rigid motion.

Before stating the embedding theorem, for given $T_1 < T_2$, $1 \leq r < \infty$ and $\Omega \subset \mathbb{R}^N$ we define $Q := (T_1, T_2) \times \Omega$ and

$$B^r(Q) := W^{1,r}\big((T_1,T_2), L^r(\Omega)\big) \cap L^r\big((T_1,T_2), W^{2,r}(\Omega)\big)$$

endowed with the usual norm

$$\|u\|_{B^r(Q)} = \bigg(\iint_Q \Big(|u|^r + |\partial_t u|^r + |\nabla u|^r + \sum_{1 \leq i,j \leq N} |\partial^2_{ij} u|^r \Big)\, dxdt \bigg)^{1/r}.$$

By $C^{\alpha,\alpha/2}(\overline{Q})$, $0 < \alpha < 1$, we mean the space of all the functions on Q such that

$$\|u\|_{C^{\alpha,\alpha/2}(\overline{Q})} := \sup_{(t,x) \in Q} |u(t,x)| + \sup_{\substack{(t_1,x_1),(t_2,x_2) \in Q \\ (t_1,x_1) \neq (t_2,x_2)}} \frac{|u(t_1,x_1) - u(t_2,x_2)|}{d^\alpha\big((t_1,x_1),(t_2,x_2)\big)} < \infty,$$

where $d(\cdot,\cdot)$ is the parabolic distance on $\mathbb{R} \times \mathbb{R}^N$ defined by

$$d\big((t_1,x_1),(t_2,x_2)\big) = \max\big\{|x_1 - x_2|, |t_1 - t_2|^{1/2}\big\}.$$

Theorem A.2 (*t-Anisotropic Embedding Theorem*)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and* $1 \leq r < \infty$.

(i) *If Ω has the property of uniform inner cone, then, when $r = (N+2)/2$,*

$$B^r(Q) \hookrightarrow L^q(Q), \quad 1 \le q < \infty$$

and for any $u \in B^r(Q)$

$$\|u\|_{L^q(Q)} \le C(N, q, Q)\|u\|_{B^r(Q)}, \quad 1 \le q < \infty;$$

when $r < (N+2)/2$,

$$B^r(Q) \hookrightarrow L^q(Q), \quad 1 \le q \le \frac{(N+2)r}{N+2-2r}$$

and for any $u \in B^r(Q)$

$$\|u\|_{L^q(Q)} \le C(N, r, Q)\|u\|_{B^r(Q)}, \quad 1 \le q \le \frac{(N+2)r}{N+2-2r}.$$

(ii) *If $\partial\Omega$ is appropriately smooth, then, when $r > (N+2)/2$,*

$$B^r(Q) \hookrightarrow C^{\alpha,\alpha/2}(\overline{Q}), \quad 0 < \alpha \le 2 - \frac{N+2}{r}$$

and for any $u \in B^r(Q)$

$$\|u\|_{C^{\alpha,\alpha/2}(\overline{Q})} \le C(N, r, Q)\|u\|_{B^r(Q)}, \quad 0 < \alpha \le 2 - \frac{N+2}{r}.$$

Next we recall a regularity result which can be found in [17]. For this purpose we set $B_\rho := \{x \in \mathbb{R}^N : |x| < \rho\}$ for any $\rho > 0$.

Theorem A.3 (*Parabolic interior estimates*)**.** *Let $1 < r < \infty$, $\rho > 0$ and set $Q_\rho = (-\rho^2, 0] \times B_\rho$. If $u \in L^r(Q_\rho)$ is a (weak) solution to*

$$\partial_t u - \Delta u + u = f \quad \text{in } Q_\rho$$

with $f \in L^r(Q_\rho)$. Then, for any $0 < \sigma < \rho$,

$$\|u\|_{B^r(Q_{\sigma,\rho})} \le C(N, \rho, \sigma) \cdot \big(\|f\|_{L^r(Q_\rho)} + \|u\|_{L^r(Q_\rho)}\big),$$

where $Q_{\sigma,\rho} := \big(-(\rho-\sigma)^2, 0\big] \times B_{\rho-\sigma}$.

Together with the embedding theorem, we have the following consequence.

Corollary A.4. *Let $\frac{N+2}{2} < r < \infty, \rho > 0$ and set $Q_\rho = (-\rho^2, 0] \times B_\rho$. If $u \in L^r(Q_\rho)$ is a (weak) solution to*

$$\partial_t u - \Delta u + u = f \quad \text{in } Q_\rho$$

with $f \in L^r(Q_\rho)$. Then, for any $0 < \sigma < \rho$,

$$\|u\|_{C^{\alpha,\alpha/2}(\overline{Q_{\sigma,\rho}})} \le C(N, r, \rho, \sigma) \cdot \big(\|f\|_{L^r(Q_\rho)} + \|u\|_{L^r(Q_\rho)}\big),$$

where $0 < \alpha \le 2 - \frac{N+2}{r}$.

Next, recall

$$B^r = W^{1,r}\big(\mathbb{R}, L^r(\mathbb{R}^N, \mathbb{R}^{2M})\big) \cap L^r\big(\mathbb{R}, W^{2,r}(\mathbb{R}^N, \mathbb{R}^{2M})\big) \quad \text{for } r \geq 1$$

denotes the Banach space equipped with the norm $\|\cdot\|_{B^r}$ defined in (1.6) and $L^r := L^r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2M})$ is equipped with the usual L^r norm. The operator L is defined by $L = \mathcal{J}\partial_t + A$ in Section 3. In order that the above mentioned regularity results are applicable to the present text, we shall give the following fundamental result in the study of the system in the form of (3.1). Recall $E := \mathscr{D}(|L|^{1/2})$ is the Hilbert space equipped with the norm $\|\cdot\|$ where $L = \mathcal{J}\partial_t + A$. Denote $\mathscr{M}_{2M\times 2M}(\mathbb{R})$ by the space of all $2M \times 2M$ real matrixes equipped with the usual vector norm.

Lemma A.5. *Let $M \in L^\infty\big(\mathbb{R}\times\mathbb{R}^N, \mathscr{M}_{2M\times 2M}(\mathbb{R})\big)$ and $H : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2M} \to \mathbb{R}$ satisfy*

$$|\nabla_z H(t,x,z)| \leq |z| + c|z|^{p-1} \tag{A.1}$$

for some $c > 0$ and $p \in \big(2, 2(N+2)/N\big)$. If $z \in E$ is a weak solution to

$$Lz + M(t,x)z = \nabla_z H(t,x,z), \tag{A.2}$$

then $z \in B^r$ for all $r \geq 2$ and

$$\|z\|_{B^r} \leq C\big(\|M\|_\infty, \|z\|, c, p, r\big).$$

Proof. Since the proof is quite similar to the proof of Lemma 8.6 on page 149 in [12], we just give a sketch here. Remark that, from [4], we have the following embedding result:

$$B^r \hookrightarrow L^q \text{ is continuous for } r > 1 \text{ and } 0 \leq \frac{1}{r} - \frac{1}{q} \leq \frac{2}{N+2}. \tag{A.3}$$

Set

$$\varphi(r) := \begin{cases} (N+2)r/(N+2-2r) & \text{if } 1 < r < \frac{N+2}{2}, \\ \infty & \text{if } r \geq \frac{N+2}{2}. \end{cases}$$

Then $B^r \hookrightarrow L^q$ for $1 < r \leq q < \varphi(r)$ and also for $q = \varphi(r)$ if $\varphi(r) < \infty$.

Now let $z \in E$ be a weak solution of (A.2) and set $w = -M(t,x)z + \nabla_z H(t,x,z)$. We rewrite (A.2) as

$$z = L^{-1}w = L^{-1}\big(-M(t,x)z + \nabla_z H(t,x,z)\big).$$

Define $\chi_z : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ by

$$\chi_z(t,x) = \begin{cases} 1 & \text{if } |z(t,x)| < 1, \\ 0 & \text{if } |z(t,x)| \geq 1. \end{cases}$$

and let

$$w_1(t,x) = -M(t,x)z + \nabla_z H\big(t,x,\chi_z(t,x)\cdot z(t,x)\big)$$

and

$$w_2(t,x) = \nabla_z H\big(t,x,(1-\chi_z(t,x))\cdot z(t,x)\big).$$

Then $w = w_1 + w_2$ and it follows from the assumptions on M and H that

$$|w_1(t,x)| \le C_1|z(t,x)|$$

with C_1 depending on $\|M\|_\infty$ and

$$|w_2(t,x)| \le C_2|z(t,x)|^{p-1}$$

with C_2 depending on the constant c in (A.1). Since E embeds continuously into L^q for $q \in [2, r_1]$ where $r_1 = 2(N+2)/N$, we have $w_1 \in L^r$ for $r \in [2, r_1]$ and $w_2 \in L^r$ for $r \in [1, q_1]$ where $q_1 = r_1/(p-1)$. Here we used that

$$\textit{meas}\big\{(t,x) \in \mathbb{R}\times\mathbb{R}^N : |z(t,x)| \ge 1\big\} \le \iint_{\mathbb{R}\times\mathbb{R}^N} |z|^2 \le \|z\|^2 < \infty.$$

Now use the fact $L : B^r \to L^r$ is an isomorphism for $r > 1$, we obtain

$$\begin{cases} z_1 := L^{-1}w_1 \in B^r \text{ for } r \in [2, r_1], \\ z_2 := L^{-1}w_2 \in B^r \text{ for } r \in [1, q_1]. \end{cases}$$

Case 1. Let us consider $q_1 \ge \frac{(N+2)}{2}$ which may occur only when $N \le 3$.

In this situation, we have $z_2 \in L^q$ for all $q \ge q_1$ as a consequence of (A.3). By interpolation we get $z_2 \in L^q$ for all $q \ge 2$. Noting that $r_1 > q_1$, we similarly obtain $z_1 \in L^q$ for all $q \ge 2$.

Case 2. Let us consider $q_1 < \frac{N+2}{2}$. In this case we define inductively $r_{k+1} = \varphi(q_k)$ and $q_{k+1} = r_{k+1}/(p-1)$. Suppose $z_1 \in B^r$ for $r \in [2, r_k]$ and $z_2 \in B^r$ for $r \in [2, q_k]$. Then we deduce that $z_1 \in L^r$ for $r \in [2, \varphi(r_k)]$ and $z_2 \in L^r$ for $r \in [2, \varphi(q_k)]$. So $z = z_1 + z_2 \in L^r$ for $r \in [2, r_{k+1}]$ since $\varphi(r_k) > r_{k+1} = \varphi(q_k)$. We claim that there exists $k_0 \ge 1$ such that $q_{k_0} \ge (N+2)/2$. Then we can go back to *Case* 1 and obtain $z \in L^q$ for all $q \ge 2$.

In order to proof the claim, by induction, we observe that

$$r_k = \frac{2(N+2)(p-2)}{(p-1)^{k-1}\big(N(p-2)-4\big)+4}.$$

Since $2 < p < 2(N+2)/N = 2 + 4/N$, we see that there exists $k_0 > 1$ such that $r_{k_0} > 0$ and either $r_{k_0+1} = \infty$ or $r_{k_0+1} < 0$. This implies that $q_{k_0} \ge (N+2)/2$ as required. □

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