Proof of a Positivity Conjecture on Schur Functions

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Abstract. In the study of Zeilberger's conjecture on an integer sequence related to the Catalan numbers, Lassalle proposed the following conjecture. Let $(t)_n$ denote the rising factorial, and let $\Lambda_{\mathbb{R}}$ denote the algebra of symmetric functions with real coefficients. If φ is the homomorphism from $\Lambda_{\mathbb{R}}$ to $\mathbb{R}$ defined by $\varphi(h_n) = 1/((t)_n n!)$ for some $t > 0$, then for any Schur function s_λ, the value $\varphi(s_\lambda)$ is positive. In this paper, we provide an affirmative answer to Lassalle's conjecture by using the Laguerre–Pólya-Schur theory of multiplier sequences.

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1 Introduction

The objective of this paper is to prove a positivity conjecture on Schur functions, which was proposed by Lassalle [6] in the study of two combinatorial sequences related to the Catalan numbers.

Let us begin with an overview of Lassalle's conjecture. Let

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

denote the n-th Catalan number. Lassalle [6] introduced a sequence of numbers A_n for $n \geq 1$, which are recursively defined by

$$(-1)^{n-1}A_n = C_n + \sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-1}A_jC_{n-j},$$

with the initial value $A_1 = 1$. He proved that the sequence $\{A_n\}_{n\geq 2}$ is positive and increasing. Josuat-Vergès [4] found a combinatorial interpretation of A_n in terms of connected matchings in the study of cumulants of the q-semicircular law. Zeilberger further conjectured that the numbers $\{2A_n/C_n\}_{n\geq 2}$ also form an increasing sequence of positive integers. Lassalle [6] proved Zeilberger's conjecture. An alternative proof was given by Amdeberhan, Moll and Vignat [1] using a probabilistic approach.

By using the theory of symmetric functions, Lassalle [6] gave a direct proof of the positivity and the monotonicity of $\{A_n\}_{n\geq 2}$, although these two properties can be deduced from Zeilberger's conjecture. For the notation and terminology on symmetric functions, see Macdonald [8] or Stanley [9]. Lassalle's proof involves the following specialization of symmetric functions. Let $\mathbb{R}$ be the field of real numbers, and let $\Lambda_{\mathbb{R}}$ be the algebra of symmetric functions with real coefficients. It is well known that the complete symmetric functions h_n ($n \geq 0$) are algebraically independent and $\Lambda_{\mathbb{R}}$ is generated by h_n. Thus any homomorphism φ from $\Lambda_{\mathbb{R}}$ to $\mathbb{R}$ is uniquely determined by the values $\varphi(h_n)$. Lassalle's specialization is given by

$$\varphi(h_n) = \frac{1}{((t)_n n!)}, \tag{1.1}$$

where $t > 0$ and $(t)_n = t(t+1)\cdots(t+n-1)$. Lassalle proved that this specialization satisfies

$$\varphi((-1)^{n-1}p_n) > 0 \qquad \text{and} \qquad \varphi(e_n) > 0,$$

where p_n and e_n denote the n-th power sum and the n-th elementary symmetric function respectively. As shown in [6], the numbers A_n are equal to $\varphi\left((-1)^{n-1}2(2n-1)!p_n\right)$ when $t = 2$.

Note that both h_n and e_n are special cases of the Schur functions. Based on the positivity of $\varphi(h_n)$ and $\varphi(e_n)$, Lassalle further considered the specialization of a general Schur function s_λ indexed by an integer partition λ. Lassalle [6] posed the following conjecture.

Conjecture 1.1 *Let* $\varphi\colon \Lambda_{\mathbb{R}} \to \mathbb{R}$ *be the specialization of* h_n *given by* (1.1). *Then* $\varphi(s_\lambda)$ *is positive for any Schur function* s_λ.

In this paper, we give an affirmative answer to Conjecture 1.1. Our proof relies on the theory of total positivity and the theory of multiplier sequences.

2 Preliminaries

In this section, we give an overview of some fundamental results on the theory of total positivity and the theory of multiplier sequences. A real sequence

$\{a_n\}_{n\geq 0}$ is said to be a totally positive sequence if all the minors of the infinite Toeplitz matrix $(a_{j-i})_{i,j\geq 1}$ are nonnegative, where we set $a_n = 0$ for $n < 0$. The following representation theorem was conjectured by Schoenberg and proved by Edrei [3], see also Macdonald [8].

Theorem 2.1 ([8, p. 98]) *Let $\{a_n\}_{n\geq 0}$ be a sequence of real numbers with $a_0 = 1$. Then $\{a_n\}_{n\geq 0}$ is totally positive if and only if its generating function*

$$f(x) = \sum_{n\geq 0} a_n x^n$$

has the form

$$\exp(\theta x)\frac{\prod_{i\geq 1}(1+\rho_i x)}{\prod_{i\geq 1}(1-\delta_i x)}, \tag{2.1}$$

where $\theta \geq 0, \rho_i \geq 0, \delta_i \geq 0$ for $i \geq 1$ and $\sum_{i\geq 1}(\rho_i + \delta_i) < \infty$.

Based on the above theorem, Karlin gave a necessary and sufficient condition to determine the strict positivity of a minor of the Toeplitz matrix $(a_{j-i})_{i,j\geq 1}$.

Theorem 2.2 ([5, p. 428]) *Suppose that $\{a_n\}_{n\geq 0}$ is a totally positive sequence. Let θ, δ_i, ρ_i be defined as in (2.1). Let K be the number of positive entries δ_i and let L be the number of positive entries ρ_i, where K and L are allowed to be infinity. Let $I = (i_1, i_2, \ldots, i_r)$ and $J = (j_1, j_2, \ldots, j_r)$ be two increasing sequences of positive numbers. Let $T(I, J)$ be the minor of $(a_{j-i})_{i,j\geq 1}$ with the row indices $i_1, i_2, \ldots, i_r$ and column indices $j_1, j_2, \ldots, j_r$. Then the following assertions hold:*

(i) For $\theta > 0$, the minor $T(I, J)$ is positive if and only if $i_k \leq j_k$ for $1 \leq k \leq r$;

(ii) For $\theta = 0$ and $K > 0$, the minor $T(I, J)$ is positive if and only if

$$j_{k-K} - L < i_k \leq j_k$$

for $1 \leq k \leq r$.

(iii) For $\theta = 0$ and $K = 0$, the minor $T(I, J)$ is positive if and only if

$$j_k - L \leq i_k \leq j_k$$

for $1 \leq k \leq r$.

As pointed out by Craven and Csordas [2], Theorem 2.1 is closely related to Pólya and Schur's transcendental characterization of multiplier sequences. A multiplier sequence is defined to be a sequence $\{\gamma_n\}_{n\geq 0}$ of real numbers such that, whenever the polynomial with real coefficients

$$\sum_{n=0}^{m} a_n x^n$$

has only real zeros, the polynomial

$$\sum_{n=0}^{m} \gamma_n a_n x^n$$

also has only real zeros. Pólya and Schur obtained the following transcendental characterization of multiplier sequences consisting of nonnegative numbers, see also Levin [7].

Theorem 2.3 ([7, p. 346]) *A sequence $\{\gamma_n\}_{n\geq 0}$ of nonnegative numbers with $\gamma_0 = 1$ is a multiplier sequence if and only if*

$$f(x) = \sum_{n\geq 0} \frac{\gamma_n}{n!} x^n$$

is of the form

$$\exp(\theta x) \prod_{i\geq 1} (1 + \rho_i x), \tag{2.2}$$

where $\theta \geq 0, \rho_i \geq 0$ for $i \geq 1$ and $\sum_{i\geq 1} \rho_i < \infty$.

To prove Lassalle's conjecture, we shall use a classic result of Laguerre on multiplier sequences, see also Levin [7].

Theorem 2.4 ([7, p. 341]) *For any $t > 0$, the sequence $\{1/(t)_n\}_{n\geq 0}$ is a multiplier sequence.*

3 Proof of Lassalle's conjecture

Before proving Conjecture 1.1, let us recall the Jacobi–Trudi identity for Schur functions, which relates Lassalle's conjecture to the theory of total positivity. Note that an integer partition λ is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of nonnegative integers. The Jacobi–Trudi identity states that a Schur function s_λ can be expressed in terms of a determinant of complete symmetric functions:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1}^{\ell}, \tag{3.1}$$

where h_k is defined to be zero if $k < 0$.

Proof of Conjecture 1.1. By Theorems 2.3 and 2.4, the generating function

$$f(x) = \sum_{n\geq 0} \frac{1}{(t)_n n!} x^n$$

is entire and has the form (2.2). Further, by Theorem 2.1, the sequence $\{1/((t)_n n!)\}_{n\geq 0}$ is totally positive. Let $T = (T_{i,j})_{i,j\geq 1}$ be the Toeplitz matrix corresponding to the sequence $\{1/((t)_n n!)\}_{n\geq 0}$, namely

$$T_{i,j} = \begin{cases} \frac{1}{(t)_{j-i}(j-i)!}, & \text{if } i \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

The Jacobi–Trudi identity shows that every $\varphi(s_\lambda)$ occurs as a minor $T(I, J)$ of T with row index set I and column index set J, where

$$\begin{aligned} I &= (1, 2, \ldots, \ell), \\ J &= (\lambda_\ell + 1, \lambda_{\ell-1} + 2, \ldots, \lambda_1 + \ell). \end{aligned}$$

Thus, $\varphi(s_\lambda) = T(I, J)$ is nonnegative.

To prove the strict positivity of $T(I, J)$, we need to consider the values of the parameters K, L and θ which appear in Theorem 2.2 for the sequence $\{1/((t)_n n!)\}_{n\geq 0}$. Since the generating function $f(x)$ is of the form (2.2), we see that $K = 0$ and $\theta \geq 0$.

While it can be shown that $\theta = 0$, we may avoid the computation by dealing with both cases with the aid of Karlin's criterion for the strict positivity of a minor of the Toeplitz matrix. In fact, if $\theta > 0$, by using (i) of Theorem 2.2, we infer that $T(I, J) > 0$, since, for $1 \leq k \leq \ell$,

$$i_k = k \leq \lambda_{\ell+1-k} + k = j_k.$$

If $\theta = 0$, then we have $L = \infty$, since $f(x)$ is not a polynomial. By (iii) of Theorem 2.2, we have $T(I, J) > 0$, since the condition

$$j_k - L \leq i_k \leq j_k$$

is satisfied for $1 \leq k \leq \ell$. In either case, we have $T(I, J) > 0$, and hence we conclude that $\varphi(s_{\lambda/\mu}) > 0$. This completes the proof. ∎

As suggested by a referee, we give a derivation of the fact that $\theta = 0$ in the above proof. Let ϱ be the order of $f(x)$, that is,

$$\varrho = \overline{\lim_{k\to\infty}} \frac{k \ln k}{\ln \frac{1}{|a_k|}} = \overline{\lim_{k\to\infty}} \frac{k \ln k}{\ln((t)_k k!)}.$$

By the Stolz-Cesàro theorem, we obtain that

$$\overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln((t)_k k!)} = \lim_{k\to\infty} \frac{(k+1)\ln(k+1) - k\ln k}{\ln((t)_{k+1}(k+1)!) - \ln((t)_k k!)}.$$

Hence

$$\varrho = \lim_{k\to\infty} \frac{\ln(1+\frac{1}{k})^k + \ln(k+1)}{\ln((t+k)(k+1))} = \frac{1}{2}.$$

By Hadamard's theorem on the representation of an entire function of finite order as an infinite product, we deduce that $\theta = 0$.

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